Minicourse on Spectral Sequences in Bounded Cohomology

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Introduction

Bounded cohomology is a variant of singular cohomology where instead of all cocycles only uniformly bounded cocycles are considered. It was first introduced by Johnson [12] in order to study Banach algebras. Only later on when Gromov [8] extended the theory to topological spaces, and proved many of its fundamental properties, it developed into its own area of research with applications in geometry and group theory. Both Ivanov [10] and Noskov [20] further generalised the theory by considering twisted coefficients. Moreover, Ivanov gave a description of bounded cohomology using methods from homological algebra.

One very powerful tool for the computation of (co)homology are spectral sequences. They where invented by Leray [14, 15] during the second world war and then further developed by Koszul [13] and Cartan [4, 5].

This minicourse is intended as an introduction to spectral sequences with a special consideration of spectral sequences in bounded cohomology. Since it is/was given as part of the *International young seminar on Bounded Cohomology and Simplicial Volume.*[1], where most of the audience is familiar with the theory of bounded cohomology but not the theory of spectral sequences, we will first spend some time introducing spectral sequences and giving some examples. Only later on we will come to the "application" of spectral sequences in bounded cohomology.

One of the result we will see in this course is the construction of the Hochschild-Serre spectral sequence in bounded cohomology with semi-normed coefficients:

Theorem (Hochschild-Serre spectral sequence in bounded cohomology)**.** *Let R be a normed ring, let*

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0$$

be a short exact sequence of groups and let V be a semi-normed $R[\Gamma]$-module. Then there exists a cohomological first-quadrant spectral sequence (E_*, d_*) converging to the bounded cohomology of Γ with coefficients in V

$$E_2^{p,q} \Longrightarrow H_b^{p+q}(\Gamma; V).$$

[1] Due to the long and unwieldy title of this seminar we henceforth refer to it as the *IYSBC-SV*.

Moreover, there are "good" cases where we have the identification

$$E_2^{p,q} \cong_R H_b^p\big(\Delta; H_b^q(\Lambda; V)\big)$$

for some $p, q \in \mathbb{N}$.

The Hochschild-Serre spectral sequence in bounded cohomology was first constructed by Noskov [21] for the case of Banach coefficients. Later Monod [18] generalised the result to work with *continuous* bounded cohomology.

We will see that this spectral sequence can for example be used to prove the characterisation of boundedly n-acyclic morphisms by Moraschini and Raptis [19].

The structure of this course In Chapter 1 will first discuss some of the classical theory of spectral sequences. We begin by defining spectral sequences and the notion of convergence of first-quadrant spectral sequences. To see how spectral sequences can helpful, we then continue with some examples of spectral sequences and their applications. As a last point in this chapter we discuss how spectral sequences can be constructed out of *filtered complexes* and *double complexes*.

Then only in Chapter 2 we discuss the above applications of spectral sequences to the theory of bounded cohomology.

Prerequisites Since this course is/was given in the context of the IYSBC-SV, we will not cover any of the basic theory of bounded cohomology. For this we refer the reader to the book of Frigerio [7].

However, for our introduction on spectral sequences we do not require any special prerequisites and only use some basic homological algebra.

A few remarks on notation The natural numbers $\mathbb{N}$ include 0. All rings are assumed to have a unit. (Co)chain complexes are, unless stated otherwise, always assumed to be $\mathbb{N}$-indexed. For a non-empty, path-connected topological space X we just write $\pi_1(X)$ for "the" fundamental group of X and omit the basepoint. Similarly, we omit the basepoint in the notation of higher homotopy groups.

A note on literature As the origin of this course is the authors masters thesis [6] and we will cover a very general case of bounded cohomology with coefficients in *semi-normed modules* over *normed rings* it might occasionally[2] happen that we refer to some results about this very general case of bounded cohomology in this thesis. However most of these results should also be found in the books of Frigerio [7] (in the case of normed modules over $\mathbb{R}$ or $\mathbb{Z}$) and Monod [18] (in the case of Banach modules over $\mathbb{R}$ or $\mathbb{C}$) and can be adapted without much modification of the proofs.

[2]This "might occasionally" should be read as "will definitely".

Contents

Chapter 1

Spectral sequences

Spectral sequences are a very helpful and powerful tool of homological algebra to compute some *graded module*, i.e., a family of modules $(H_n)_{n\in\mathbb{N}}$, most notably the the (co)homology of a (co)chain complex.

We first introduce the notion of spectral sequences and the convergence of first-quadrant spectral sequences. In order to see that spectral sequences have indeed useful applications we will then discuss some examples of spectral sequences and their applications. Finally we discuss how one can construct spectral sequences using *filtered cochain complexes* and *double complexes*.

Overview of this chapter

1.1 The notion of spectral sequences

We begin by introducing the notion of spectral sequence of R-modules. Although we generally define what a spectral sequence is, we restrict our coverage of the further theory to the case of *first-quadrant spectral sequences*, which is sufficient for most applications and greatly reduces the complexity of the theory. The more general theory is for example studied in the books of McCleary [17], Rotman [23, Chapter 10], or Weibel [25, Chapter 5], where the latter discusses the most general case of spectral sequences in *abelian categories*.

1.1.1 Spectral sequences

Before we come to the definition of spectral sequences we first introduce some auxiliary definitions.

Definition 1.1.1 (bigraded module, morphism of bigraded modules)**.** Let R be a ring.

- A family $E = (E^{p,q})_{p,q\in\mathbb{Z}}$ of R-modules is called a *bigraded R-module*. For $p, q \in \mathbb{Z}$ we call (p, q) the *bidegree* of $E^{p,q}$ and $p + q$ the *total degree* of $E^{p,q}$.

- For two bigraded R-modules E and F a *morphism of bigraded R-modules* $f\colon E \to F$ is a family $(f^{p,q}\colon E^{p,q} \to F^{p,q})_{p,q\in\mathbb{Z}}$ of R-module homomorphisms.

- If $f\colon E \to F$ and $g\colon F \to G$ are two morphisms of bigraded R-modules we define the composition $g \circ f\colon E \to G$ to be the family

$$g \circ f := (g^{p,q} \circ f^{p,q}\colon E^{p,q} \to G^{p,q})_{p,q\in\mathbb{Z}}.$$

- A morphism of bigraded R-modules $f\colon E \to F$ is called an *isomorphism* if $f^{p,q}$ is an isomorphism for each $p, q \in \mathbb{Z}$. If there exists an isomorphism $f\colon E \to F$ we call E and F *isomorphic* and write $E \cong_R F$.

Besides morphisms of bigraded modules that keep the bidegree constant we can also consider morphisms that change the bidegree uniformly. This leads to the following definition.

Definition 1.1.2 (bigraded morphisms of bigraded modules)**.** Let R be a ring

- Let E, F be bigraded R-modules and let $r, s \in \mathbb{Z}$. A *bigraded morphism of bigraded R-modules $f\colon E \to F$ of bidegree* (r, s), or *bigraded morphism of bidegree* (r, s) for short, is a family

$$(f^{p,q}\colon E^{p,q} \to F^{p+r,q+s})_{p,q\in\mathbb{Z}}$$

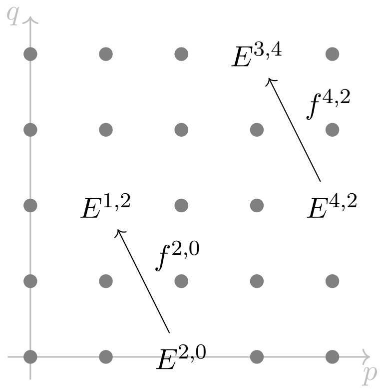


Figure 1.1: Visualisation of a bigraded morphism $f\colon E \to E$ of bidegree $(-1, 2)$

of R-linear morphisms.

- If $f\colon E \to F$ and $g\colon F \to G$ are bigraded morphisms of bidegree (r, s) and (r', s'), respectively, we define their composition $g \circ f\colon E \to G$ to be the bigraded morphism of bidegree $(r + r', s + s')$ given by the family

$$(g^{p+r,q+s} \circ f^{p,q}\colon E^{p,q} \to G^{p+r+r',q+s+s'})_{p,q\in\mathbb{Z}}.$$

With this definition morphisms of bigraded modules are simply bigraded morphisms of bidegree $(0, 0)$.

If we are now given a bigraded endomorphism $d\colon E \to E$ of bidegree (r, s) such that each "line of slope (r, s)"

$$\dots \xrightarrow{d^{p-2r,q-2s}} E^{p-r,q-s} \xrightarrow{d^{p-r,q-s}} E^{p,q} \xrightarrow{d^{p,q}} E^{p+r,q+s} \xrightarrow{d^{p+r,q+s}} \dots$$

is a complex we obtain the notion of a *differential bigraded module.*

Definition 1.1.3 (differential bigraded module)**.** Let R be a ring and let $r, s \in \mathbb{Z}$.

- A *differential bigraded R-module of bidegree* (r, s) is a pair (E, d) of a bigraded R-module E together with a bigraded morphism $d\colon E \to E$ of bidegree (r, s), called *differential* such that $d \circ d = 0$, i.e., $(d \circ d)^{p,q} = 0$ for all $p, q \in \mathbb{Z}$.
- If (E, d) is a differential bigraded R-module of bidegree (r, s) we define its *homology* to be the bigraded R-module

$$H(E, d) := \left(H^{p,q}(E, d) := \frac{\ker d^{p,q}}{\operatorname{im} d^{p-r,q-s}}\right)_{p,q\in\mathbb{Z}}.$$

- For two differential bigraded R-modules $(E, d_E), (F, d_F)$ of the same bidegree, a *morphism of differential bigraded R-modules* $f\colon (E, d_E) \to (F, d_F)$ is a morphism of bigraded R-modules $f\colon E \to F$ which commutes with the differentials, i.e., such that

$$d_F \circ f = f \circ d_E.$$

By the compatibility of a morphism of differential bigraded modules

$$f\colon (E, d_E) \longrightarrow (F, d_F)$$

with the differentials, it is easy to see that f induces a well-defined morphism of bigraded modules

$$H(f)\colon H(E, d_E) \longrightarrow H(F, d_F).$$

Example 1.1.4 (differential bigraded module of bidegree $(1,0)$)**.** Let R be a ring and let (E, d) be a differential bigraded R-module of bidegree $(1, 0)$. Since we have by definition that $d^{p+1,q} \circ d^{p,q} = 0$ for each $p, q \in \mathbb{Z}$, each "row" $E^{*,q}$ turns, with $d^{*,q}$ as coboundary operator, into an $\mathbb{Z}$-indexed R-cochain complex.

Conversely if we are given a family $\big((C^*_q, \delta^*_q)\big)_{q\in\mathbb{Z}}$ of $\mathbb{Z}$-indexed R-cochain complexes, we can assemble them into a differential bigraded module of bidegree $(1, 0)$, such that the rows are given by these cochain complexes, by letting

$$E^{p,q} = C^p_q \qquad \text{and} \qquad d^{p,q} = \delta^p_q.$$

It is easy to see that the homology of this differential bigraded module is simply given by the cohomology of the cochain complexes (C^*_q, δ^*_q), i.e., that

$$H^{p,q}(E, d) = H^p(C^*_q, \delta^*_q).$$

In particular we can turn a single R-cochain complex (C^*, δ^*) into a differential bigraded module of bidegree $(1, 0)$ such that the 0-th row $E^{*,0}$ is given by this cochain complex and all other entries and differentials are trivial.

Similarly we can turn every differential bigraded module of bidegree $(-1, 0)$ into a family of chain complexes and vice-versa.

We finally come to the definition of a spectral sequence.

Definition 1.1.5 (spectral sequence)**.** Let R be a ring and let $a \in \mathbb{N}$. A *(cohomological) spectral sequence (starting with E_a)* consists of a family $(E_r, d_r)_{r\in\mathbb{N}_{\geq a}}$ of differential bigraded R-modules (E_r, d_r) of bidegree $(r, 1-r)$ together with isomorphisms $H(E_r, d_r) \cong_R E_{r+1}$ for each $r \in \mathbb{N}_{\geq a}$.

