# Yet Another Talk on Spectral Sequences

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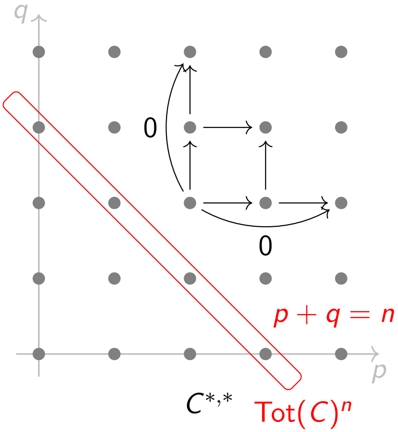
# Motivation

Let $C$ be a first quadrant double cochain complex, i.e., a bigraded module $(C^{p,q})_{p,q\in\mathbb{N}}$ together with differentials

$$d_v^{p,q}\colon C^{p,q} \longrightarrow C^{p,q+1} \quad \text{and} \quad d_h^{p,q}\colon C^{p,q} \longrightarrow C^{p+1,q}$$

such that

$$d_v \circ d_v = 0, \quad d_h \circ d_h = 0$$
$$\text{and} \quad d_v \circ d_h + d_h \circ d_v = 0.$$




We define the total complex Tot($C$) of $C$ to be given by

$$\mathrm{Tot}(C)^n = \bigoplus_{p+q=n} C^{p,q} \quad \text{and} \quad d = d_v + d_h.$$

# Motivation

## Question

How can we compute the cohomology of $\mathrm{Tot}(C)$?

## Wishful dream

Taking cohomology and total complex "commutes"! Meaning something like

$$H^n\big(\mathrm{Tot}(C)\big) = \bigoplus_{p+q=n} H^p\big(H^q(C, d_v), d_h\big).$$

To simplify notation let us define

$$T = \mathrm{Tot}(C), \qquad E_1^{p,q} = H^q(C^{p,*}, d_v) \quad \text{and} \quad E_2^{p,q} = H^p(E_1^{*,q}, d_h).$$

# Easy example

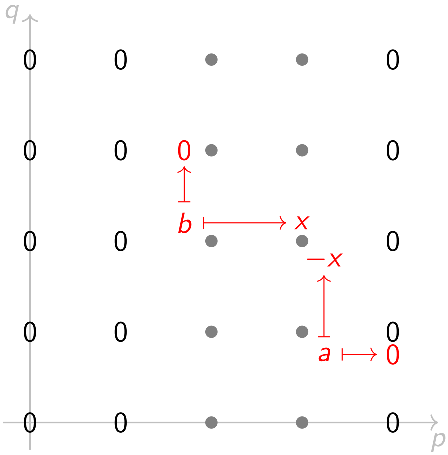
Let us consider the case that $C$ consists only of two non-trivial columns. An element in $H^{p+q}(T)$ can be represented by $a + b \in C^{p,q} \oplus C^{p-1,q+1}$.

Obviously $b$ defines an element in $E_1^{p-1,q+1}$ and we obtain moreover a map

$$0 \longrightarrow E_2^{p,q} \longrightarrow H^{p+q}(T) \longrightarrow E_2^{p-1,q+1} \longrightarrow 0$$

$$[a+b] \longmapsto [b]$$

$$[a] \longmapsto [a]$$




$C^{*,*}$

# Identification of $E_2^{p,q}$

In general we have

$$E_1^{p,q} = \frac{\left\{a \in C^{p,q} \,\middle|\, d_v(a) = 0\right\}}{\left\{d_v(x) \,\middle|\, x \in C^{p,q-1}\right\}},$$

and thus we get

$$\begin{aligned} E_2^{p,q} &= \frac{\left\{a \in C^{p,q} \,\middle|\, d_v(a) = 0, \exists_{b \in C^{p+1,q-1}} : d_h(a) = d_v(b)\right\}}{\left\langle \left\{d_v(x) \,\middle|\, x \in C^{p,q-1}\right\} \cup \left\{d_h(c) \,\middle|\, c \in C^{p,q-1}, d_v(c) = 0\right\} \right\rangle} \\ &\cong \frac{\left\{(a,b) \in C^{p,q} \times C^{p+1,q-1} \,\middle|\, d_v(a) = 0, d_h(a) + d_v(b) = 0\right\}}{\left\langle \left\{(0,c) \,\middle|\, d_v(c) = 0\right\} \cup \left\{(d_v(x), d_h(x))\right\} \cup \left\{(d_h(y), 0) \,\middle|\, d_v(y) = 0\right\} \right\rangle}. \end{aligned}$$

