Spectral Sequences – Exercise solutions

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Exercise 1 (Short questions)**.** Are the following statements true or false? Give either a short proof or a counterexample.

1) Two spectral sequences converging towards the same graded module have to have isomorphic E_∞-pages.

2) A spectral sequence can converge towards two different (non-isomorphic) terms.

3) A spectral sequence whose E_∞-page has at most one non-trivial entry $E_\infty^{p,q}$ with $p+q=n$ for each $n \in \mathbb{N}$ hat as unique term it converges to.

4) If a spectral sequence converges towards zero, E_∞ has to be trivial.

5) If a spectral sequence converges towards zero, E_n has to be trivial for some $n \in \mathbb{N}$.

Hint: It is always useful to examine the extension problems given by convergence.

Solution:

1) This statement is false: Consider the following two spectral sequences:

$$\begin{array}{c|ccccc} q & & & & & \\ & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & \mathbb{Z} & 0 & 0 & 0 & 0 \\ \hline & 0 & 0 & 0 & 0 & 0 & p \end{array} \qquad \begin{array}{c|ccccc} q & & & & & \\ & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ \hline & 0 & \mathbb{Z} & 0 & 0 & 0 & p \end{array}$$

E_2 $\qquad$ E_2

They are both degenerate and it is easy to see that they converge towards the same term.

2) This is true: Consider the following spectral sequence

$$
\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z}/2 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & 0
\end{array}
$$

E_2

In this case convergence towards H^* only asks for the existence of a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H^1 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

However there are at least two different such sequences:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

3) This is also true: Converging towards H^* is given by the extension problems

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & F^nH^n & \longrightarrow & E_\infty^{n,0} & \longrightarrow & 0 \\
0 & \longrightarrow & F^nH^n & \longrightarrow & F^{n-1}H^n & \longrightarrow & E_\infty^{n-1,1} & \longrightarrow & 0 \\
 & & & & \vdots & & & & \\
0 & \longrightarrow & F^2H^n & \longrightarrow & F^1H^n & \longrightarrow & E_\infty^{1,n-1} & \longrightarrow & 0 \\
0 & \longrightarrow & F^1H^n & \longrightarrow & H^n & \longrightarrow & E_\infty^{0,n} & \longrightarrow & 0.
\end{array}
$$

If there is only one non-trivial term $E_\infty^{p,q}$ with $p+q=n$, all of these extension problems become trivial and the only solution is given by $H^n \cong E_\infty^{p,q}$.

4) This is again true: All of the extension problems given by convergence are of the form

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow E_\infty^{p,q} \longrightarrow 0.$$

Thus all entries of the E_∞-page need to be trivial.

5) This is again false: Consider the spectral sequence with

$$E_2^{0,2n-1} = E_2^{n,n} = \mathbb{Z} \quad \text{for all } n \in \mathbb{N}_{>1} \qquad \text{and} \qquad E_2^{p,q} = 0 \quad \text{otherwise.}$$

We set all of the differentials to be trivial, except for

$$d_n^{0,2n-1} \colon E_n^{0,2n-1} \longrightarrow E_n^{n,n},$$

which should be the identity (or any isomorphism). Then the terms at position $(0, 2n-1)$ and (n, n) will vanish on the $(n+1)$-th page, and thus $E_\infty = 0$. However on the n-th page $E_n^{n,n}$ will always be non-trivial.

Exercise 2 (Homological algebra)**.** In this exercise we want to prove some basic statements form homological algebra using spectral sequences. Use the two spectral sequences of a double cochain complex in order to prove the following lemmata.

Four lemmata: Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[r, "g"] \arrow[d, "j"] & C \arrow[r, "h"] \arrow[d, "k"] & D \arrow[d, "\ell"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & D' \end{tikzcd}$$

be a commutative diagram with exact rows.

1. If i and k are epimorphisms and ℓ is a monomorphism, prove that j is an epimorphism.
2. If j and ℓ are monomorphisms and i is an epimorphism, prove that k is a monomorphism.

Hint: A double complex that is exact in one direction is always nice ...

Nine lemma: Let

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

be a commutative diagram with exact rows. If the first two columns are exact, prove that also the thrid is exact.