For $r \in \mathbb{N}_{\geq a}$ we call E_r the *r-th page* of the spectral sequence and the isomorphism $E_{r+1} \cong_R H(E_r, d_r)$ the *r-th page-turning isomorphism.*

Usually we will denote a spectral sequence just by the family of (E_*, d_*) of differential bigraded modules, which, for simplicity, we assume to be $\mathbb{N}$-indexed instead of $\mathbb{N}_{\geq a}$-indexed. However one should keep in mind that the page-turning isomorphisms are also part of the spectral sequence.

Dual to the notion of a cohomological spectral sequence there is also the notion of a *homological* spectral sequence where the differential on the r-th page is supposed to have bidegree $(-r, r-1)$. We will focus our discussion of the theory to cohomological spectral sequences, however one can easily obtain the corresponding definitions and statements for homological spectral sequences by dualizing.

Additionally to the notion of spectral sequences there is also the notion of a morphism of spectral sequences:

Definition 1.1.6 (morphism of spectral sequences)**.** Let R be a ring and let (E_*, d_*) and $(\widetilde{E}_*, \widetilde{d}_*)$ be two spectral sequences. A *morphism of spectral sequences* $f\colon (E_*, d_*) \to (\widetilde{E}_*, \widetilde{d}_*)$ is a family

$$\big(f_r\colon (E_r, d_r) \to (\widetilde{E}_r, \widetilde{d}_r)\big)_{r\in\mathbb{N}}$$

of morphisms of differential bigraded R-modules, that are compatible with the page-turning isomorphisms, i.e., for each $r \in \mathbb{N}$ we have a commutative diagram

$$\begin{tikzcd} H(E_r, d_r) \arrow[r, "\cong"] \arrow[d, "H(f_r)"'] & E_{r+1} \arrow[d, "f_{r+1}"] \\ H(\widetilde{E}_r, \widetilde{d}_r) \arrow[r, "\cong"'] & \widetilde{E}_{r+1}, \end{tikzcd}$$

where the horizontal maps are the page-turning isomorphisms.

Example 1.1.7 (spectral sequence of a cochain complex)**.** Let R be a ring and let $C = (C^*, \delta^*)$ be an R-cochain complex. We start with (E_1, d_1) as the differential bigraded module of bidegree $(1, 0)$ with (C^*, δ^*) as 0-th row as described in Example 1.1.4 and want to construct a spectral sequence starting with this differential bigraded module.

Since we are required to have page-turning isomorphisms $E_{r+1} \cong_R H(E_r, d_r)$ we define the second page to be $E_2 \coloneqq H(E_1, d_1)$, which is as seen in Example 1.1.4 given by the cohomology of (C^*, δ^*) in the 0-th row and trivial in all other entries. As E_2 is supposed to be a differential bigraded module of bidegree $(2, 1)$ and we only have a single non-trivial row, the differential d_2 has to be trivial. Now the triviality of d_2 gives that $H(E_2, d_2) = E_2$, and thus we define, again with the page-turning isomorphism in mind, $E_3 \coloneqq E_2$. Again for bidegree reasons we get that the differential d_3 has to be trivial. By repeating this argument inductively we see that with $E_{r+1} \coloneqq E_r$ and $d_{r+1} = 0$ for $r \in \mathbb{N}_{\geq 1}$ we obtain a spectral sequence $(E_r, d_r)_{r\geq\mathbb{N}_{\geq 1}}$.

$$\begin{tikzcd}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
C^0 \arrow[r, "\delta^0"] & C^1 \arrow[r, "\delta^1"] & C^2 \arrow[r, "\delta^2"] & C^3 \arrow[r, "\delta^3"] & C^4
\end{tikzcd}$$

q p

E_1

Figure 1.2: The first page of the spectral sequence described in Example 1.1.7

This example is already an example of a *first-quadrant spectral sequence*.

Definition 1.1.8 (first-quadrant spectral sequence)**.** Let R be a ring and $a \in \mathbb{N}$. A spectral sequence (E_*, d_*) starting at E_a is called a *first-quadrant spectral sequence*, if we have for $p, q \in \mathbb{Z}$ that $E_a^{p,q} = 0$ whenever $p < 0$ or $q < 0$.

By the page-turning isomorphisms $E_{r+1} \cong_R H(E_r, d_r)$ we obtain that $E_{r+1}^{p,q}$ is always a subquotient of $E_r^{p,q}$. Thus for ever first-quadrant spectral sequence (E_*, d_*) we have that $E_*^{p,q} = 0$ whenever $p < 0$ or $q < 0$.

1.1.2 Convergence of first-quadrant spectral sequences

We now come to the *convergence* of spectral sequences. Since the convergence of general spectral sequences is rather complicated we only discuss the convergence of first-quadrant spectral sequences.

We begin by noticing that the entries of a first-quadrant spectral sequences eventually “stabilise”:

Remark 1.1.9 (∞-page of a first-quadrant spectral sequence)**.** Let R be a ring, let (E_*, d_*) be a first-quadrant spectral sequence and let $p, q, r \in \mathbb{N}$. We consider

$$
\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
H^0(C) & H^1(C) & H^2(C) & H^3(C) & H^4(C)
\end{array}
$$

$$E_2 = E_3 = E_4 = \dots$$

Figure 1.3: The higher pages of the spectral sequence described in Example 1.1.7

the two differentials involving $E_r^{p,q}$:

$$d_r^{p,q} \colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1} \qquad \text{and} \qquad d_r^{p-r,q+r-1} \colon E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q}.$$

For $r \geq q+2$ we have that the codomain of $d_r^{p,q}$ lies outside the first quadrant, and thus is trivial. Similarly for $r \geq p+1$ we have that the domain of $d_r^{p-r,q+r-1}$ lies outside the first quadrant. Thus for $r \geq \max\{p+1, q+2\}$ both differentials are trivial and thereby give

$$H^{p,q}(E_r, d_r) = \ker d_r^{p,q} / \operatorname{im} d_r^{p-r,q+r-1} = E_r^{p,q}/0 \cong_R E_r^{p,q}.$$

So the page-turning isomorphism gives an isomorphism $E_{r+1}^{p,q} \cong_R E_r^{p,q}$. Hence the value of $E_*^{p,q}$ "stabilises" eventually.

We define $E_\infty^{p,q} := E_{r_0}^{p,q}$ to be this stable vale, where $r_0 \in \mathbb{N}$ is the smallest value such that the page-turning isomorphisms induce $E_r^{p,q} \cong_R E_{r+1}^{p,q}$ for all $r \in \mathbb{N}_{\geq r_0}$. Moreover we call the bigraded R-module

$$E_\infty = \left(E_\infty^{p,q}\right)_{p,q \in \mathbb{Z}}$$

the ∞-*page of the spectral sequence* (E_*, d_*).

Example 1.1.10. Let R be a ring, let (C^*, δ^*) be an R-cochain complex and let (E_*, d_*) be the spectral sequence constructed out of (C^*, δ^*) in Example 1.1.7. By construction we have that all differentials d_r are trivial for $r \in \mathbb{N}_{\geq 2}$. Thus we have $E_\infty = E_2$.

Definition 1.1.11 (filtration)**.** Let R be a ring and let H^* be a graded R-module, i.e., a family $(H^n)_{n\in\mathbb{N}}$ of R-modules. A *decreasing filtration F of H^** is a family $(F^pH^n)_{p\in\mathbb{Z}}$ of submodules of H^n, for each $n \in \mathbb{N}$, such that $F^{p+1}H^n \subseteq F^pH^n$ for each $n \in \mathbb{N}$ and $p \in \mathbb{Z}$.

If H^* and $\widetilde{H}^*$ are two graded R-modules with filtrations F and $\widetilde{F}$ respectively, we call a morphism of graded R-modules $f^*\colon H^* \to \widetilde{H}^*$, i.e., a family

$$(f^n\colon H^n \longrightarrow \widetilde{H}^n)_{n\in\mathbb{N}}$$

of R-module homomorphisms, compatible with the filtrations F and $\widetilde{F}$ if

$$g(F^pH^n) \subseteq \widetilde{F}^p\widetilde{H}^n$$

for all $n \in \mathbb{N}$ and $p \in \mathbb{Z}$.

Moreover, we call a decreasing filtration F of H^* *canonically bounded* if we have $F^{n+1}H^n = 0$ and $F^0H^n = H^n$ for each $n \in \mathbb{N}$.

$$0 = F^{n+1}H^n \subseteq F^nH^n \subseteq \cdots \subseteq F^1H^n \subseteq F^0H^n = H^n$$

Definition 1.1.12 (convergence of a first-quadrant spectral sequence)**.** Let R be a ring, let (E_*, d_*) be a cohomological first-quadrant spectral sequence and let H^* be a graded R-module. We say *the spectral sequence (E_*, d_*) converges to H^** if there exists a canonically bounded decreasing filtration F of H^* such that

$$E_\infty^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$$

for all $p, q \in \mathbb{N}$. If the spectral sequence (E_*, d_*) starts with E_a we denote the convergence to H^* by

$$E_a^{p,q} \Longrightarrow H^{p+q}.$$

Example 1.1.13. Let R be a ring and let (C^*, δ^*) be a R-cochain complex. Then the spectral sequence (E_*, d_*) constructed out of (C^*, δ^*) in Example 1.1.7 converges towards the cohomology of (C^*, δ^*)

$$E_1^{p,q} \Longrightarrow H^{p+q}(C^*, \delta^*).$$

Let us simply write H^* instead of $H^*(C^*, \delta^*)$. We consider the filtration F of H^* given by

$$F^pH^n = \begin{cases} H^n & \text{if } p \leq 0 \\ 0 & \text{if } p > 0 \end{cases}$$

for $p \in \mathbb{Z}$ and $n \in \mathbb{N}$. This filtration is obviously decreasing and canonically bounded. Moreover it is easy to see that for $p, q \in \mathbb{N}$ we have

$$F^pH^{p+q}/F^{p+1}H^{p+q} \cong_R \begin{cases} H^q & \text{if } p = 0 \\ 0 & \text{if } p > 0, \end{cases}$$

which agrees with $E_\infty^{p,q}$ as

$$E_\infty^{p,q} = E_2^{p,q} = \begin{cases} H^q & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$$

by Example 1.1.10.

This example is in fact only one example of a *collapsing spectral sequence*.

Example 1.1.14 (collapsing spectral sequences)**.** Let R be a ring and let (E_r, d_r) be a first-quadrant spectral sequence and let $r \in \mathbb{N}_{\geq 2}$. We say that (E_r, d_r) *collapses at the r-th page*, if E_r only has a single row $E_r^{*,p}$ or column $E_r^{q,*}$ that is non-trivial.

Since $r \geq 2$ the condition of only having only a single non-trivial row or column gives that all differential on E_r and higher pages are trivial and thus gives $E_\infty = E_r$. Now a similar argument as in Example 1.1.13 can be used to easily show that such a collapsing spectral sequence converges towards $(E_r^{n-q,q})_{n\in\mathbb{N}}$ if $E_r^{*,q}$ is the non-trivial row of E_r, or towards $(E_r^{p,n-p})_{p\in\mathbb{N}}$ if $E_r^{p,*}$ is the non-trivial column of E_r.

