Dihedral subgroups of quasi-finite groups

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(the proof is extracted from Ol′shanskii's book «Geometry of Defining Relations in Groups» and is simplified in order to understand why S_3 cannot be a subgroup of a quasi-finite group)

Lemma 1. *Suppose that a group H is generated by involutions i and j. Then*
1) if $a = ij$ has order n then $H \simeq D(n)$;
2) if n is even then $a^{n/2}$ is an involution in $Z(H)$;
3) if n is odd then all involutions of H are conjugated by powers of a.

Definition 2. *If x and y are elements of a group then $x^y := y^{-1}xy$.*

Theorem 3. *Let x and y be distinct involutions of a quasi-finite group G. Then the order of xy is even.*

Proof. The proof is split into several steps. First, observe that G is not cyclic, so elements of G have finite orders.

Step 0. (general fact) If N is a proper normal subgroup of G then $N \subseteq Z(G)$. Indeed, we have $N_G(N) = G$. On the other hand, it is well-known that $N_G(N)/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$. Since N is finite, its automorphism group is finite. Therefore, $|G : C_G(N)| = |N_G(N) : C_G(N)| < \infty$ and hence $|C_G(N)| = \infty$. This implies that $C_G(N) = G$ and $N \subseteq Z(G)$.

Suppose now that x and y are involutions of G and $|xy|$ is odd.

Step 1. We may assume that $Z(G) = 1$. Indeed, consider $\overline{G} = G/Z(G)$. Since $xy \neq yx$, we have $|Z(G)| < \infty$ and hence $|\overline{G}| = \infty$. Clearly, all proper subgroups of $\overline{G}$ are finite and images of x and y are involutions of $\overline{G}$ whose product has an odd order. By Step 0, $\overline{G}$ is simple, so $Z(\overline{G}) = 1$.

Step 2. Denote $M = x \cdot x^G = \{x \cdot x^g \mid g \in G\}$. Then $|M| = \infty$ and for every $m \in M$ it is true that $m^x = m^{-1}$. Indeed, we have $|M| = |x^G| = |G : C_G(x)| = \infty$ and $m = xx^g = xg^{-1}xg$ for some $g \in G$. Now, we see that $m^x = xxg^{-1}xgx = g^{-1}xgx$ and so $m \cdot m^x = xg^{-1}xg \cdot g^{-1}xgx = 1$.

Step 3. Denote $N = \{xy^m \mid m \in M \text{ and } ord(xy^m) \text{ is odd}\}$. Then $|N| = \infty$. Indeed, as above we see that the set $\{xy^m \mid m \in M\}$ is infinite. Suppose that $|N| < \infty$. Then there exist infinitely many distinct elements xy^{m_i} of even orders $2k_i$, respectively. By Lemma 1, each $a_i = (xy^{m_i})^{k_i} \neq 1$ commutes with x. Since $|C_G(x)| < \infty$, we can find infinitely many elements a_i equal to a fixed element $g \in G$. However, if $a_i = g$ then $xy^{m_i} \in C_G(g)$ and hence $|G_G(g)| = \infty$; a contradiction with $Z(G) = 1$.

Step 4. If $g \in N$ then by the same argument as above we see that $g^x = g^{-1}$. By assumption $ord(xy)$ is odd, so by Lemma 1 there exists $a \in \langle x, y\rangle$ such that $x = y^a$. Consider arbitrary $c \in N$ and let $c = xy^m$, where $m \in M$. Since $ord(c)$ is odd, Lemma 1 implies that there exists $s \in \langle c\rangle$ such that $x = (y^m)^s = y^{ms}$. Therefore, we find that $y^{msa^{-1}} = y$ and hence $msa^{-1} \in C_G(y)$ or, equivalently, $ms \in C_G(y)a$. Since $|C_G(y)a| = |C_G(y)| < \infty$, we have some fixed $g \in C_G(y)a$ equals to $m_i s_i$ for infinitely many i. Observe that $g \neq 1$, since otherwise $a \in C_G(y)$ and $x = y^a = y$. Then $g = m_1 s_1 = m_i s_i$ and hence $(m_1 s_1)^x = (m_i s_i)^x$. Since s_1 is a power of c_1 and $c_1^x = c_1^{-1}$, we have $s_1^x = s_1^{-1}$. Similarly, we find that $s_i^x = s_i^{-1}$. So $m_1^{-1}s_1^{-1} = (m_1 s_1)^x = (m_i s_i)^x = m_i^{-1}s_i^{-1}$. Taking inverses of both sides, we find $s_1 m_1 = s_i m_i$. We claim that $m_1 m_i^{-1} \in C_G(g)$. Note that

$$g \cdot m_1 m_i^{-1} = m_1(s_1 m_1)m_i^{-1} = m_1(s_i m_i)m_i^{-1} = m_1 s_i;$$

$$m_1 m_i^{-1} \cdot g = m_1 m_i^{-1}(m_i s_i) = m_1 s_i.$$

Since $m_1 m_i^{-1}$ are distinct elements, we get a contradiction with $|C_G(g)| < \infty$. □