Snake lemma: Let

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' & \end{tikzcd}$$

be a commutative diagram with exact rows. Prove that there is an exact sequence

$$\ker a \longrightarrow \ker b \longrightarrow \ker c \longrightarrow \operatorname{coker} a \longrightarrow \operatorname{coker} b \longrightarrow \operatorname{coker} c.$$

Zig-zag lemma Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of cochain complexes. Prove that there exists a long exact sequence

$$\dots \longrightarrow H^k(A) \longrightarrow H^k(B) \longrightarrow H^k(C) \longrightarrow H^{k+1}(A) \longrightarrow \dots$$

Solution:

- We consider the two spectral sequences associated to the double complex (up to signs of the differentials)

$$\begin{tikzcd}
\ker f' \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & D' \arrow[r] & \operatorname{coker} h' \\
\ker f \arrow[r] & A \arrow[r, "f"] \arrow[u, "i"] & B \arrow[r, "g"] \arrow[u, "j"] & C \arrow[r, "h"] \arrow[u, "k"] & D \arrow[r] \arrow[u, "\ell"] & \operatorname{coker} h
\end{tikzcd}$$

  Since by construction the rows are exact, the spectral sequence obtained by taking the horizontal differentials first turns out to be trivial on the first page. The second spectral sequence, taking the vertical differentials, has as first page

$$\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\bullet & \operatorname{coker} i & \operatorname{coker} j & \operatorname{coker} k & \operatorname{coker} \ell & \bullet \\
\bullet & \ker i & \ker j & \ker k & \ker \ell & \bullet \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

E_1

  1. In the case that i and k are epimorphisms and ℓ is a monomorphism the first

page looks as follows:

$$
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\bullet & 0 & \operatorname{coker} j & 0 & \operatorname{coker} \ell & \bullet \\
\bullet & \ker i & \ker j & \ker k & 0 & \bullet \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

E_1

Thus *every* differential involving the entry $\operatorname{coker} j$ has to be trivial. But as the spectral sequence converges towards 0 the E_∞-page needs to be trivial. Hence $\operatorname{coker} j$ needs to be trivial, i.e. j is an epimorphism.

2. In the case that j and ℓ are monomorphisms and i is an epimorphism the first page looks as follows:

$$
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\bullet & 0 & \operatorname{coker} j & \operatorname{coker} k & \operatorname{coker} \ell & \bullet \\
\bullet & \ker i & 0 & \ker k & 0 & \bullet \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

E_1

This time every differential involving $\ker k$ is trivial. Thus $\ker k = 0$, so k is a monomorphism.

- We consider the spectral sequence associated to the double complex

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & A'' \arrow[r] \arrow[u] & B'' \arrow[r] \arrow[u] & C'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[u] & B' \arrow[r] \arrow[u] & C' \arrow[r] \arrow[u, "g"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & C \arrow[r] \arrow[u, "f"] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

As before we obtain two spectral sequences. Since the complex is exact in the horizontal direction they both converge towards zero. The spectral sequence taking the vertical differentials first has as first page

$$\begin{array}{c|ccccc}
q & & & & & \\
 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & \operatorname{coker} g & 0 \\
 & 0 & 0 & 0 & \ker g / \operatorname{im} f & 0 \\
 & 0 & 0 & 0 & \ker f & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & & & p
\end{array}$$

E_1

as the first two columns are exact. As all differentials on this and all higher pages are trivial, this is also the E_∞-page. But as the spectral sequence converges towards zero we obtain that the three terms have to be trivial, and thus the last column is exact as well.

- We consider the spectral sequence associated to the double complex

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & \operatorname{coker} g' \\
\ker f \arrow[r] & A \arrow[r, "f"'] \arrow[u, "a"] & B \arrow[r, "g"'] \arrow[u, "b"] & C \arrow[r] \arrow[u, "c"] & 0
\end{tikzcd}$$

As in the previous part the horizontal cohomology is trivial, thus the two spectral sequences converge to zero. The first page of the spectral sequence obtained by taking the vertical differentials first is

$$
\begin{tikzcd}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & \operatorname{coker} a \arrow[r, "p"] & \operatorname{coker} b & \operatorname{coker} c & \operatorname{coker} g \\
\ker f & \ker a & \ker b \arrow[r, "i"] & \ker c & 0 \\
0 & 0 & 0 & 0 & 0
\end{tikzcd}
$$