Remark 1.1.15 (extension problems given by convergence)**.** Let R be a ring and let (E_*, d_*) be a first-quadrant spectral sequence converging to a graded R-module H^*. Then for $n \in \mathbb{N}$ there are submodules

$$0 = F^{n+1}H^n \subseteq F^nH^n \subseteq \cdots \subseteq F^1H^n \subseteq F^0H^n = H^n$$

with

$$E_\infty^{p,n-p} \cong_R F^pH^n/F^{p+1}H^n$$

for all $p \in \{0, \ldots, n\}$. Now we can rewrite these isomorphism as extension problems, i.e., short exact sequences,

$$0 \longrightarrow F^{n+1}H^n = 0 \longrightarrow F^nH^n \longrightarrow E_\infty^{n,0} \longrightarrow 0$$

$$0 \longrightarrow F^nH^n \longrightarrow F^{n-1}H^n \longrightarrow E_\infty^{n-1,1} \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F^1H^n \longrightarrow F^0H^n = H^n \longrightarrow E_\infty^{0,n} \longrightarrow 0.$$

If all of these extension problems can be solved, we can determine the module H^n. But even in the case were one can *not* solve all of these problems one might still obtain some information about H^n: For example, if all the modules $E_\infty^{*,n-*}$ are finitely generated we can inductively deduce that also the module H^n is finitely generated.

Now assume we are given a morphism $f\colon (E_*, d_*) \to (\widetilde{E}_*, \widetilde{d}_*)$ of two first-quadrant spectral sequences that are both convergent

$$\begin{aligned} E_2^{p,q} &\Longrightarrow H^{p+q} \\ \widetilde{E}_2^{p,q} &\Longrightarrow \widetilde{H}^{p+q}. \end{aligned}$$

Although the morphism f induces a morphism $f_\infty\colon E_\infty \to \widetilde{E}_\infty$ of the ∞-pages, the convergence of both spectral sequences does not guarantee that f induces a morphism $H^* \to \widetilde{H}^*$.

However, if we are given a morphism of graded modules $H^* \to \widetilde{H}^*$ we can say whether this morphism is compatible with f:

Definition 1.1.16 (morphism of convergent spectral sequences)**.** Let R be a ring and let (E_*, d_*) and $(\widetilde{E}_*, \widetilde{d}_*)$ be two first-quadrant spectral sequences convergent towards graded R-modules H^* and $\widetilde{H}^*$, respectively. We call a morphism $f\colon (E_*, d_*) \to (\widetilde{E}_*, \widetilde{d}_*)$ of spectral sequences and a morphism $g^*\colon H^* \to \widetilde{H}^*$ of graded R-modules *compatible* if:

1) There are canonically bounded filtrations F and $\widetilde{F}$ of H^* and $\widetilde{H}^*$, respectively, such that g^* is compatible with these filtrations.

2) The filtrations F and $\widetilde{F}$ witness the convergence of (E_*, d_*) and $(\widetilde{E}*, \widetilde{d}_*)$, respectively.

3) For each $n \in \mathbb{N}$ and $p \in \{0, \dots, n\}$ the diagram

$$\begin{tikzcd} 0 \arrow[r] & F^{p+1}H^n \arrow[r] \arrow[d, "g^n"'] & F^pH^n \arrow[r] \arrow[d, "g^n"] & E_\infty^{p,n-p} \arrow[r] \arrow[d, "f_\infty^{p,n-p}"] & 0 \\ 0 \arrow[r] & \widetilde{F}^{p+1}\widetilde{H}^n \arrow[r] & \widetilde{F}^p\widetilde{H}^n \arrow[r] & \widetilde{E}_\infty^{p,n-p} \arrow[r] & 0 \end{tikzcd}$$

commutes, where the rows are the extension problems of Remark 1.1.15.

If $f\colon (E_*, d_*) \to (\widetilde{E}_*, \widetilde{d}_*)$ and $g^*\colon H^* \to \widetilde{H}^*$ are compatible we also say that

$$\begin{tikzcd} E_2^{p,q} \arrow[r, Rightarrow] \arrow[d, "f_2^{p,q}"'] & H^{p+q} \arrow[d, "g^{p+q}"] \\ \widetilde{E}_2^{p,q} \arrow[r, Rightarrow] & \widetilde{H}^{p+q} \end{tikzcd}$$

commutes.

Proposition 1.1.17 (five-term exact sequence)**.** *Let R be a ring and let (E_*, d_*) be a first-quadrant spectral sequence converging to a graded R-module H^*. Then we have $H^0 \cong E_2^{0,0}$ and an exact sequence*

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1 \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \longrightarrow E_\infty^{2,0} \longrightarrow 0,$$

where $E_\infty^{2,0}$ is isomorphic to a submodule of H^2.

Moreover, this exact sequence is natural in the following sense: If we are given compatible morphisms

$$\begin{tikzcd} E_2^{p,q} \arrow[r, Rightarrow] \arrow[d, "f_2^{p,q}"'] & H^{p+q} \arrow[d, "g^{p+q}"] \\ \widetilde{E}_2^{p,q} \arrow[r, Rightarrow] & \widetilde{H}^{p+q} \end{tikzcd}$$

we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & E_2^{1,0} \arrow[r] \arrow[d, "f_2^{1,0}"] & H^1 \arrow[r] \arrow[d, "g^1"] & E_2^{0,1} \arrow[r, "d_2^{0,1}"] \arrow[d, "f_2^{0,1}"] & E_2^{2,0} \arrow[r] \arrow[d, "f_2^{2,0}"] & E_\infty^{2,0} \arrow[r] \arrow[d, "f_\infty^{2,0}"] & 0 \\ 0 \arrow[r] & \widetilde{E}_2^{1,0} \arrow[r] & \widetilde{H}^1 \arrow[r] & \widetilde{E}_2^{0,1} \arrow[r, "d_2'^{0,1}"'] & \widetilde{E}_2^{2,0} \arrow[r] & \widetilde{E}_\infty^{2,0} \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact.

Proof. Let F be a canonically bounded filtration of H^* witnessing the convergence of (E_*, d_*). This filtration gives in particular submodules

$$0 = F^1H^0 \subseteq F^0H^0 = H^0,$$

$$0 = F^2H^1 \subseteq F^1H^1 \subseteq F^0H^1 = H^1$$

and

$$0 = F^3H^2 \subseteq F^2H^2 \subseteq F^1H^2 \subseteq F^0H^2 = H^2.$$

By the convergence we have isomorphisms

$$\begin{aligned} & E_\infty^{0,0} \cong_R F^0H^0/F^1H^0 \cong_R H^0, \\ & E_\infty^{1,0} \cong_R F^1H^1/F^2H^1 \cong_R F^1H^1, \\ & E_\infty^{2,0} \cong_R F^2H^2/F^3H^2 \cong_R F^2H^2, \\ \text{and}\qquad & E_\infty^{0,1} \cong_R F^0H^1/F^1H^1 = H^1/F^1H^1, \end{aligned}$$

and these isomorphisms give

$$H^0 \cong_R E_\infty^{0,0} \cong_R E_2^{0,0},$$

as all differentials involving $E_r^{0,0}$ are trivial for $r \geq 2$ (see Remark 1.1.9), that

$$E_\infty^{2,0} \cong_R F^2H^2 \subseteq H^2$$

is isomorphic to a submodule of H^2 and the exact sequence

$$0 \longrightarrow E_\infty^{1,0} \cong_R F^1H^1 \longrightarrow H^1 \longrightarrow E_\infty^{0,1} \longrightarrow 0.$$

Moreover, since the differentials

$$\begin{aligned} d_2^{-2,2}\colon E_2^{-2,2} = 0 \longrightarrow E_2^{0,1} \\ \text{and} \qquad d_2^{2,0}\colon E_2^{2,0} \longrightarrow E_2^{4,-1} = 0 \end{aligned}$$

are trivial the page-turning isomorphisms give

$$E_3^{0,1} \cong_R \ker d_2^{0,1} \qquad \text{and} \qquad E_3^{2,0} \cong_R E_2^{2,0} / \operatorname{im} d_2^{0,1}.$$

So we also obtain the exact sequence

$$0 \longrightarrow E_3^{0,1} \longrightarrow E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \longrightarrow E_3^{2,0} \longrightarrow 0.$$

Now in our computation in Remark 1.1.9 we have seen that $E_r^{p,q} \cong_R E_\infty^{p,q}$ whenever $r \geq \max\{p+1, q+2\}$, thus the above two exact sequences can be rewritten as

$$0 \longrightarrow E_\infty^{1,0} \cong_R E_2^{1,0} \longrightarrow H^1 \xrightarrow{\varphi} E_\infty^{0,1} \longrightarrow 0$$

and

$$0 \longrightarrow E_3^{0,1} \cong_R E_\infty^{0,1} \xrightarrow{\psi} E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \longrightarrow E_\infty^{2,0} \cong_R E_3^{2,0} \longrightarrow 0.$$

By combining these two sequences we obtain the claimed exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1 \xrightarrow{\psi\circ\varphi} E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \longrightarrow E_\infty^{2,0} \longrightarrow 0.$$

The naturality follows from the compatibility of f with the morphism g^*, the differentials and the page-turning isomorphisms. □

1.2 Classical examples of spectral sequences

Now that we have introduced the notion of spectral sequences we give some classical examples of spectral sequences in algebraic topology, group cohomology and homological algebra.

1.2.1 The Künneth spectral sequence

We begin with an example of a spectral sequence in homological algebra: the Künneth spectral sequence. This spectral sequence relates the homology of two chain complexes with the homology of their *tensor product*.

Definition 1.2.1 (tensor product of chain complexes)**.** Let R be a ring and let C_* and D_* be R-chain complexes. Then the *tensor product* $C_* \otimes D_*$ is defined as the R-chain complex with chain modules

$$(C_* \otimes D_*)_n \coloneqq \bigoplus_{i+j=n} C_i \otimes_R D_j$$

and differentials

$$\begin{aligned} d\colon (C_* \otimes D_*)_n &\longrightarrow (C_* \otimes D_*)_{n-1} \\ C_i \otimes_R D_j \ni v \otimes w &\longmapsto d_i^C(v) \otimes w + (-1)^i \cdot v \otimes d_j^D(w) \end{aligned}$$

for each $n \in \mathbb{N}$.

Theorem 1.2.2 (Künneth spectral sequence [17, Theorem 2.20])**.** *Let R be a ring and let C_* and D_* be R-chain complexes where C_* consists only of flat R-modules. Then there exists a converging first-quadrant spectral sequence*

$$E_2^{p,q} = \bigoplus_{i+j=q} \operatorname{Tor}_R^p\big(H_i(C_*), H_j(D_*)\big) \Longrightarrow H_{p+q}(C_* \otimes D_*).$$

Using this spectral sequences one can easily derive the algebraic Künneth theorem (without the splitting property).

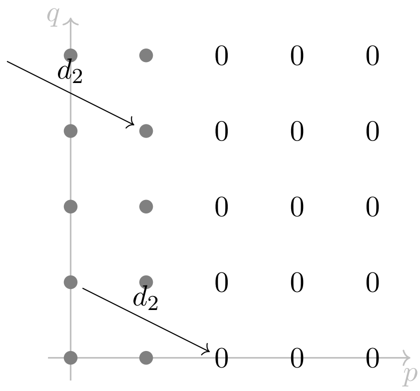
Theorem 1.2.3 (algebraic Künneth theorem)**.** *Let R be a principal ideal domain and let C_* and D_* be R-chain complexes where C_* consists only of flat R-modules. Then we have for each $n \in \mathbb{N}$ an exact sequence*

$$0 \to \bigoplus_{i+j=n} H_i(C_*) \otimes_R H_j(D_*) \to H_n(C_* \otimes D_*) \to \bigoplus_{i+j=n-1} \operatorname{Tor}_R^1\big(H_i(C_*), H_j(D_*)\big) \to 0.$$

Proof. We consider the converging first-quadrant spectral sequence

$$E_2^{p,q} = \bigoplus_{i+j=q} \operatorname{Tor}_R^p\big(H_i(C_*), H_j(D_*)\big) \Longrightarrow H_{p+q}(C_* \otimes D_*).$$

given by Theorem 1.2.2. Since R is a principal ideal domain we know that the terms $\operatorname{Tor}_R^n(\,\cdot\,,\,\cdot\,)$ vanish for $n \geq 2$, and thus $E_2^{p,q}$ can only have non-trivial entries for $p \in \{0, 1\}$.



