Spectral Sequences – Exercise sheet

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Exercise 1 (Short questions)**.** Are the following statements true or false? Give either a short proof or a counterexample.

1) Two spectral sequences converging towards the same graded module have to have isomorphic E_∞-pages.

2) A spectral sequence can converge towards two different (non-isomorphic) terms.

3) A spectral sequence whose E_∞-page has at most one non-trivial entry $E_\infty^{p,q}$ with $p+q=n$ for each $n \in \mathbb{N}$ hat as unique term it converges to.

4) If a spectral sequence converges towards zero, E_∞ has to be trivial.

5) If a spectral sequence converges towards zero, E_n has to be trivial for some $n \in \mathbb{N}$.

Hint: It is always useful to examine the extension problems given by convergence.

Exercise 2 (Homological algebra)**.** In this exercise we want to prove some basic statements form homological algebra using spectral sequences. Use the two spectral sequences of a double cochain complex in order to prove the following lemmata.

Four lemmata: Let

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \ell} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \xrightarrow[h']{} & D' \end{array}$$

be a commutative diagram with exact rows.

1. If i and k are epimorphisms and ℓ is a monomorphism, prove that j is an epimorphism.
2. If j and ℓ are monomorphisms and i is an epimorphism, prove that k is a monomorphism.

Hint: A double complex that is exact in one direction is always nice ...

Nine lemma: Let

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

be a commutative diagram with exact rows. If the first two columns are exact, prove that also the thrid is exact.

Snake lemma: Let

$$\begin{array}{ccccccccc}
 & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & &
\end{array}$$

be a commutative diagram with exact rows. Prove that there is an exact sequence

$$\ker a \longrightarrow \ker b \longrightarrow \ker c \longrightarrow \operatorname{coker} a \longrightarrow \operatorname{coker} b \longrightarrow \operatorname{coker} c.$$

Zig-zag lemma Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of cochain complexes. Prove that there exists a long exact sequence

$$\ldots \longrightarrow H^k(A) \longrightarrow H^k(B) \longrightarrow H^k(C) \longrightarrow H^{k+1}(A) \longrightarrow \ldots$$

Exercise 3 (Cohomology of $\mathbb{C}P^\infty$)**.** Apply the Serre spectral sequence to the circle bundle

$$S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$$

in order to compute the cohomology of $\mathbb{C}P^n$.
Hint: You will need that $H^k(S^n) \cong \mathbb{Z}$ for $k = 0, n$ and $H^k(S^n) = 0$ for $k \neq 0, n$.

Exercise 4 (five term exact sequence)**.** Let $n \in \mathbb{N}$ and let $E_2^{p,q}$ be a cohomological spectral sequence converging towards H^*. Assume that $E_\infty^{p,q} \cong 0$ for all $p, q > 0$ with $p + q = n - 1$. Prove that there exists a natural five-term exact sequence

$$0 \longrightarrow E_n^{n-1,0} \longrightarrow H^{n-1} \longrightarrow E_n^{0,n-1} \longrightarrow E_n^{n,0} \longrightarrow H^n.$$

Exercise 5 (Grothendieck spectral sequence)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be additive, left exact functors between abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and F takes injective objects to G-acyclic objects. Show that for every object A of $\mathcal{A}$ there is a spectral sequence

$$E_2^{p,q} = (R^pG \circ R^qF)(A) \implies R^{p+q}(G \circ F)(A).$$

Hint: Use a Cartan-Eilenberg resolution.

Exercise 6 (Universal coefficient theorem)**.** Let R be a ring, let C_* be a chain complex of projective R-modules and let M be an R-module.

1. Prove that there is a spectral sequence

$$E_2^{p,q} = \operatorname{Ext}_R^p\big(H_q(C_*), M\big) \Longrightarrow H^{p+q}\big(\operatorname{Hom}_R(C_*, M)\big).$$

   Hint: Take a Hom_R of C_* with a resolution of M to get a double complex. Then proceed as in the construction of the Grothendieck spectral sequence.

2. Deduce the universal coefficient theorem:

   Let R be a principal ideal domain. Then for each $n \in \mathbb{N}_{>0}$ there is a short exact sequence

$$0 \longrightarrow \operatorname{Ext}_R^1\big(H_{n-1}(C_*), M\big) \longrightarrow H^n\big(\operatorname{Hom}_R(C_*, M)\big) \longrightarrow \operatorname{Hom}_R\big(H_n(C_*), M\big) \longrightarrow 0.$$

Exercise 7 (Cohomology of the loops space)**.** Let $n \in \mathbb{N}$ and let X be a simply connected space. Prove inductively, that if $H^i(X) = 0$ for all $1 \le i \le n$ we have

$$H^{n+1}(X) \cong H^n(\Omega X),$$

where ΩX denotes the *loop space of* X.
Hint: Use the *path space fibration* $PX \to X$ with fiber ΩX and the fact that PX is contractible, i.e., has the cohomology of a point.

Exercise 8 ($\pi_4(S^3)$)**.** In this exercise we want to compute $\pi_4 = \pi_4(S^3)$.