E_1

As all higher differentials are trivial and the spectral sequence converges toward 0, the second page looks like

$$
\begin{tikzcd}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & \ker p \arrow[drr, "d_2"] & 0 & 0 & 0 \\
0 & 0 & 0 & \operatorname{coker} i & 0 \\
0 & 0 & 0 & 0 & 0
\end{tikzcd}
$$

E_2

On the one hand this gives that the sequences

$$0 \longrightarrow \ker p \longrightarrow \operatorname{coker} a \longrightarrow \operatorname{coker} b \longrightarrow \operatorname{coker} c$$

and

$$\ker a \longrightarrow \ker b \longrightarrow \ker c \longrightarrow \operatorname{coker} i \longrightarrow 0$$

are exact. On the other hand this give that

$$d_2 \colon \ker p \longrightarrow \operatorname{coker} i$$

has to be an isomorphism. Then combining the above two sequences yields the desired kernel/cokernel sequence.

- We consider the short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ of cochain complexes as a double complex. As previously, the cohomology in horizontal direction is trivial. Thus the two spectral sequences converge to zero. The spectral sequence obtained by taking vertical differentials first has as first page

$$\begin{tikzcd}
0 \arrow[r] & H^4(A) \arrow[r, "i^*"] & H^4(B) \arrow[r, "p^*"] & H^4(C) \arrow[r] & 0 \\
0 \arrow[r] & H^3(A) \arrow[r, "i^*"] & H^3(B) \arrow[r, "p^*"] & H^3(C) \arrow[r] & 0 \\
0 \arrow[r] & H^2(A) \arrow[r, "i^*"] & H^2(B) \arrow[r, "p^*"] & H^2(C) \arrow[r] & 0 \\
0 \arrow[r] & H^1(A) \arrow[r, "i^*"] & H^1(B) \arrow[r, "p^*"] & H^1(C) \arrow[r] & 0 \\
0 \arrow[r] & H^0(A) \arrow[r, "i^*"] & H^0(B) \arrow[r, "p^*"] & H^0(C) \arrow[r] & 0
\end{tikzcd}$$

E_1

Then the second page will be

$$\begin{tikzcd}
0 & \ker i^* & 0 & \operatorname{coker} p^* & 0 \\
0 & \ker i^* \arrow[drr, "d_2"] & 0 & \operatorname{coker} p^* & 0 \\
0 & \ker i^* & 0 & \operatorname{coker} p^* & 0 \\
0 & \ker i^* & 0 & \operatorname{coker} p^* & 0 \\
0 & \ker i^* & 0 & \operatorname{coker} p^* & 0
\end{tikzcd}$$

E_2

and all the differentials $d_2 \colon \ker i^* \to \operatorname{coker} p^*$ have to be isomorphisms, which allows us to combine the exact sequences

$$0 \longrightarrow \ker i^* \longrightarrow H^n(A) \longrightarrow H^n(B) \longrightarrow H^n(C) \longrightarrow \operatorname{coker} p^* \longrightarrow 0$$

to a long exact sequence.

Exercise 3 (Cohomology of $\mathbb{C}P^\infty$)**.** Apply the Serre spectral sequence to the circle bundle

$$S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$$

in order to compute the cohomology of $\mathbb{C}P^n$.
Hint: You will need that $H^k(S^n) \cong \mathbb{Z}$ for $k = 0, n$ and $H^k(S^n) = 0$ for $k \neq 0, n$.

Solution: For simplicity we just write H^k instead of $H^k(\mathbb{C}P^n)$. Applying the Serre spectral sequence we obtain as second page

$$\begin{array}{c|ccccc}
q & & & & & \\
 & 0 & 0 & \cdots & 0 & 0 \\
 & 0 & 0 & \cdots & 0 & 0 \\
 & 0 & 0 & \cdots & 0 & 0 \\
 & \mathbb{Z} & H^1 & \cdots & H^{2n} & 0 \\
 & \mathbb{Z} & H^1 & \cdots & H^{2n} & 0 \\
\hline
 & & & & & p
\end{array}$$

E_2

where we used that $\mathbb{C}P^n$ has dimension $2n$ and thus $H^k = 0$ for $k \geq 2n+1$.