E_2

For degree reasons we now obtain that each differential d_r for $r \in \mathbb{N}_{\geq 2}$ is trivial, and thus have $E_\infty = E_2$. Finally from the convergence of the spectral sequence we obtain for each $n \in \mathbb{N}$ an exact sequence

$$0 \longrightarrow E_\infty^{0,n} \longrightarrow H_n(C_* \otimes D_*) \longrightarrow E_\infty^{1,n-1} \longrightarrow 0,$$

where

$$E_\infty^{1,n-1} = E_2^{1,n-1} = \bigoplus_{i+j=n-1} \operatorname{Tor}_R^1(H_i(C_*), H_j(D_*)$$

and

$$E_\infty^{0,n} = E_2^{0,n} = \bigoplus_{i+j=n} \operatorname{Tor}_R^0\big(H_i(C_*), H_j(D_*)\big) = \bigoplus_{i+j=n} H_i(C_*) \otimes_R H_j(D_*),$$

which gives the claim. □

In the special case that D_* only consists of a single module in degree 0 this gives the universal coefficient theorem (again without the splitting property).

Theorem 1.2.4 (universal coefficient theorem)**.** *Let R be a principal ideal domain, let C_* be a R-chain complex consisting of flat R-modules, and let A be an R-module. Then we have for each $n \in \mathbb{N}$ an exact sequence*

$$0 \longrightarrow H_n(C_*) \otimes_R A \longrightarrow H_n(C_* \otimes_R A) \longrightarrow \operatorname{Tor}_R^1\big(H_{n-1}(C_*), A\big) \longrightarrow 0.$$

Since by definition the singular chain complex of a topological space consists of free modules we can apply both the universal coefficient theorem and the algebraic Künneth theorem. We note, however, that in order to obtain the Künneth theorem in algebraic topology [22, Theorem 9.37], which relates the singular homology of a product $X \times Y$ to the homology of X and Y, one still requires the Eilenberg-Zilber theorem [22, Theorem 9.33] to relate the singular chain complex of $X \times Y$ with the tensor product of the singular chain complexes of X and Y.

1.2.2 The Serre spectral sequence

Next we come to an example of a spectral sequence in algebraic topology: the Serre spectral sequence. This spectral sequence relates the (co)homology of the *total space* of a *fibration* to the (co)homology of the *base space* and the *fiber* of the fibration.

Definition 1.2.5 (fibration, fiber)**.** A continuous map $\pi\colon E \to B$ of topological spaces is called a *(Hurewicz) fibration* if it has the *homotopy lifting property* with respect to all topological spaces, i.e., given a topological space Y, a homotopy $h\colon Y \times [0,1] \to B$ and a continuous map $f\colon Y \to E$ with $\pi \circ f = h(\,\cdot\,,0)$

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "{(\mathrm{id}_Y,0)}"'] & E \arrow[d, "\pi"] \\ Y \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "H"] & B, \end{tikzcd}$$

there exists a homotopy $H\colon Y \times [0,1] \to E$ such that

$$H(\,\cdot\,,0) = h(\,\cdot\,,0) \qquad \text{and} \qquad \pi \circ H = h.$$

If $\pi\colon E \to B$ is a fibration, we call E the *total space* and B the *base space* of the fibration. Moreover, for $x \in B$ we call

$$F_x := \pi^{-1}\{x\} = \big\{ e \in E \bigm| \pi(e) = x \big\}$$

the *fiber over* x.

In the case that $\pi\colon E \to B$ is a fibration with B path-connected, then for any $x, y \in B$ we have that the fibers F_x and F_y are homotopy equivalent [9, Proposition 4.61]. With this result in mind we will assume in the following for simplicity that the base space B is always a path connected pointed space, i.e., has a designated base point $b_0 \in B$, and write a fibration as

$$F \hookrightarrow E \xrightarrow{\pi} B$$

where F is the fiber over the base-point $b_0 \in B$.

Example 1.2.6 (path space fibration)**.** Let X be a path-connected pointed topological space with basepoint $x_0 \in X$. The *path space of* X is defined as

$$PX := \{\gamma\colon [0,1] \to X \mid \gamma \text{ is continuous with } \gamma(0) = x_0\}\,,$$

equipped with the compact-open topology. The continuous map

$$\begin{aligned} PX &\longrightarrow X \\ \gamma &\longmapsto \gamma(1) \end{aligned}$$

is a fibration [9, Proposition 4.64], the so-called *path space fibration*, and we call its fiber

$$\Omega X := \{\gamma\colon [0,1] \to X \mid \gamma \text{ is continuous with } \gamma(0) = \gamma(1) = x_0\}$$

the *loop space of* X.

One special property of this fibration $\Omega X \hookrightarrow PX \to X$ is that the total space PX is contractible by continuously truncating paths.

After this short recap on the notation of fibrations we come the the formulation of the Serre spectral sequence. We will only give a simplified version where we assume that the base space is simply connected and only use integer coefficients in (co)homology. Therefore we also use the shorthand notation $H^*(X)$ and $H_*(X)$, instead of $H^*(X;\mathbb{Z})$ and $H_*(X;\mathbb{Z})$, respectively, for the (co)homology with integer coefficients. The general statement can for example be found in the book of McCleary [17, Chapter 5].

Theorem 1.2.7 (Serre spectral sequence [17, Theorem 5.1, Theorem 5.2])**.** *Let*

$$F \hookrightarrow E \longrightarrow B$$

be a fibration where B is simply connected, i.e., $\pi_1(B) \cong 0$, and F is connected. Then there exists a cohomological first-quadrant spectral sequence

$$E_2^{p,q} \cong_{\mathbb{Z}} H^p\big(B; H^q(F)\big) \Longrightarrow H^{p+q}(E).$$

Similarly, there exists a homological first-quadrant spectral sequence

$$E^2_{p,q} \cong_{\mathbb{Z}} H_p\big(B; H_q(F)\big) \Longrightarrow H_{p+q}(E).$$

As an application we can consider the case of the path space fibration.

Corollary 1.2.8. *Let $n \in \mathbb{N}_{\geq 1}$ and let X be a simply connected pointed space with $H_i(X) \cong 0$ for $i \in \{1, \dots, n\}$. Then we have*

$$H_i(\Omega X) \cong_{\mathbb{Z}} 0$$

for $i \in \{1, \dots, n-1\}$ and

$$H_n(\Omega X) \cong_{\mathbb{Z}} H_{n+1}(X).$$

Similarly, if $H^i(X) \cong 0$ for $i \in \{1, \dots, n\}$ then we obtain

$$H^i(\Omega X) \cong_{\mathbb{Z}} 0 \qquad \text{for } i \in \{1, \dots, n-1\}$$

and

$$H^n(\Omega X) \cong_{\mathbb{Z}} H^{n+1}(X).$$

Proof. Let us consider the path space fibration $\Omega X \hookrightarrow PX \to X$ and the corresponding converging spectral sequence

$$E^2_{p,q} \cong_{\mathbb{Z}} H_p\big(X; H_q(\Omega X)\big) \Longrightarrow H_{p+q}(PX).$$

Since X and ΩX are both path-connected we have

$$E^2_{0,q} \cong_{\mathbb{Z}} H_0\big(X; H_q(\Omega X)\big) \cong_{\mathbb{Z}} H_q(\Omega X)$$

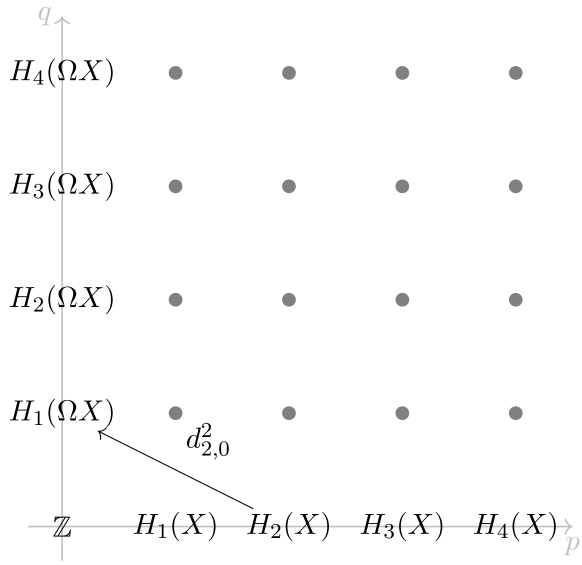
and

$$E^2_{p,0} \cong_{\mathbb{Z}} H_p\big(X; H_0(\Omega X)\big) \cong_{\mathbb{Z}} H_p(X)$$

for all $p, q \in \mathbb{N}$. Moreover, since the path space PX is contractible, we have

$$H_i(PX) \cong_{\mathbb{Z}} 0$$

for $i \in \mathbb{N}_{\geq 1}$. Thus the convergence of the spectral sequence gives $E^\infty_{p,q} \cong 0$ for all $p, q \in \mathbb{N}$ with $p + q \neq 0$.



E^2

Now let us precede by induction on n. First we assume $n = 1$. Since $d^2_{2,0}$ is the last non-trivial differential involving $E^*_{2,0}$ and $E^*_{0,1}$ we obtain

$$\ker d^2_{2,0} = E^\infty_{2,0} \cong_{\mathbb{Z}} 0,$$

which gives the injectivity of $d^2_{2,0}$, and

$$H_1(\Omega X)/\operatorname{im} d^2_{2,0} \cong_{\mathbb{Z}} E^2_{0,1}/\operatorname{im} d^2_{2,0} = E^\infty_{0,1} \cong_{\mathbb{Z}} 0,$$

which gives the surjectivity of $d^2_{2,0}$. Hence $d^2_{2,0}\colon H_2(X) \to H_1(\Omega X)$ is an isomorphism.

Now let us assume we have proven the claim for $n \in \mathbb{N}_{\geq 1}$ an that $H_i(X) \cong_{\mathbb{Z}} 0$ for $i \in \{1, \ldots, n+1\}$. By using the induction hypothesis we obtain

$$H_i(\Omega X) \cong_{\mathbb{Z}} 0 \qquad \text{for } i \in \{1, \ldots, n-1\}$$

and

$$H_n(\Omega X) \cong_{\mathbb{Z}} H_{n+1}(X) \cong_{\mathbb{Z}} 0.$$

Since we have

$$E^2_{p,q} \cong H_p\big(X; H_q(\Omega X)\big)$$

this gives that the rows $E^2_{*,q}$ are all trivial for $q \in \{1, \ldots, n\}$.

$$
\begin{array}{c|ccccc}
q & & & & & \\
H_{n+2}(\Omega X) & & \bullet & \bullet & \bullet & \bullet \\
H_{n+1}(\Omega X) & & \bullet & \bullet & \bullet & \bullet \\
 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & \mathbb{Z} & 0 & 0 & 0 & H_{n+2}(X) \quad p
\end{array}
$$

$$E^2$$

Using this we can deduce that the *only* non-trivial differential involving $E^*_{n+2,0}$ and $E^*_{0,n+1}$ is $d^{n+2}_{n+2,0}$: For $r < n+2$ either the domains and codomains lie inside one of trivial rows or outside the first quadrant and for $r > n+2$ both the domains and the codomains are outside the first quadrant.

Now the triviality of the differential for $r < n+2$ gives (inductively) that we have

$$H_{n+1}(\Omega X) \cong_{\mathbb{Z}} E^2_{0,n+1} \cong_{\mathbb{Z}} E^3_{0,n+1} \cong_{\mathbb{Z}} \ldots \cong_{\mathbb{Z}} E^{n+2}_{0,n+1}$$

and

$$H_{n+2}(X) \cong_{\mathbb{Z}} E^2_{n+2,0} \cong_{\mathbb{Z}} E^3_{n+2,0} \cong_{\mathbb{Z}} \ldots \cong_{\mathbb{Z}} E^{n+2}_{n+2,0}.$$

Moreover a similar argument as in the above case shows that $d^{n+2}_{n+2,0}$ induces an isomorphism $H_{n+2}(X) \cong_{\mathbb{Z}} H_{n+1}(\Omega X)$.

