Grothendieck Pairs and Natural Profinite Invariants

Daniel Echtler

Heinrich Heine University Düsseldorf

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Profinite Invariance

Question

Let Γ be a finitely generated residually finite group. Which properties of Γ are determined by its finite quotiens, or equivalently the profinite completion $\widehat{\Gamma}$?

So if Γ and Λ are two finitely generated residually finite groups with $\widehat{\Gamma} \cong \widehat{\Lambda}$, which properties do Γ and Λ share?

We call such a property a *profinite property* or a *profinite invariant*.

Example

- Being abelian is a profinite property.
- More generally satisfying any group law is a profinite property.
- Using the classification of finitely generated abelian groups, one can show that the abelianization is a profinite invariant.

Grothendieck Pairs

When proving that a property P is *not* profinite, we need to find two finitely generated residually finite groups Γ, Λ with $\widehat{\Gamma} \cong \widehat{\Lambda}$ such that one has property P and the other one does not.

One way of find such examples are *Grothendieck pairs*, i.e., group homomorphisms $\phi\colon \Lambda \to \Gamma$ such that $\widehat{\phi}\colon \widehat{\Lambda} \to \widehat{\Gamma}$ is an isomorphism.

Example (Examples where Grothendieck paris where used)

- Amenability is not a profinite property (Kionke–Schesler 2023).
- Property FA and other fixed point properties are not profinite (Bridson 2024).
- Having vanishing stable commutator length is not profinite (Fournier-Facio 2026, preprint).

Given a profinite invariant I and a Grothendieck pair $\phi\colon \Lambda \to \Gamma$, we only know that $I(\Lambda)$ and $I(\Gamma)$ agree. However, they do not need to agree "via ϕ".

Natural Profinite Invariants

Definition

Let $F\colon \text{Group} \to \mathcal{D}$ be a functor. We say that F is a *natural profinite invariant*, if for every Grothendieck pair $\phi\colon \Lambda \to \Gamma$ the induced morphism $F(\phi)\colon F(\Lambda) \to F(\Gamma)$ is an isomorphism.

Example

- By definition the profinite completion is a natural profinite invariant.
- One can show by hand that the abelianization is a natural profinite invariant.
- Because there are non-trivial Grothendieck pairs, the Forgetful functor Group $\to$ Set is *not* a natural profinite invariant. However, it is a profinite invariant.
- The higher homology groups $H_{n\geq 2}(\,\cdot\,;\mathbb{Z})\colon \text{Group} \to \text{Ab}$ are neither profinite invariants nor natural profinite invariants.

Functor	profinite?	natural profinite?
$\widehat{\cdot}\colon \mathsf{Group} \to \mathsf{ProFin}$	✓	✓
$(\,\cdot\,)_{\mathrm{ab}}\colon \mathsf{Group} \to \mathsf{Ab}$	✓	✓
$\mathrm{Forget}\colon \mathsf{Group} \to \mathsf{Set}$	✓	✗
$H_{n\geq 2}(\,\cdot\,;\mathbb{Z})\colon \mathsf{Group} \to \mathsf{Ab}$	✗	✗
$\mathrm{Bohr}\colon \mathsf{Group} \to \mathsf{CHGroup}$	✗	✓

What is the *Bohr compactification*?