As the differentials on all higher pages are all trivial (they either start or end outside the first two rows), we have $E_3 = E_\infty$. Moreover, the convergence towards $H^*(S^{2n+1})$ gives short exact sequences

$$0 \longrightarrow E_3^{k-1,1} \longrightarrow H^k(S^{2n+1}) \longrightarrow E_3^{k,0} \longrightarrow 0$$

for each $k \in \mathbb{N}_{>1}$. As S^{2n+1} has only non-trivial cohomology in degrees 0 and $2n+1$, the only non-trivial entries on E_3 can be $E_3^{0,0} \cong \mathbb{Z}$, $E_3^{2n,1}$ and $E_3^{2n+1,0}$. However we already know that $E_2^{2n+1,0} = H^{2n+1}(\mathbb{C}P^n) = 0$. Thus also $E_3^{2n+1,0}$ has to be trivial. So the third page of our spectral sequence needs to be

$$\begin{array}{c|ccccc}
q & & & & & \\
 & 0 & 0 & \cdots & 0 & 0 \\
 & 0 & 0 & \cdots & 0 & 0 \\
 & 0 & 0 & \cdots & 0 & 0 \\
 & 0 & 0 & \cdots & \mathbb{Z} & 0 \\
 & \mathbb{Z} & 0 & \cdots & 0 & 0 \\
\hline
 & & & & & p
\end{array}$$

E_3

Using

$$0 = E_3^{k-1,1} = \ker(d_2^{k-1,1}) \big/ \operatorname{im}(d_2^{k-3,2}) = \ker(d_2^{k-1,1})$$

we see that $d_2^{k-1,1}$ needs to be injective for all $k \in \{1, \dots, 2n\}$. Moreover,

$$0 = E_3^{k+1,0} = \ker(d_2^{k+1,0}) \big/ \operatorname{im}(d_2^{k-1,1}) = H^{k+1} \big/ \operatorname{im}(d_2^{k-1,1})$$

shows that $d_2^{k-1,1}$ needs to be surjective for all $k \in \{0, \dots, 2n+1\}$. Thus the second page has to be

$$\begin{tikzcd}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} \arrow[drr, "\cong"] & 0 \arrow[drr, "\cong"] & \mathbb{Z} \arrow[drr, "\cong"] & 0 \arrow[dr] & \cdots \\
\mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots
\end{tikzcd}$$

E_2

and so

$$H^k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & \text{for } k \leq 2n \text{ even} \\ 0 & \text{else.} \end{cases}$$

Exercise 4 (five term exact sequence)**.** Let $n \in \mathbb{N}$ and let $E_2^{p,q}$ be a cohomological spectral sequence converging towards H^*. Assume that $E_\infty^{p,q} \cong 0$ for all $p, q > 0$ with $p + q = n - 1$. Prove that there exists a natural five-term exact sequence

$$0 \longrightarrow E_n^{n-1,0} \longrightarrow H^{n-1} \longrightarrow E_n^{0,n-1} \longrightarrow E_n^{n,0} \longrightarrow H^n.$$

Solution: Because $d_n^{0,n-1}$ is the last non-trivial differential involving $E_*^{0,n-1}$ and $E_*^{n,0}$ we have an exact sequence

$$0 \longrightarrow E_\infty^{0,n-1} \longrightarrow E_n^{0,n-1} \xrightarrow{d_n^{0,n-1}} E_n^{n,0} \longrightarrow E_\infty^{n,0} \longrightarrow 0. \tag{1}$$