The proof of the statement for cohomology works analogously using the corresponding cohomological spectral sequence. □

Now this can be used to prove the Hurewicz theorem:

Theorem 1.2.9 (Hurewicz theorem)**.** *Let X be a path-connected pointed topological space. Then we have*

$$H_1(X) \cong_{\mathbb{Z}} \pi_1(X)_{\mathrm{ab}}.$$

If, moreover, X is n-connected for some $n \in \mathbb{N}_{\geq 1}$, i.e., $\pi_i(X) \cong 0$ for all $i \in \{1, \ldots, n\}$, then we have

$$H_i(X) \cong_{\mathbb{Z}} 0$$

for all $i \in \{1, \dots, n\}$ *and*

$$H_{n+1}(X) \cong_{\mathbb{Z}} \pi_{n+1}(X).$$

Sketch of proof. By considering loops $S^1 \to X$ as singular 1-simplices $\Delta^1 \to X$ one obtains a well defined group homomorphism $\pi_1(X) \to H_1(X)$. One can check that this group homomorphism is surjective and has the commutator subgroup of $\pi_1(X)$ as kernel, and thus induces the isomorphism $H_1(X) \cong_{\mathbb{Z}} \pi_1(X)_{\mathrm{ab}}$ [22, pp. 80-84].

Before we come to the second part we recall that for each $n \in \mathbb{N}_{\geq 1}$ there is an isomorphism

$$\pi_n(\Omega X) \cong_{\mathbb{Z}} \pi_{n+1}(X).$$

This can for example be seen using the long exact sequence of homotopy groups [9, Theorem 4.41] of the fibration $\Omega X \hookrightarrow PX \to X$ and the contractibility of PX. In particular, if X is $(n+1)$-connected then the loop space ΩX is n-connected.

Now let us prove the second claim by induction on n. If $n = 1$ and X is n-connected, i.e., if X is simply connected, we have

$$H_1(X) \cong_{\mathbb{Z}} \pi_1(X)_{\mathrm{ab}} \cong_{\mathbb{Z}} 0$$

as well as

$$\begin{aligned} \pi_2(X) &\cong_{\mathbb{Z}} \pi_1(\Omega X) \\ &\cong_{\mathbb{Z}} \pi_1(\Omega X)_{\mathrm{ab}} && (\pi_1(\Omega X) \cong_{\mathbb{Z}} \pi_2(X) \text{ is abelian}) \\ &\cong_{\mathbb{Z}} H_1(\Omega X) && (\text{by the first part}) \\ &\cong_{\mathbb{Z}} H_2(X). && (\text{by Corollary 1.2.8}) \end{aligned}$$

Now let $n \in \mathbb{N}_{\geq 1}$, let X be an $(n+1)$ connected space and assume we have already show the claim for n-connected spaces. Since X is in particular n-connected we have by induction

$$H_i(X) \cong_{\mathbb{Z}} 0$$

for all $i \in \{1, \dots, n\}$ and

$$H_{n+1}(X) \cong_{\mathbb{Z}} \pi_{n+1}(X) \cong_{\mathbb{Z}} 0.$$

Moreover, since the $(n+1)$-connectedness of X gives the n-connectedness of ΩX we have for $H_{n+2}(X)$ that

$$\begin{aligned} H_{n+2}(X) &\cong_{\mathbb{Z}} H_{n+1}(\Omega X) && (\text{by Corollary 1.2.8}) \\ &\cong_{\mathbb{Z}} \pi_{n+1}(\Omega X) && (\text{by induction}) \\ &\cong_{\mathbb{Z}} \pi_{n+2}(X), \end{aligned}$$

which concludes the proof. $\square$

1.2.3 The Hochschild-Serre spectral sequence

As a final example of a spectral sequence we consider the Hochschild-Serre spectral sequence in group (co)homology.

Theorem 1.2.10 (Hochschild-Serre spectral sequence [25, Theorem 6.8.2])**.** *Let*

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0$$

be a short exact sequence of groups and let V be an $\mathbb{Z}[\Gamma]$-module. Then there exists a natural converging cohomological first-quadrant spectral sequence

$$E_2^{p,q} = H^p\big(\Delta; H^q(\Lambda; V)\big) \Longrightarrow H^{p+q}(\Gamma; V).$$

Similarly, there is a natural converging homological first-quadrant spectral sequence

$$E^2_{p,q} = H_p\big(\Delta; H_q(\Lambda; V)\big) \Longrightarrow H_{p+q}(\Gamma; V).$$

As usual for first-quadrant spectral sequences we obtain a five-term exact sequence.

Corollary 1.2.11 (five-term exact sequence for group cohomology)**.** *Let*

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0$$

be a short exact sequence of groups and let V be an $\mathbb{Z}[\Gamma]$-module. Then there exists an exact sequence

$$0 \longrightarrow H^1(\Delta; V^\Lambda) \longrightarrow H^1(\Gamma; V) \longrightarrow H^1(\Lambda; V)^\Delta \longrightarrow H^2(\Delta; V^\Lambda) \longrightarrow H^2(\Gamma; V).$$

Proof. Using the Hochschild-Serre spectral sequence the five-term exact sequence of a converging first-quadrant spectral sequence Proposition 1.1.17 gives the exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(\Gamma; V) \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow H^2(\Gamma; V),$$

where we have used that $E_\infty^{2,0}$ is isomorphic to a submodule of $H^2(\Gamma; V)$. Now the identification of the second page and the computation of group cohomology in degree zero as invariants gives the claim. □

In order to obtain a similar result for group homology we need the following.

Lemma 1.2.12 (group homology in degree 0 [25, Definition 6.1.2])**.** *Let Γ be a group and let V be be an $\mathbb{Z}[\Gamma]$-module. Then we we group homology of Γ with coefficients in V is isomorphic to the* coinvariants *of V*

$$H_0(\Gamma; V) \cong_{\mathbb{Z}} V_\Gamma := V \big/ \{g \cdot v - v \mid g \in \Gamma, v \in V\}.$$

Using this we obtain dual to Corollary 1.2.11 the following corollary.

Corollary 1.2.13 (five-term exact sequence in group homology)**.** *Let*

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0$$

be a short exact sequence of groups and let V be an $\mathbb{Z}[\Gamma]$-module. Then there exists an exact sequence

$$H_2(\Gamma; V) \longrightarrow H_2(\Delta; V_\Lambda) \longrightarrow H_1(\Lambda; V)_\Delta \longrightarrow H_1(\Gamma; V) \longrightarrow H_1(\Delta; V_\Lambda) \longrightarrow 0.$$

One application of this exact sequence is *Hopf's formula* which allows to describe the second group homology, with trivial $\mathbb{Z}$ coefficients, using a *representation* of the group.

Theorem 1.2.14 (Hopf's formula [25, Theorem 6.8.8])**.** *Let Γ be a free group, let $\Lambda \subseteq \Gamma$ be a free subgroup and let $\Delta = \Lambda/\Gamma$. Then we have*

$$H_2(\Delta; \mathbb{Z}) \cong_R \frac{\Lambda \cap [\Gamma, \Gamma]}{[\Gamma, \Lambda]},$$

where Δ acts trivial on $\mathbb{Z}$ and $[\Gamma, \Lambda]$ denotes the subgroup of Γ generated by the commutators $[h, g]$ with $h \in \Lambda$ and $g \in \Gamma$.

Sketch of proof. We consider the five term exact sequence of the short exact sequence

$$0 \longrightarrow \Lambda \hookrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0$$

in group homology with trivial $\mathbb{Z}$ coefficients:

$$H_2(\Gamma; \mathbb{Z}) \longrightarrow H_2(\Delta; \mathbb{Z}) \longrightarrow H_1(\Lambda; \mathbb{Z})_\Delta \longrightarrow H_1(\Gamma; \mathbb{Z}) \longrightarrow H_1(\Delta; \mathbb{Z}) \longrightarrow 0.$$

Moreover, using the naturality of the spectral sequence one can give an explicit description of the morphism $H_1(\Lambda; \mathbb{Z})_\Delta \to H_1(\Gamma; \mathbb{Z})$.

Now group homology in degree 1 with trivial $\mathbb{Z}$ coefficients is given by the abelianisation [25, Theorem 6.1.11] and since Γ is a free group we have

$$H_2(\Gamma; V) \cong_{\mathbb{Z}} 0$$

[25, Corollary 6.2.7]. Thus we have the exact sequence

$$0 \longrightarrow H_2(\Delta; \mathbb{Z}) \longrightarrow (\Lambda_{\mathrm{ab}})_\Delta \longrightarrow \Gamma_{\mathrm{ab}} \longrightarrow \Delta_{\mathrm{ab}} \longrightarrow 0.$$

Furthermore, by identifying the action $\Delta \curvearrowright \Lambda_{\mathrm{ab}}$ one can compute

$$(\Lambda_{\mathrm{ab}})_\Delta \cong_{\mathbb{Z}} \frac{\Lambda}{[\Gamma, \Lambda]}$$

and use the explicit description of the morphism $(\Lambda_{\mathrm{ab}})_\Delta \to \Gamma_{\mathrm{ab}}$ to obtain that

$$H_2(\Delta; \mathbb{Z}) \cong_{\mathbb{Z}} \ker\big((\Lambda_{\mathrm{ab}})_\Delta \to \Gamma_{\mathrm{ab}}\big) \cong_{\mathbb{Z}} \ker\left(\frac{\Lambda}{[\Gamma, \Lambda]} \to \frac{\Gamma}{[\Gamma, \Gamma]}\right) = \frac{\Lambda \cap [\Gamma, \Gamma]}{[\Gamma, \Lambda]}. \qquad \square$$

1.3 Constructions of spectral sequences

Now that we have seen that the theory of spectral sequences can indeed be helpful we will discuss some constructions of spectral sequences. First we construct a spectral sequence out of a *filtered complex* and then apply this construction to obtain two spectral sequences of a *double complex*.

1.3.1 Spectral sequence of a filtered complex

Definition 1.3.1 (filtered cochain complex)**.** Let R be a ring. A *filtered R-cochain complex* is a pair (C, F) where $C = (C^*, \delta^*)$ is an R-cochain complex and F is a filtration of the graded R-module C^* that is compatible with the coboundary operators, i.e., we have

$$\delta^n(F^pC^n) \subseteq F^pC^{n+1}$$

for all $n \in \mathbb{N}$ and $p \in \mathbb{Z}$.

A *morphism of filtered R-cochain complexes* $f\colon (C, F) \to (\widetilde{C}, \widetilde{F})$ is a morphisms of R-cochain complexes $f\colon C \to \widetilde{C}$ that is, as morphism of graded R-modules $f\colon C^* \to \widetilde{C}^*$, compatible with the filtrations F and $\widetilde{F}$.

We call a filtered cochain complex (C, F) *canonically bounded* if the filtration F of C^* is canonically bounded.

Remark 1.3.2 (alternative description of filtered cochain complexes)**.** Let R be a ring and let (C, F) be a filtered R-cochain complex. By the compatibility of the filtration F with the coboundary operator of C we obtain for each $p \in \mathbb{Z}$ a sub-cochain complex

$$F^pC := (F^pC^*, \delta^*)$$

of C. Moreover, since F is a decreasing filtration this gives a decreasing sequence

$$\cdots \subseteq F^{p+1}C \subseteq F^pC \subseteq \cdots \subseteq C$$

of sub-cochain complexes of C. Conversely it is easy to see that each such decreasing sequence $(F^pC)_{p\in\mathbb{Z}}$ of sub-cochain complexes of C give C the structure of a filtered cochain complex.