# Higher differentials

Using the identification

$$E_2^{p,q} \cong \frac{\left\{(a,b) \in C^{p,q} \times C^{p+1,q-1} \;\middle|\; d_v(a) = 0, d_h(a) + d_v(b) = 0\right\}}{\left\langle \{(0,c) \mid d_v(c) = 0\} \cup \{(d_v(x), d_h(x))\} \cup \{(d_h(y), 0) \mid d_v(y) = 0\} \right\rangle}$$

we see that

$$\begin{aligned} d_2^{p,q} \colon E_2^{p,q} &\longrightarrow E_2^{p+2,q-1} \\ [a,b] &\longmapsto [d_h(b), 0] \end{aligned}$$

gives a well defined differential.

# A five term exact sequence

In the special case $q = 0$ we have

$$E_2^{p,0} \cong \frac{\left\{a \in C^{p,0} \;\middle|\; d_v(a) = 0, d_h(a) = 0\right\}}{\left\{d_h(y) \;\middle|\; d_v(y) = 0\right\}}.$$

So in particular $E_2^{0,0} = H^0(T)$ and we obtain

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & E_2^{1,0} & \longrightarrow & H^1(T) & \longrightarrow & E_2^{0,1} & \xrightarrow{d_2} & E_2^{2,0} & \longrightarrow & H^2(T). \\
 & & [a] & \longmapsto & [a] & & & & [a] & \longmapsto & [a] \\
 & & & & & & [x,y] & \longmapsto & [d_h(y)] & & \\
 & & & & [x+y] & \longmapsto & [x,0] & & & &
\end{array}$$

Here the first map is injective and the kernel of the last one is the image of $d_2$. Moreover we can combined the two sequences to an exact sequence.

# What is a spectral sequence?

A (cohomological, first quadrant) spectral sequence consists of:

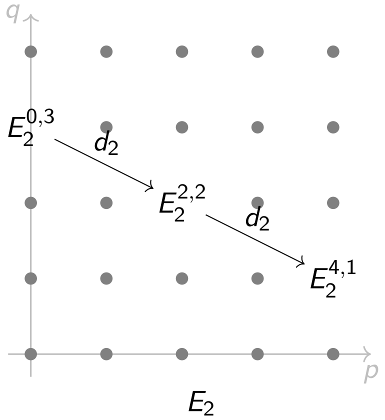
- For each $n \in \mathbb{N}$ a differential bigraded module

$$(E_n, d_n) = \left((E_n^{p,q})_{p,q\in\mathbb{N}}, (d_n^{p,q})_{p,q\in\mathbb{N}}\right),$$

  called the <u>$n$-th page</u> of the spectral sequence.
- Isomorphisms

$$E_{n+1}^{p,q} \cong H^{p,q}(E_n, d_n) = \frac{\ker d_n^{p,q}}{\operatorname{im} d_n^{p-n,q+n-1}},$$

  called <u>page turning isomorphisms</u>.



# Example

Let $C$ be a double cochain complex. Then

$$^{①}E_0^{p,q} = C^{p,q}, \qquad ^{①}E_1^{p,q} = H^q(C^{p,*}, d_v) \quad \text{and} \quad ^{①}E_2^{p,q} = H^p(E_1^{*,q}, d_h)$$

define the first few pages of a spectral sequence. The differentials on $E_0$ and $E_1$ are $d_v$ and $d_h$ respectively. On the $E_2$ the differentials are the $d_2$ we defined previously.