By the convergence towards H^n we have extension problems

$$\begin{array}{c} 0 \longrightarrow 0 \longrightarrow F^0H^{n-1} \longrightarrow E_\infty^{n-1,0} \longrightarrow 0 \\ 0 \longrightarrow F^0H^{n-1} \longrightarrow F^1H^{n-1} \longrightarrow E_\infty^{n-2,1} \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow F^{n-2}H^{n-1} \longrightarrow H^{n-1} \longrightarrow E_\infty^{0,n-1} \longrightarrow 0 \end{array}$$

and the assumption on $E_\infty^{p,q}$ gives that most of these extension problems degenerate. Thus we obtain another exact sequence

$$0 \longrightarrow E_\infty^{n-1,0} \longrightarrow H^{n-1} \longrightarrow E_\infty^{0,n-1} \longrightarrow 0, \tag{2}$$

where in fact $E_\infty^{n-1,0} = E_n^{n-1,0}$. Finally again the the convergence gives extension problems

$$\begin{array}{c} 0 \longrightarrow 0 \longrightarrow F^0H^n \longrightarrow E_\infty^{n,0} \longrightarrow 0 \\ 0 \longrightarrow F^0H^n \longrightarrow F^1H^n \longrightarrow E_\infty^{n-1,1} \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow F^{n-1}H^n \longrightarrow H^n \longrightarrow E_\infty^{0,n} \longrightarrow 0, \end{array}$$

where $F^0H^n \subseteq H^n$. In particular we get the exact sequence

$$0 \longrightarrow E_\infty^{n,0} \longrightarrow H^n. \tag{3}$$

Combining the three exact sequences (1), (2) and (3) gives the desired sequence

$$0 \longrightarrow E_n^{n-1,0} \longrightarrow H^{n-1} \longrightarrow E_n^{0,n-1} \xrightarrow{d_n^{0,n-1}} E_n^{n,0} \longrightarrow H^n.$$

Exercise 5 (Grothendieck spectral sequence)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be additive, left exact functors between abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and F takes injective objects to G-acyclic objects. Show that for every object A of $\mathcal{A}$ there is a spectral sequence

$$E_2^{p,q} = (R^pG \circ R^qF)(A) \implies R^{p+q}(G \circ F)(A).$$

Hint: Use a Cartan-Eilenberg resolution.

Solution: Let A be an object of $\mathcal{A}$, and let $(C^p)_{p\in\mathbb{N}}$ be an injective resolution of A. Then $\big(F(C^p)\big)_{p\in\mathbb{N}}$ is a chain complex whose homology gives the right derived functors $R^pF(A)$. Taking a Cartan-Eilenberg resolution $I^{*,*}$ of $F(C^*)$ an applying the functor G we obtain a double complex $\big(G(I^{p,q})\big)_{p,q\in\mathbb{N}}$.

Associated to this double complex we obtain two spectral sequences. Taking the vertical differentials first we obtain

$${}^vE_1^{p,q} = H^q\big(G(I^{p,*}), d_v\big) = R^qG\big(F(C^p)\big),$$

as $I^{p,*}$ is an injective resolution of $F(C^p)$. By assumption F maps injective objects to G-acyclic ones, thus we get

$${}^vE_1^{p,q} \cong \begin{cases} G\big(F(C^p)\big) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

and thus

$${}^vE_2^{p,q} \cong \begin{cases} R^p(G \circ F)(A) & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, this spectral sequence degenerates and converges towards $R^{p+q}(G \circ F)(A)$.

On the other hand, taking the horizontal differentials first we get

$$\begin{aligned} {}^hE_1^{p,q} &= H^q\big(G(I^{*,p}), d_h\big) \\ &\cong G(Z^{q,p}) \big/ G(B^{q,p}) \\ &\cong G\big(H^q(I^{p,*}, d_v)\big), \end{aligned}$$

where we have used, that the short exact sequence

$$0 \longrightarrow \operatorname{im}(d_v^{p,q+1}) \longrightarrow \ker(d_v^{p,q}) \longrightarrow H^q(I^{p,*}, d^v) \longrightarrow 0$$

is split (all objects are injecitve!), and thus stays exact after applying G. Hence we finally obtain

$$\begin{aligned} {}^hE_2^{p,q} &\cong H^p\Big(G\big(H^q(I^{p,*}, d_v), d_h\big)\Big) \\ &\cong R^pG\big(R^qF(A)\big) \\ &= (R^pG \circ R^qF)(A), \end{aligned}$$

as $H^q(I^{*,*}, d_v)$ is an injective resolution of $H^q\big(F(C^*)\big) = R^qF(A)$.