Remark 1.3.3 (induced filtration on cohomology)**.** Let R be a ring and let (C, F) be a filtered R-cochain complex. Then the filtration F induces a filtration (which we will also denote by F) on the cohomology $H^*(C)$ of C with

$$F^pH^n(C) := \operatorname{im}\big(H^n(F^pC) \to H^n(C)\big),$$

where $H^n(F^pC) \to H^n(C)$ is the map induced by inclusion. Moreover, if (C, F) is a canonically bounded filtered cochain complex the induced filtration on $H^*(C)$ is canonically bounded as well.

Theorem 1.3.4 (spectral sequence of filtered cochain complex)**.** *Let R be a ring and let (C, F) be a filtered cochain complex. Then there exists a spectral sequence (E_*, d_*) starting with*

$$E_1^{p,q} \cong_R H^{p+q}(F^pC/F^{p+1}C)$$

for all $p, q \in \mathbb{Z}$.

Moreover, if (C, F) is a canonically bounded *filtered cochain complex this is a first-quadrant spectral sequence and we have*

$$E_\infty^{p,q} \cong_R {}^{F^pH^{p+q}(C)}\!\diagup_{F^{p+1}H^{p+q}(C)}$$

for all $p, q \in \mathbb{N}$, i.e., we have

$$E_1^{p,q} \Longrightarrow H^{p+q}(C).$$

Proof. We divide the proof into several steps:

Introduction of notation: For $p, q \in \mathbb{Z}$ and $r \in \mathbb{N} \cup \{-1\}$ we define

$$\begin{aligned} Z_r^{p,q} &:= F^pC^{p+q} \cap (\delta^{p+q})^{-1}\big(F^{p+r}C^{p+q+1}\big), \\ B_r^{p,q} &:= F^pC^{p+q} \cap \delta^{p+q-1}\big(F^{p-r}C^{p+q-1}\big), \\ Z_\infty^{p,q} &:= F^pC^{p+q} \cap \ker \delta^{p+q} \\ \text{and} \qquad B_\infty^{p,q} &:= F^pC^{p+q} \cap \operatorname{im} \delta^{p+q-1}. \end{aligned}$$

Since F is a decreasing filtration it is easy to see that for all $p, q \in \mathbb{Z}$ we obtain a chain of inclusions

$$B_0^{p,q} \subseteq B_1^{p,q} \subseteq \cdots \subseteq B_\infty^{p,q} \subseteq Z_\infty^{p,q} \subseteq \cdots \subseteq Z_1^{p,q} \subseteq Z_0^{p,q}. \tag{1}$$

Moreover we have

$$\begin{aligned} Z_{r-1}^{p+1,q-1} &= F^{p+1}C^{p+q} \cap (\delta^{p+q})^{-1}\big(F^{p+r}C^{p+q+1}\big) \\ &\subseteq F^pC^{p+q} \cap (\delta^{p+q})^{-1}\big(F^{p+r}C^{p+q+1}\big) \qquad (F \text{ is decreasing}) \\ &= Z_r^{p,q} \end{aligned} \tag{2}$$

as well as

$$\begin{aligned} \delta^{p+q}\big(Z_r^{p,q}\big) &= \delta^{p+q}\Big(F^pC^{p+q} \cap (\delta^{p+q})^{-1}\big(F^{p+r}C^{p+q+1}\big)\Big) \\ &= \delta^{p+q}\big(F^pC^{p+q}\big) \cap F^{p+r}C^{p+q+1} \\ &= B_r^{p+r,q+1-r} \end{aligned} \tag{3}$$

for all $p, q \in \mathbb{Z}$ and $r \in \mathbb{N}$.

Construction of the differential bigraded modules (E_r, d_r)**:** Let $r \in \mathbb{N}$. Since for $p, q \in \mathbb{Z}$ the inclusions (1) and (2) give $Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q} \subseteq Z_r^{p,q}$ we can define

$$E_r^{p,q} := Z_r^{p,q} \big/ Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}$$

and denote by $\eta_r^{p,q}\colon Z_r^{p,q} \longrightarrow E_r^{p,q}$ the canonical projection. For the construction of the differential $d_r^{p,q}$ we consider the diagram

$$\begin{tikzcd} Z_r^{p,q} \arrow[r, "\delta^{p+q}"] \arrow[d, "\eta_r^{p,q}"'] & Z_r^{p+r,q+1-r} \arrow[d, "\eta_r^{p+r,q+1-r}"] \\ E_r^{p,q} \arrow[r, dashed, "d_r^{p,q}"'] & E_r^{p+r,q+1-r}. \end{tikzcd}$$

Since we have

$$\begin{aligned} \delta^{p+q}\big(Z_{r-1}^{p+1,q-1} + Z_{r+1}^{p,q}\big) &= B_{r-1}^{p+r,q+1-r} + B_{r+1}^{p+r+1,q-r} && \text{(by (3))} \\ &\subseteq Z_{r-1}^{p+r,q+1-r} + B_{r+1}^{p+r+1,q-r} && \text{(by (1))} \\ &= \ker \eta_r^{p+r,q+1-r} \end{aligned}$$

we obtain, with $Z_{r-1}^{p+1,q-1} + Z_{r+1}^{p,q} \subseteq Z_r^{p,q}$ from (1) and (2), that

$$\begin{aligned} Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q} &\subseteq Z_{r-1}^{p+1,q-1} + Z_{r+1}^{p,q} && \text{(by (1))} \\ &\subseteq \ker(\eta_r^{p+r,q+1-r} \circ \delta^{p+q}|_{Z_r^{p,q}}). \end{aligned}$$

Thus an application of the universal property of the quotient gives the existence of a well-defined R-linear map

$$d_r^{p,q}\colon E_r^{p,q} \longrightarrow E_r^{p+r,q+1-r}$$

making the above diagram commute.

For the differential property, i.e., $d_r \circ d_r = 0$, we consider the commutative diagram

$$\begin{tikzcd} Z_r^{p-r,q+r-1} \arrow[r, "\delta^{p+q-1}"] \arrow[d, "\eta_r^{p-r,q+r-1}"'] & Z_r^{p,q} \arrow[r, "\delta^{p+q}"] \arrow[d, "\eta_r^{p,q}"'] & Z_r^{p+r,q+1-r} \arrow[d, "\eta_r^{p+r,q+1-r}"] \\ E_r^{p-r,q+r-1} \arrow[r, "d_r^{p-r,q+r-1}"'] & E_r^{p,q} \arrow[r, "d_r^{p,q}"'] & E_r^{p+r,q+1-r}. \end{tikzcd}$$

By the surjectivity of $\eta_r^{p-r,q+r-1}$ and $\delta^{p+q} \circ \delta^{p+q-1} = 0$ the commutativity of this diagram gives $d_r^{p,q} \circ d_r^{p-r,q+r-1} = 0$.

Construction of the page-turning isomorphisms: Let $r \in \mathbb{N}$. For $p, q \in \mathbb{Z}$ we first compute $\ker d_r^{p,q}$. By construction we have that

$$\ker d_r^{p,q} = \eta_r^{p,q}\Big(\ker\big(\eta_r^{p+r,q+1-r} \circ \delta^{p,q}|_{Z_r^{p,q}}\big)\Big),$$

and in the above step we have seen that

$$\begin{aligned} Z_{r-1}^{p+1,q-1} + Z_{r+1}^{p,q} &\subseteq \ker\big(\eta_r^{p+r,q+1-r} \circ \delta^{p,q}|_{Z_r^{p,q}}\big) \\ &= (\delta^{p,q}|_{Z_r^{p,q}})^{-1}\big(\ker \eta_r^{p+r,q+1-r}\big) \\ &= Z_r^{p,q} \cap (\delta^{p,q})^{-1}\big(Z_{r-1}^{p+r+1,q-r} + B_{r-1}^{p+r,q+1-r}\big). \end{aligned}$$

We now claim that this is in fact an equality: Let $c \in Z_r^{p,q}$ such that

$$\delta^{p,q}(c) \in Z_{r-1}^{p+r+1,q-r} + B_{r-1}^{p+r,q+1-r}.$$

Since $B_{r-1}^{p+r,q+1-r} = \delta^{p,q}(Z_{r-1}^{p+1,q-1})$ by (3), we can assume without loss of generality that

$$\delta^{p+q}(c) \in Z_{r-1}^{p+r+1,q-r} = F^{p+r+1}C^{p+q+1} \cap (\delta^{p+q+1})^{-1}\big(F^{p+2r}C^{p+q+2}\big).$$

Because $\delta^{p+q+1} \circ \delta^{p+q} = 0$ we thus obtain

$$\begin{aligned} &c \in Z_r^{p,q} \cap (\delta^{p,q})^{-1}\big(F^{p+r+1}C^{p+q+1}\big) \\ &\quad = F^pC^{p+q} \cap (\delta^{p,q})^{-1}\big(F^{p+r}C^{p+q+1}\big) \cap (\delta^{p,q})^{-1}\big(F^{p+r+1}C^{p+q+1}\big) \\ &\quad = F^pC^{p+q} \cap (\delta^{p,q})^{-1}\big(F^{p+r+1}C^{p+q+1}\big) \qquad (F \text{ is decreasing}) \\ &\quad = Z_{r+1}^{p,q}, \end{aligned}$$

which gives in total the inclusion

$$\ker\big(\eta_r^{p+r,q+1-r} \circ \delta^{p,q}|_{Z_r^{p,q}}\big) \subseteq Z_{r-1}^{p+1,q-1} + Z_{r+1}^{p,q},$$

and thus the claimed equality.

Now back in our computation of $\ker d_r^{p,q}$ this gives

$$\begin{aligned} \ker d_r^{p,q} &= \eta_r^{p,q}\Big(\ker\big(\eta_r^{p+r,q+1-r} \circ \delta^{p,q}|_{Z_r^{p,q}}\big)\Big) \\ &= \eta_r^{p,q}\big(Z_{r-1}^{p+1,q-1} + Z_{r+1}^{p,q}\big) \\ &= \eta_r^{p,q}\big(Z_{r+1}^{p,q}\big). \qquad (\text{since } Z_{r-1}^{p+1,q-1} \in \ker \eta_r^{p,q}) \end{aligned}$$

Using this we obtain that

$$Z_{r+1}^{p,q} \xrightarrow{\eta_r^{p,q}} \ker d_r^{p,q} \xrightarrow{\text{can. proj.}} H^{p,q}(E_r, d_r)$$

is a surjective map, which induces an isomorphism

$$H^{p,q}(E_r, d_r) \cong_R Z^{p,q}_{r+1} \big/ (\eta^{p,q}_r|_{Z^{p,q}_{r+1}})^{-1}\big(\operatorname{im} d^{p-r,q+r-1}_r\big).$$

For the "denominator" we first compute

$$\begin{aligned}
\operatorname{im} d^{p-r,q+r-1}_r &= \operatorname{im} d^{p-r,q+r-1}_r \circ \eta^{p-r,q+r-1}_r && (\eta^{p-r,q+r-1}_r \text{ is surjective})\\
&= \operatorname{im} \eta^{p,q}_r \circ \delta^{p+q-1}|_{Z^{p-r,q+r-1}_r} && (\text{by construction of } d^{p-r,q+r-1}_r)\\
&= \eta^{p,q}_r\big(\delta^{p+q-1}(Z^{p-r,q+r-1}_r)\big)\\
&= \eta^{p,q}_r\big(B^{p,q}_r\big), && (\text{by (3)})
\end{aligned}$$

which gives

$$\begin{aligned}
(\eta^{p,q}_r|_{Z^{p,q}_{r+1}})^{-1}\big(\operatorname{im} d^{p-r,q+r-1}_r\big) &= Z^{p,q}_{r+1} \cap (\eta^{p,q}_r)^{-1}\big(\operatorname{im} d^{p-r,q+r-1}_r\big)\\
&= Z^{p,q}_{r+1} \cap \big(B^{p,q}_r + \ker \eta^{p,q}_r\big)\\
&= Z^{p,q}_{r+1} \cap \big(B^{p,q}_r + Z^{p+1,q-1}_{r-1} + B^{p,q}_{r-1}\big)\\
&= Z^{p,q}_{r+1} \cap \big(B^{p,q}_r + Z^{p+1,q-1}_{r-1}\big)\\
&\qquad\qquad (B^{p,q}_{r-1} \subseteq B^{p,q}_r \text{ by (1)})\\
&= B^{p,q}_r + Z^{p,q}_{r+1} \cap Z^{p+1,q-1}_{r-1}.\\
&\qquad\qquad (B^{p,q}_r \subseteq Z^{p,q}_{r+1} \text{ by (1)})
\end{aligned}$$