Similarly, by transposing the double cochain complex, i.e., setting $D^{p,q} = C^{q,p}$, $d_v^D = d_h^C$ and $d_h^D = d_v^C$, we obtain another spectral sequence with

$$^{②}E_0^{p,q} = C^{q,p}, \qquad ^{②}E_1^{p,q} = H^q(C^{*,p}, d_h) \quad \text{and} \quad ^{②}E_2^{p,q} = H^p(E_1^{q,*}, d_v).$$

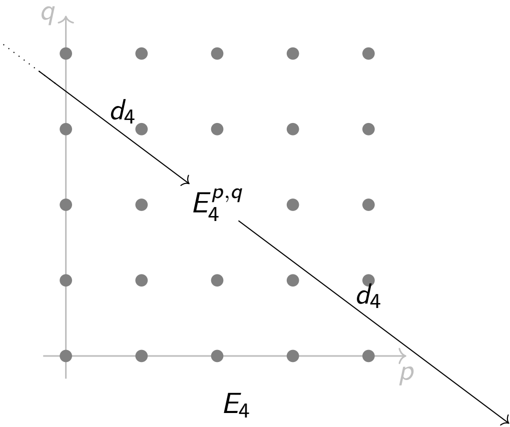
Fixing $p, q \in \mathbb{N}$, there is always an $N \in \mathbb{N}$ such that all differentials

$$E_n^{p-n,q+n-1} \xrightarrow{d_n^{p-n,q+n-1}} E_n^{p,q} \xrightarrow{d_n^{p,q}} E_n^{p+n,q-n+1},$$

with $n \geq N$, are trivial.

Thus the page turning isomorphisms give

$$E_\infty^{p,q} := E_N^{p,q} \cong E_{N+1}^{p,q} \cong \dots .$$




This leads to the definition of the $\infty$-page of a spectral sequence.

# Convergence of a spectral sequence

A spectral sequence is said to converge towards a graded module $(H_n)_{n\in\mathbb{N}}$ if there are short exact sequences

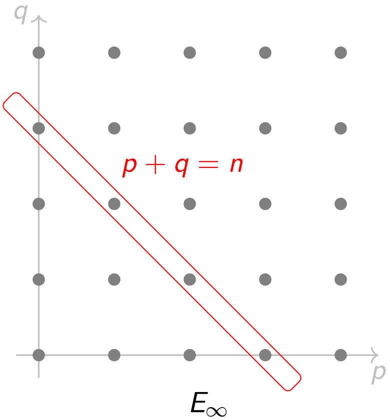
$$0 \longrightarrow 0 \longrightarrow F^nH^n \longrightarrow E_\infty^{n,0} \longrightarrow 0$$

$$0 \longrightarrow F^nH^n \longrightarrow F^{n-1}H^n \longrightarrow E_\infty^{n-1,1} \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F^1H^n \longrightarrow H^n \longrightarrow E_\infty^{0,n} \longrightarrow 0.$$

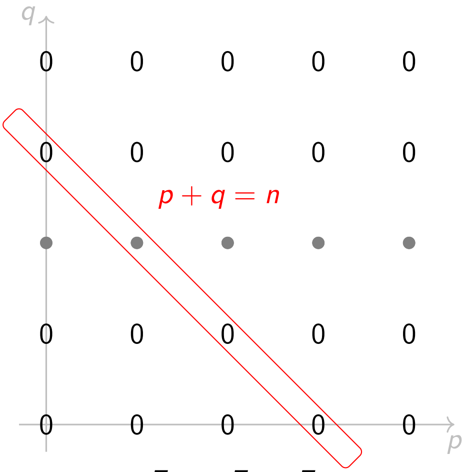
In this case we write $E_r^{p,q} \Longrightarrow H^{p+q}$.



# Example: collapsing spectral sequence

We say a spectral sequence <u>collapses at $E_r$</u> $(r \geq 2)$ if there is exactly one nonzero column or row on $E_r$.