Since both of these spectral sequences converge towards the same term, we obtain the desired spectral sequence

$$E_2^{p,q} = (R^pG \circ R^qF)(A) \implies R^{p+q}(G \circ F)(A).$$

Exercise 6 (Universal coefficient theorem)**.** Let R be a ring, let C_* be a chain complex of projective R-modules and let M be an R-module.

1. Prove that there is a spectral sequence
$$E_2^{p,q} = \operatorname{Ext}^p_R\big(H_q(C_*), M\big) \Longrightarrow H^{p+q}\big(\operatorname{Hom}_R(C_*, M)\big).$$
Hint: Take a Hom_R of C_* with a resolution of M to get a double complex. Then proceed as in the construction of the Grothendieck spectral sequence.

2. Deduce the universal coefficient theorem:
Let R be a principal ideal domain. Then for each $n \in \mathbb{N}_{>0}$ there is a short exact sequence
$$0 \longrightarrow \operatorname{Ext}^1_R\big(H_{n-1}(C_*), M\big) \longrightarrow H^n\big(\operatorname{Hom}_R(C_*, M)\big) \longrightarrow \operatorname{Hom}_R\big(H_n(C_*), M\big) \longrightarrow 0.$$

Solution:

1. Taking a injective resolution (I_*) of M we define the double complex $\operatorname{Hom}_R(C_p, I_q)_{p,q\in\mathbb{N}}$ with the obvious differentials (up to sign). As the C_p are projective the complexes $\operatorname{Hom}_R(C_p, I_*)$ are still exact. Thus the vertical spectral sequence converges trivially towards $\operatorname{Hom}_R(C_p, M)$. For the other spectral sequence we see that
$$E_1^{p,q} \cong H^q\big(\operatorname{Hom}_R(C_*, I_p)\big) \cong \operatorname{Hom}_R\big(H_q(C_*), I_p\big)$$
and thus
$$E_2^{p,q} \cong \operatorname{Ext}^p\big(H_q(C_*), M\big).$$

2. For R a principal ideal domain, all $\operatorname{Ext}^{n\geq 2}_R$-terms are trivial. Thus the second page of the above spectral sequence has only two non-trivial columns: $\operatorname{Hom}_R\big(H_*(C), M\big)$ as the zeroth column and $\operatorname{Ext}^1_R\big(\mathbb{H}_*(C), M\big)$ as the first column. As all differentials fall outside this two column band this is also the E_∞-page. Then convergence towards $H^n\big(\operatorname{Hom}(C_*, M)\big)$ gives the desired short exact sequences.

Exercise 7 (Cohomology of the loops space)**.** Let $n \in \mathbb{N}$ and let X be a simply connected space. Prove inductively, that if $H^i(X) = 0$ for all $1 \leq i \leq n$ we have

$$H^{n+1}(X) \cong H^n(\Omega X),$$

where ΩX denotes the *loop space of* X.
Hint: Use the *path space fibration* $PX \to X$ with fiber ΩX and the fact that PX is contractible, i.e., has the cohomology of a point.

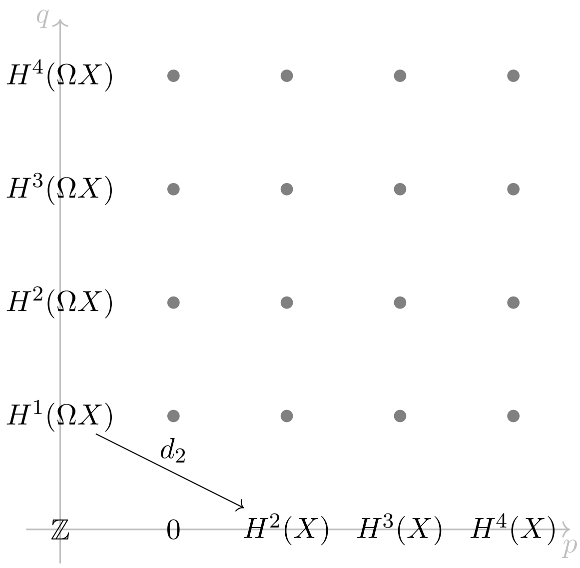
Solution: Applying the Serre spectral sequence to the path space fibration we obtain a spectral sequence

$$E_2^{p,q} = H^p\big(X; H^q(\Omega X)\big) \Longrightarrow H^{p+q}(PX) \cong \begin{cases} \mathbb{Z} & \text{if } p+q=0 \\ 0 & \text{esle.} \end{cases}$$