Now the decreasing property of F gives

$$\begin{aligned}
Z^{p,q}_{r+1} \cap Z^{p+1,q-1}_{r-1} &= F^pC^{p+q} \cap (\delta^{p,q})^{-1}\big(F^{p+r+1}C^{p+q+1}\big)\\
&\quad \cap F^{p+1}C^{p+q} \cap (\delta^{p+q})^{-1}\big(F^{p+r}C^{p+q+1}\big)\\
&= F^{p+1}C^{p+q} \cap (\delta^{p+q})^{-1}\big(F^{p+r+1}C^{p+q+1}\big)\\
&= Z^{p+1,q-1}_r
\end{aligned}$$

and thus we have

$$(\eta^{p,q}_r|_{Z^{p,q}_{r+1}})^{-1}\big(\operatorname{im} d^{p-r,q+r-1}_r\big) = B^{p,q}_r + Z^{p+1,q-1}_r,$$

which finally gives

$$\begin{aligned}
H^{p,q}(E_r, d_r) &\cong_R Z_{r+1} \big/ (\eta^{p,q}_r|_{Z^{p,q}_{r+1}})^{-1}\big(\operatorname{im} d^{p-r,q+r-1}_r\big)\\
&= Z_{r+1} \big/ B^{p,q}_r + Z^{p+1,q-1}_r\\
&= E^{p,q}_{r+1}.
\end{aligned}$$

Identification of the first page: We first compute the 0-th page E_0:

Since the decreasing filtration F is compatibility with the differentials δ^* we have

$$\delta^n(F^pC^n) \subseteq F^pC^{n+1} \subseteq F^{p-1}C^{n+1}$$

for all $p \in \mathbb{Z}$ and $n \in \mathbb{N}$. For $p, q \in \mathbb{Z}$ and $r \in \{-1, 0\}$ this gives

$$F^pC^{p+q} \subseteq (\delta^{p+q})^{-1}\big(F^{p+r}C^{p+q+1}\big),$$

and thus

$$Z_r^{p,q} = F^pC^{p+q} \cap (\delta^{p+q})^{-1}\big(F^{p+r}C^{p+q+1}\big) = F^pC^{p+q}$$

as well as

$$\begin{aligned} B_r^{p,q} &= F^pC^{p+q} \cap \underbrace{\delta^{p+q-1}\big(F^{p-r}C^{p+q-1}\big)}_{\subseteq F^{p-r}C^{p+q} \subseteq F^pC^{p+q}} \\ &= \delta^{p+q-1}\big(F^{p-r}C^{p+q-1}\big) \\ &\subseteq F^{p-r}C^{p+q}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} E_0^{p,q} &= Z_r^{p,q} \big/ Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q} \\ &= F^pC^{p+q} \big/ F^{p+1}C^{p+q} + \underbrace{B_{r-1}^{p,q}}_{\subseteq F^{p+1}C^{p+q}} \\ &= F^pC^{p+q} \big/ F^{p+1}C^{p+q}. \end{aligned}$$

Now, since by construction the differential $d_0 \colon E_0 \to E_0$ is induced by the coboundary operators δ^*, the page-turning isomorphism gives

$$E_1^{p,q} \cong_R H^{p,q}(E_0, d_0) = H^{p+q}\left(F^pC \big/ F^{p+1}C\right).$$

Convergence in the bounded case: Now assume that (C, F) is a bounded filtered cochain complex. First we show that the constructed spectral sequence is indeed a first-quadrant spectral sequence:

Let $p, q \in \mathbb{Z}$ and first assume $p < 0$. Then our identification of the E_1-page gives

$$E_1^{p,q} \cong_R H^{p+q}(F^pC/F^{p+1}C) \cong_R 0,$$

as we have that $F^{p+1}C = F^pC = C$, since $p, p+1 \leq 0$. In the case that $q < 0$ we can use our original definition of $E_1^{p,q}$ as quotient of

$$\begin{aligned} Z_r^{p,q} &= F^pC^{p+q} \cap (\delta^{p+q})^{-1}\big(F^{p+1}C^{p+q+1}\big) \\ &\subseteq F^pC^{p+q}. \end{aligned}$$

Since $p+q < p$ we obtain $F^pC^{p+q} = 0$ which gives the triviality of $E_1^{p,q}$.

For the convergence of the spectral sequence we first note that for $p, q \in \mathbb{N}$ and $r \in \mathbb{N}$ large enough the canonical boundedness of F gives

$$Z_r^{p,q} = F^pC^{p+q} \cap (\delta^{p+q})^{-1}\big(\underbrace{F^{p+r}C^{p+q+1}}_{=0 \text{ for } r \text{ large enough}}\big) = F^pC^{p+q} \cap \ker \delta^{p,q} = Z_\infty^{p,q}$$

and

$$B_r^{p,q} = F^pC^{p+q} \cap \delta^{p+q-1}\big(\underbrace{F^{p-r}C^{p+q-1}}_{=C^{p+q+1} \text{ for } r \text{ large enough}}\big) = F^pC^{p+q} \cap \operatorname{im} \delta^{p+q-1} = B_\infty^{p,q}.$$

Thus, for $p, q \in \mathbb{N}$, we can assume without loss of generality that the module on the ∞-page E_∞ is given by

$$E_\infty^{p,q} = Z_\infty^{p,q} \big/ Z_\infty^{p+1,q-1} + B_\infty^{p,q}.$$

Similar as before we denote by

$$\eta_\infty^{p,q} \colon Z_\infty^{p,q} \longrightarrow E_\infty^{p,q}$$

the canonical projection and moreover we also denote by

$$\pi \colon \ker \delta^{p+q} \longrightarrow H^{p+q}(C)$$

the canonical projection.

Now we have by definition that $Z_\infty^{p,q} \subseteq \ker \delta^{p+q}$ and

$$\begin{aligned} \pi(Z_\infty^{p,q}) &= \pi\big(F^pC^{p+q} \cap \ker \delta^{p+q}\big) \\ &= \operatorname{im}\big(H^{p+q}(F^pC) \to H^{p+q}(C)\big) \\ &= F^pH^{p+q}(C). \end{aligned}$$

Next we consider the diagram

$$\begin{tikzcd} Z_\infty^{p,q} \arrow[rr, "\pi"] \arrow[d, "\eta_\infty^{p,q}"'] & & F^pH^{p+q}(C) \arrow[d, "\text{can. proj.}"] \\ E_\infty^{p,q} \arrow[r, dashed, "\varphi"'] & F^pH^{p+q}(C) \big/ F^{p+1}H^{p+q}(C). & \end{tikzcd}$$

Since

$$\begin{aligned}
\pi\bigl(\ker \eta_\infty^{p,q}\bigr) &= \pi\bigl(Z_\infty^{p+1,q-1} + \underbrace{B_\infty^{p,q}}_{\subseteq\, \operatorname{im}\delta^{p+q-1} = \ker\pi}\bigr)\\
&= \pi\bigl(Z_\infty^{p+1,q-1}\bigr)\\
&= F^{p+1}H^{p+q}(C) && \text{(same computation as above)}
\end{aligned}$$

we can again apply the universal property of the quotient to obtain a well-defined R-linear map

$$\varphi\colon E_\infty^{p,q} \longrightarrow F^pH^{p+q}(C) \big/ F^{p+1}H^{p+q}$$

making the above diagram commute. Because $\pi|_{Z_\infty^{p,q}}\colon Z_\infty^{p,q} \to F^pC^{p+q}(C)$ is surjective, also φ is surjective. Moreover, φ is also injective as

$$\begin{aligned}
\ker\varphi &= \eta_\infty^{p,q}\Bigl(Z_\infty^{p,q} \cap \pi^{-1}\bigl(F^{p+1}H^{p+q}(C)\bigr)\Bigr)\\
&= \eta_\infty^{p,q}\Bigl(Z_\infty^{p,q} \cap \bigl(\ker\pi + \ker\eta_\infty^{p,q}\bigr)\Bigr) && \text{(since } \pi(\ker\eta_\infty^{p,q}) = F^{p+1}H^{p+q}(C))\\
&= \eta_\infty^{p,q}\Bigl(F^pC^{p+q} \cap \ker\delta^{p+q} \cap \operatorname{im}\delta^{p+q-1}\Bigr)\\
& && \text{(by construction } \ker\pi = \operatorname{im}\delta^{p+q-1})\\
&= \eta_\infty^{p,q}\bigl(F^pC^{p+q} \cap \operatorname{im}\delta^{p+q-1}\bigr) && \text{(as } \operatorname{im}\delta^{p+q-1} \subseteq \ker\delta^{p+q})\\
&= \eta_\infty^{p,q}\bigl(B_\infty^{p,q}\bigr)\\
&= 0. && (B_\infty^{p,q} \subseteq \ker\eta_\infty^{p,q})
\end{aligned}$$

Hence φ is the claimed isomorphism

$$E_\infty^{p,q} \cong_R F^pH^{p+q}(C) \big/ F^{p+1}H^{p+q}(C). \qquad \square$$

Remark 1.3.5 (naturality of the construction)**.** The above construction of a filtered cochain complex is functorial in the following sense:

Let R be a ring, let (C, F) and $(\widetilde{C}, \widetilde{F})$ be filtered R-cochain complexes and let (E_*, d_*), $(\widetilde{E}_*, \widetilde{d}_*)$ be the spectral sequences given by Theorem 1.3.4 from (C, F) and $(\widetilde{C}, \widetilde{F})$, respectively. Moreover, let $f\colon C \to \widetilde{C}$ be a morphism of cochain complexes that is compatible with the filtrations F and $\widetilde{F}$, i.e., for each $p \in \mathbb{Z}$ the morphism f restricts to a morphism $F^pC \to \widetilde{F}^p\widetilde{C}$ of cochain complexes.

Then f induces a morphism of spectral sequences $g\colon (E_*, d_*) \to (\widetilde{E}_*, \widetilde{d}_*)$ such that the identification of the first page gives for all $p, q \in \mathbb{Z}$ a commutative diagram

$$\begin{array}{ccc}
E_1^{p,q} & \xrightarrow{\cong} & H^{p+q}(F^pC/F^{p+q}C)\\
{\scriptstyle g_1^{p,q}}\big\downarrow & & \big\downarrow{\scriptstyle H^{p+q}(\varphi^p)}\\
\widetilde{E}_1^{p,q} & \xrightarrow[\cong]{} & H^{p+q}(\widetilde{F}^p\widetilde{C}/\widetilde{F}^{p+q}\widetilde{C}),
\end{array}$$

where $\varphi^p\colon F^pC/F^{p+1}C \to \widetilde{F}^p\widetilde{C}/\widetilde{F}^{p+1}\widetilde{C}$ is also induced by f.

If the filtered cochain complexes (C, F), $(\widetilde{C}, \widetilde{F})$ are additionally both canonically bounded, the induced morphism of spectral sequences is compatible with the induced morphism in cohomology, i.e., we have that

$$\begin{tikzcd}
E_1^{p,q} \cong_R H^{p+q}(F^pC/F^{p+1}C) \arrow[r, Rightarrow] \arrow[d, "H^{p+q}(\varphi^p)"'] & H^{p+q}(C) \arrow[d, "H^{p+q}(f)"] \\
\widetilde{E}_1^{p,q} \cong_R H^{p+q}(\widetilde{F}^p\widetilde{C}/\widetilde{F}^{p+q}\widetilde{C}) \arrow[r, Rightarrow] & H^{p+q}(\widetilde{C})
\end{tikzcd}$$

commutes.