In this case all of the extension problems required for convergence are trivial. Thus the spectral sequence converges towards $H^*$ where $H^n$ is the unique nonzero $E_r^{p,q}$ with $p + q = n$.




$$E_\infty = E_r = E_{r+1} = \dots$$

# Example: two columns

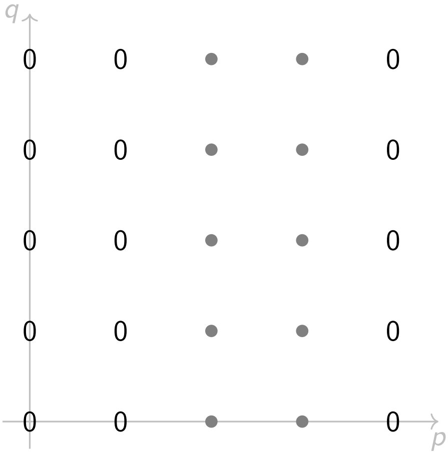
Consider again the case that $C$ is a double complex consisting of only two non-trivial columns.

We have already seen that there are short exact sequences

$$0 \longrightarrow E_2^{p,q} \longrightarrow H^{p+q}(T) \longrightarrow E_2^{p-1,q+1} \longrightarrow 0.$$

As $E_2 = E_\infty$ these sequences just describe the convergence of the spectral sequence:

$$E_2^{p,q} \Longrightarrow H^{p+q}(T).$$




$E_2 = E_\infty$

# The five term exact sequence

## Lemma

*Suppose* $E_2^{p,q} \Longrightarrow H^{p+q}$. *Then* $H^0 \cong E_2^{0,0}$ *and there is an exact sequence*

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1 \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \longrightarrow H^2.$$

## Proof.

By convergence towards $H^*$ we have $H^0 \cong E_\infty^{0,0}$ and a short exact sequence

$$0 \longrightarrow E_\infty^{1,0} \longrightarrow H^1 \longrightarrow E_\infty^{0,1} \longrightarrow 0.$$

Moreover, there is an exact sequence

$$0 \longrightarrow \ker d_2 \longrightarrow E_2^{1,0} \xrightarrow{d_2} E_2^{2,0} \longrightarrow E_2^{2,0} / \operatorname{im} d_2 \longrightarrow 0.$$

Combining these two exact sequences and using $E_3^{2,0} \cong F^2H^2 \subseteq H^2$ gives the claim. □

# The two spectral sequences of a double complex

**Theorem**

*Let $C$ be a double cochain complex. Then there are two spectral sequences*

$$^{①}E_1^{p,q} = H^q(C^{p,*}, d_v) \quad \text{and} \quad ^{②}E_1^{p,q} = H^q(C^{*,p}, d_h),$$

*with $d_1$ induced by $d_h$ and $d_v$ respectively. Both of these spectral sequences converge towards the cohomology of the total complex*

$$^{①}E_1^{p,q} = H^q(C^{p,*}, d_v) \Longrightarrow H^{p+q}\big(\mathrm{Tot}(C)\big)$$
$$^{②}E_1^{p,q} = H^q(C^{*,p}, d_h) \Longrightarrow H^{p+q}\big(\mathrm{Tot}(C)\big).$$

# Example: The Hochschild-Serre spectral sequence

**Theorem**

*Let $0 \to \Lambda \to \Gamma \to \Delta \to 0$ be a short exact sequence of groups. Then there exists a spectral sequence*

$$E_2^{p,q} = H^p\big(\Delta; H^q(\Lambda;\mathbb{Z})\big) \Longrightarrow H^{p+q}(\Gamma;\mathbb{Z}).$$

**Sketch of proof.**

The double complex

$$C^{p,q} = \mathrm{Hom}_{\mathrm{Set}}\big(\Delta^{q+1}, \mathrm{Hom}_{\mathrm{Set}}(\Gamma^{p+1},\mathbb{Z})^{\Lambda}\big)^{\Delta}$$

gives two spectral sequences converging towards the same term. The first one degenerates with

$${}^{①}E_2^{p,0} = H^p(\Gamma;\mathbb{Z}).$$

Thus we obtain for the second one

$${}^{②}E_2^{p,q} \cong H^p\big(\Delta; H^q(\Lambda;\mathbb{Z})\big) \Longrightarrow H^{p+q}(\Gamma;\mathbb{Z}).$$

□