Since X is simply connected, ΩX is path connected. Thus

$$E_2^{p,0} \cong H^p(X) \qquad \text{and} \qquad E_2^{0,q} \cong H^q(\Omega X).$$

Now for the base case $n = 1$ the second page of the spectral sequence is



E_2

Since PX is contractible all E_∞-terms (except for $E_\infty^{0,0}$) are trivial. In particular

$$0 = E_\infty^{0,1} \cong \ker d_2^{0,1} \qquad \text{and} \qquad 0 = E_\infty^{2,0} \cong H^2(X) / \operatorname{im} d_2^{0,1},$$

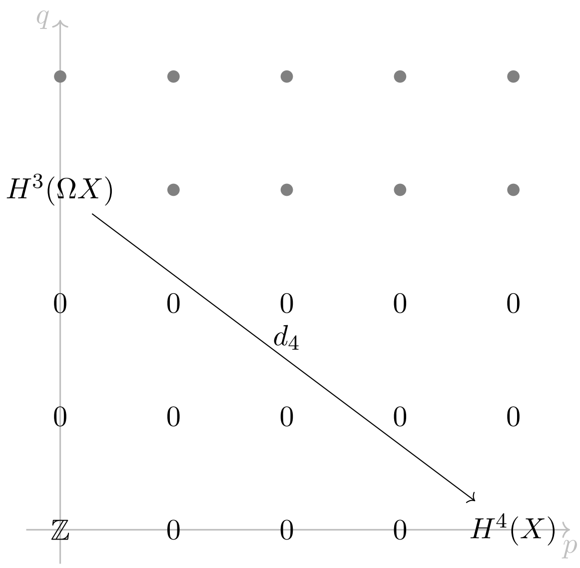
which shows that $d_2^{0,1} \colon H^1(\Omega X) \to H^2(X)$ is an isomorphism.

For the induction step we now assume that $H^i(X) \cong 0$ for $1 \leq i \leq n$ and that $H^i(\Omega X) \cong H^{i+1}(X) \cong 0$ for $1 \leq i \leq n-1$. Using the identification

$$E_2^{p,q} = H^p(X; H^q(\Omega X))$$

we obtain that the rows $1, \ldots, n-1$ are all trivial. This behaviour also persists on the higher pages of the spectral sequence.

Then the $(n+1)$-th page looks as follows (here for $n = 3$)



E_4

As before, convergence gives

$$0 = E_\infty^{0,n} \cong \ker d_{n+1}^{0,n} \qquad \text{and} \qquad 0 = E_\infty^{n+1,0} \cong H^{n+1}(X) / \operatorname{im} d_{n+1}^{0,n},$$

which proves that $d_{n+1}^{0,n} \colon H^n(\Omega X) \to H^{n+1}(X)$ is an isomorphism.

Exercise 8 ($\pi_4(S^3)$)**.** In this exercise we want to compute $\pi_4 = \pi_4(S^3)$.

1. First we want to see that S^3 is *not* a $K(\mathbb{Z}, 3)$: Use the path space fibration
$$\Omega K(\mathbb{Z},3) \longrightarrow PK(\mathbb{Z},3) \longrightarrow K(\mathbb{Z},3)$$
in order to show
$$H^4\big(K(\mathbb{Z},3)\big) \cong H^5\big(K(\mathbb{Z},3)\big) \cong 0 \qquad \text{and} \qquad H^6\big(K(\mathbb{Z},3)\big) \cong \mathbb{Z}/2.$$
Hint: Using the long exact sequence for homotopy groups one can see $\Omega K(\mathbb{Z},3) = K(\mathbb{Z},2) = \mathbb{C}P^\infty$. Also use that the differentials on higher pages a compatible with the ring structure of $H^*(\mathbb{C}P^\infty)$.