1. First we want to see that S^3 is *not* a $K(\mathbb{Z}, 3)$: Use the path space fibration

$$\Omega K(\mathbb{Z}, 3) \longrightarrow PK(\mathbb{Z}, 3) \longrightarrow K(\mathbb{Z}, 3)$$

   in order to show

$$H^4\big(K(\mathbb{Z}, 3)\big) \cong H^5\big(K(\mathbb{Z}, 3)\big) \cong 0 \qquad \text{and} \qquad H^6\big(K(\mathbb{Z}, 3)\big) \cong \mathbb{Z}/2.$$

   Hint: Using the long exact sequence for homotopy groups one can see $\Omega K(\mathbb{Z}, 3) = K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$. Also use that the differentials on higher pages a compatible with the ring structure of $H^*(\mathbb{C}P^\infty)$.

2. From the Postnikov tower of S^3 we obtain a fibration

$$K(\pi_4, 4) \longrightarrow X_4 \longrightarrow K(\mathbb{Z}, 3).$$

Use the *homological* Serre spectral sequence in order to prove $\pi_4 \cong \mathbb{Z}/2$.

Hint: Using the computation of the the first part and the universal coefficient theorem one can deduce $H_5\big(K(\mathbb{Z}, 3)\big) \cong \mathbb{Z}/2$. Moreover, the space X_4 is obtained from S^3 by killing the homotopy groups in dimensions ≥ 5 by attaching cells of dimension ≥ 6. Thus the homology of X_4 and S^3 agree in degrees ≤ 5.

Exercise 9. Formulate and prove the cohomological versions of Exercises 3–7.

Useful Theorems and Definitions

Theorem (Cartan-Eilenberg resolution)**.** *Let $\mathcal{A}$ be an abelian category and let $(K^p)_{p\in\mathbb{N}}$ be a cochain complex. A* Cartan-Eilenberg resolution of K^* *is given by a double complex $(I^{p,q})_{p,q\in\mathbb{N}}$ and a morphism of complexes $\varepsilon\colon K^* \to I^{*,0}$ with the following properties:*

- *The complex $I^{p,*}$ is an injective resolution of K^p.*
- *The complex* $\ker(d^{p,*})$ *is an injective resolution of* $\ker(d^p_K)$*.*
- *The complex* $\operatorname{im}(d^{p,*})$ *is an injective resolution of* $\operatorname{im}(d^p_K)$*.*
- *The complex $H^p(I^{*,*}, d_v)$ is an injective resolution of $H^p(K^*)$.*

Some of the above exercises are of topological flavour. Thus some notions from algebraic topology will be useful. For example the following (simplified version) of the *Serre spectral sequence*:

Theorem (Serre spectral sequence)**.** *Let $p\colon E \to B$ be a fibration with simply connected base B and connected fiber F. Then there exists a spectral sequence*

$$E_2^{p,q} = H^p\big(B; H^q(F)\big) \Longrightarrow H^{p+q}(E).$$

Here $H^*(\,\cdot\,)$ denotes singular cohomology (possibly with coefficients).

Also the notion of *Eilenberg-MacLane spaces* and of *Postnikov towers* will be needed.

Definition (Eilenberg-MacLane space)**.** Let $n \in \mathbb{N}_{>0}$ and let G be a group (which is abelian if $n > 1$). Then an *Eilenberg-MacLane space of type* $K(G,n)$ is a connected topological space X such that

$$\pi_k(X) \cong \begin{cases} G & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

Such Eilenberg-MacLane spaces are unique up to weak homotopy equivalence.

By abuse of notation we usually denote such an Eilenberg-MacLane space by $K(G,n)$.

Definition (Postnikov tower)**.** Let X be a path-connected topological space. A *Postnikov tower* is an inverse system of spaces

$$\dots \longrightarrow X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{p_{n-1}} \dots \xrightarrow{p_2} X_1 \xrightarrow{p_1} \{*\}$$

with a sequence of maps $\varphi_n\colon X \to X_n$ compatible with the inverse system such that

1. The map φ_n induces an isomorphism $\pi_i(X) \to \pi_i(X_n)$ for every $i \leq n$.
2. $\pi_i(X_n) = 0$ for $i > n$.
3. Each map $p_n\colon X_n \to X_{n-1}$ is a fibration.

Combined these conditions also give that the fiber F_n of p_n is a $K\big(\pi_n(X), n\big)$.

It can be proven that Postnikov tower exist for every path-connected space.

Theorem (Hurewicz theorem)**.** *Let X be a path connected topological space. Then there is an isomorphism*

$$\big(\pi_1(X)\big)^{ab} \cong H_1(X).$$

Moreover, if $n \in \mathbb{N}_{>1}$ and X is n-connected, i.e., $\pi_i(X) \cong 0$ for all $i \leq n$, there is an isomorphism

$$\pi_{n+1}(X) \cong H_{n+1}(X).$$

Remark (proving Hurewicz)**.** Using the homological version of Exercise 7 together with the long exact sequnece of homotopy groups one can give a proof of the Hurewicz theorem.