2. From the Postnikov tower of S^3 we obtain a fibration
$$K(\pi_4, 4) \longrightarrow X_4 \longrightarrow K(\mathbb{Z},3).$$
Use the *homological* Serre spectral sequence in order to prove $\pi_4 \cong \mathbb{Z}/2$.
Hint: Using the computation of the the first part and the universal coefficient theorem one can deduce $H_5\big(K(\mathbb{Z},3)\big) \cong \mathbb{Z}/2$. Moreover, the space X_4 is obtained from S^3 by killing the homotopy groups in dimensions ≥ 5 by attaching cells of dimension ≥ 6. Thus the homology of X_4 and S^3 agree in degrees ≤ 5.

Solution:

1. Applying the Serre spectral sequence gives
$$E_2^{p,q} = H^p\big(K(\mathbb{Z},3); H^q(\mathbb{C}P^\infty)\big) \Longrightarrow H^{p+q}\big(PK(\mathbb{Z},3)\big) \cong H^{p+q}(\{*\}).$$
Using Hurewicz and the universal coefficient theorem we can identify the second page with

$$\begin{tikzcd}
\mathbb{Z}\langle X^2\rangle \arrow[ddrrr, "d_3^{0,4}"] & 0 & 0 & \mathbb{Z}\langle X^2\rangle \otimes \mathbb{Z} & A & B & C \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z}\langle X\rangle \arrow[ddrrr, "d_3^{0,2}"] & 0 & 0 & \mathbb{Z}\langle X\rangle \otimes \mathbb{Z} \arrow[ddrrr, "d_3^{3,2}"] & A & B & C \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} & A & B & C
\end{tikzcd}$$

E_3

where $A \cong H^4\big(K(\mathbb{Z},3)\big), B \cong H^5\big(K(\mathbb{Z},3)\big)$ and $C \cong H^6\big(K(\mathbb{Z},3)\big)$. Because of the direciton of the differentials this is also the third page.

Since the spectral sequence converges towards the homology of a contractible space, the E_∞-page is trivial, except for $E_\infty^{0,0}$. As all differentials involving A are trivial, we get $A \cong 0$. Now let us investigate the differentials of the third page a bit more. We know that $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[X]$ where $\deg X = 2$. Because $E_\infty^{0,2} = E_\infty^{3,0} = 0$ we get that $d_3^{0,2}$ has to be an isomorphism. In particular it sends the generator X to a generator a of $\mathbb{Z}$.

As $d_3^{0,4}$ and $d_3^{3,2}$ are the last non-trivial differentials involving the entries at $(3,2)$ and $(6,0)$ we obtain that

$$\mathbb{Z}\langle X^2\rangle \xrightarrow{d_3^{0,4}} \mathbb{Z}\langle X\rangle \otimes \mathbb{Z} \xrightarrow{d_3^{3,2}} C \longrightarrow 0$$

is exact. Using that d_3 is compatible with the ring structure (i.e. is a derivation) we get

$$d_2^{0,4}(X^2) = 2X \otimes d_2^{0,2}(X) = 2X \otimes a.$$

In particular we get that $d_2^{0,2}$ is injective and that the sequnece above is a actually a short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \longrightarrow C \longrightarrow 0.$$

Hence $C \cong \mathbb{Z}/2$.

Finally as $d_2^{0,2}$ is injective we have $E_4^{0,2} = 0$. So also all higher differentials involving B are trivial and thus $B \cong 0$.

2. The homological Serre spectral sequence gives

$$E^2_{p,q} = H_p\big(K(Z,3); H_q\big(K(\pi_4,4)\big)\big) \Longrightarrow H_{p+q}(X_4).$$

Using Hurewicz and the (homological) universal coefficient theorem we get

$$
\begin{array}{c|cccccc}
q & & & & & & \\
\pi_4 & \bullet & \bullet & \bullet & \bullet & \bullet & \\
0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & \\
\hline
\mathbb{Z} & 0 & 0 & \mathbb{Z} & \bullet & \mathbb{Z}/2 & p
\end{array}
$$

E^2

The differential

$$d^5_{5,0} \colon \mathbb{Z}/2 \longrightarrow \pi_4$$

is the only non-trivial differential involving the entries at $(0, 4)$ and $(5, 0)$. Since $H_4(X_4) \cong H_5(X_4) \cong 0$ we obtain that this differential has to be an isomorphism, so $\pi_4 \cong \mathbb{Z}/2$.