1.3.2 Spectral sequences of a double complex

Now that we have a first tool at hand to construct spectral sequences, we apply it to construct two spectral sequences of a *double complex*.

Definition 1.3.6 (double complex)**.** Let R be a ring. A *R-double complex* consists of a bigraded R-module M and two bigraded morphisms of bigraded modules $d_h\colon M \to M$ of bidegree $(1, 0)$ and $d_v\colon M \to M$ of bidegree $(0, 1)$ such that we have

$$d_h \circ d_h = 0, \qquad d_v \circ d_v = 0$$

and

$$d_v \circ d_h = d_h \circ d_v.$$

For two R-double complexes $M, \widetilde{M}$, we call a morphism of bigraded modules $f\colon M \to \widetilde{M}$ a *morphism of double complexes*, if it commutes with the two differentials, i.e., we have

$$\widetilde{d}_v \circ f = f \circ d_v \qquad \text{and} \qquad \widetilde{d}_h \circ f = f \circ d_h,$$

where d_h, d_v are the differentials of M and $\widetilde{d}_h, \widetilde{d}_v$ are the differentials of $\widetilde{M}$.

Moreover we say that M is a *first-quadrant double complex*, if M is a double complex such that $M^{p,q} = 0$ for all $p, q \in \mathbb{Z}$ with $p < 0$ or $q < 0$.

Remark 1.3.7 (the two homologies of a double complex)**.** Let R be a ring and let M be a R-double complex. Since by definition both (M, d_h) and (M, d_v) are differential bigraded modules we have two possibilities of taking homology: $H(M, d_h)$ and $H(M, d_v)$.

Moreover, as d_v and d_h commute with each other we obtain that d_v induces for each $p, q \in \mathbb{Z}$ a well-defined R-linear map

$$\bar{d}_v^{p,q}\colon H^{p,q}(M, d_h) \longrightarrow H^{p,q+1}(M, d_h)$$

such that $\bar{d}_v$ turns $H(M, d_h)$ into a differential bigraded module of bidegree $(0, 1)$.
Similar, the differential d_h induces a bigraded morphism

$$\bar{d}_h \colon H(M, d_v) \to H(M, d_v),$$

turning $H(M, d_v)$ into a differential bigraded module of bidegree $(1, 0)$.

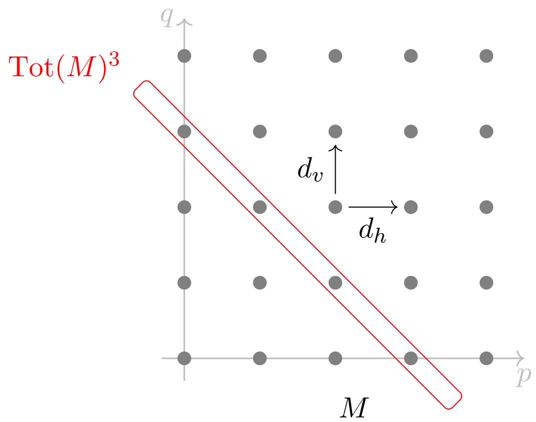


Figure 1.4: Visualisation of a double complex and its double complex

Definition 1.3.8 (total complex)**.** Let R be a ring and let M be a double complex. The *total complex* $\mathrm{Tot}(M)$ *of* M is defined to be the R-cochain complex with the cochain modules

$$\Big(\mathrm{Tot}(M)^n := \bigoplus_{p+q=n} M^{p,q}\Big)_{n \in \mathbb{N}}$$

and the coboundary operators

$$\big(\delta^n \colon \mathrm{Tot}(M)^n \to \mathrm{Tot}(M)^{n+1}\big)_{n \in \mathbb{N}}$$

given by

$$\begin{aligned} \delta^n \colon \mathrm{Tot}(M)^n &\longrightarrow \mathrm{Tot}(M)^{n+1} \\ M^{p,q} \ni m &\longmapsto d_v^{p,q}(m) + (-1)^p \cdot d_h^{p,q}(m) \end{aligned}$$

for each $n \in \mathbb{N}$.

Due to the conditions on the differentials d_v and d_h it is easy to see that the total complex $\mathrm{Tot}(M)$ is indeed a cochain complex.

Remark 1.3.9 (two filtrations of the total complex)**.** Let R be a ring and let M be a R-double complex. Then the total complex $\operatorname{Tot}(M)$ of M has two canonical filtrations:

The *column-wise filtration* given by

$$^{①}F^p \operatorname{Tot}(M)^n = \bigoplus_{\substack{r+s=n \\ r\geq p}} M^{r,s},$$

for $p \in \mathbb{Z}$ and $n \in \mathbb{N}$, and the *row-wise filtration* given by

$$^{②}F^p \operatorname{Tot}(M)^n = \bigoplus_{\substack{r+s=n \\ s\geq p}} M^{r,s},$$

for $p \in \mathbb{Z}$ and $n \in \mathbb{N}$.

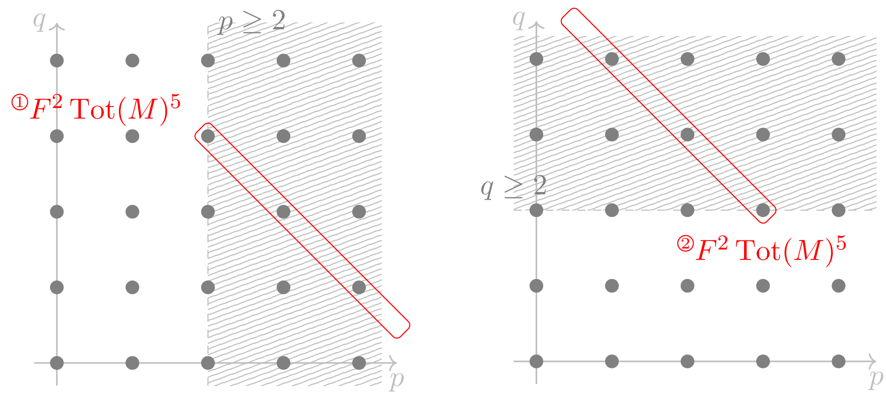


Figure 1.5: Visualisation of the column-wise and the row-wise filtration

We can now consider the spectral sequences associated to these filtrations.

Theorem 1.3.10 (spectral sequences of a double complex)**.** *Let R be a ring and let M be a R-double complex. Then there are two spectral sequences $({}^{①}E_*, {}^{①}d_*)$ and $({}^{②}E_*, {}^{②}d_*)$ starting with*

$${}^{①}E_1^{p,q} \cong_R H^{p,q}(M, d_v) \qquad \text{and} \qquad {}^{②}E_1^{p,q} \cong_R H^{q,p}(M, d_h),$$

where the differentials ${}^{①}d_1$ and ${}^{②}d_1$ are given by $\bar{d}_h$ and $\bar{d}_v$ respectively.

Moreover, if M is a first-quadrant double complex, both of these spectral sequences are first-quadrant spectral sequences converging to the cohomology of the total complex

$$\begin{aligned} {}^{①}E_1^{p,q} \cong_R H^{p,q}(M, d_v) &\Longrightarrow H^{p+q}\big(\operatorname{Tot}(M)\big) \\ {}^{②}E_1^{p,q} \cong_R H^{q,p}(M, d_h) &\Longrightarrow H^{p+q}\big(\operatorname{Tot}(M)\big). \end{aligned}$$

Proof. Consider the column-wise filtration ${}^{①}F$ of $\mathrm{Tot}(M)$. By Theorem 1.3.4 there exists a spectral sequence $({}^{①}E_*, {}^{①}d_*)$ starting with

$$
{}^{①}E_1^{p,q} \cong_R H^{p+q}\left({}^{①}F^p \,\mathrm{Tot}(M) \big/ {}^{①}F^{p+1}\,\mathrm{Tot}(M) \right).
$$

Using the definition of the filtration we could use this to directly compute the first page. However, since we also want to identify the the differentials we make another computation:

With the same notation as in the proof of Theorem 1.3.10, we know that the page ${}^{①}E_1$ is given by

$$
\begin{aligned}
{}^{①}E_1^{p,q} &= Z_1^{p,q} \big/ Z_0^{p+1,q-1} + B_0^{p,q} \\
&= Z_1^{p,q} \big/ Z_0^{p+1,q-1} + d^{p+q}(Z_0^{p,q-1}).
\end{aligned}
$$

Now we can compute for $p, q \in \mathbb{Z}$ that

$$
\begin{aligned}
Z_1^{p,q} &= {}^{①}F^p \,\mathrm{Tot}(M)^{p+q} \cap (\delta^{p+q})^{-1}\big({}^{①}F^{p+1}\,\mathrm{Tot}(M)^{p+q+1}\big) \\
&= \bigoplus_{\substack{r+s=p+q \\ r\geq p}} M^{r,s} \cap (\delta^{p+q})^{-1}\Big(\bigoplus_{\substack{r+s=p+q \\ r\geq p+1}} M^{r,q} \Big) \\
&= \bigoplus_{\substack{r+s=p+q \\ r\geq p}} M^{r,s} \cap \bigoplus_{r+s=p+q} \left\{ m \in M^{r,s} \;\middle|\; \delta^{p+q}(m) \in \bigoplus_{\substack{r'+s'=p+q \\ r'\geq p+1}} M^{p',q'} \right\} \\
&= \bigoplus_{\substack{r+s=p+q \\ r\geq p}} \left\{ m \in M^{r,s} \;\middle|\; \underbrace{d_v^{r,s}(m)}_{\in M^{r,s+1}} + (-1)^r \cdot \underbrace{d_h^{r,s}(m)}_{\in M^{r+1,s}} \in \bigoplus_{\substack{r'+s'=p+q \\ r'\geq p+1}} M^{p',q'} \right\} \\
&= \ker d_v^{p,q} + \bigoplus_{\substack{r+s=p+q \\ r\geq p+1}} M^{r,s},
\end{aligned}
$$

and similar computations show

$$
\begin{aligned}
Z_0^{p+1,q-1} &= {}^{①}F^{p+1}\,\mathrm{Tot}(M)^{p+q} \cap (\delta^{p+q})^{-1}\big({}^{①}F^{p+1}\,\mathrm{Tot}(M)^{p+q+1}\big) \\
&= \bigoplus_{\substack{r+s=p+q \\ r\geq p+1}} M^{r,s}
\end{aligned}
$$

as well as

$$
\begin{aligned}
Z_0^{p,q-1} &= {}^{①}F^{p}\,\mathrm{Tot}(M)^{p+q-1} \cap (\delta^{p+q-1})^{-1}\big({}^{①}F^{p}\,\mathrm{Tot}(M)^{p+q}\big) \\
&= \bigoplus_{\substack{r+s=p+q-1 \\ r\geq p}} M^{r,s}.
\end{aligned}
$$

Thus we have

$$\begin{aligned}
Z_0^{p+1,q-1} + \delta^{p+q-1}\big(Z_0^{p,q-1}\big) &= \bigoplus_{\substack{r+s=p+q\\ r\geq p+1}} M^{r,s} + \delta^{p+q-1}\Big(\bigoplus_{\substack{r+s=p+q-1\\ r\geq p}} M^{r,s}\Big)\\
&= \bigoplus_{\substack{r+s=p+q\\ r\geq p+1}} M^{r,s} + \bigoplus_{\substack{r+s=p+q-1\\ r\geq p}} \big(\operatorname{im} d_v^{r,s} + \operatorname{im} d_h^{r,s}\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by construction of }\delta\text{)}\\
&= \bigoplus_{\substack{r+s=p+q\\ r\geq p+1}} M^{r,s} + \operatorname{im} d_v^{p,q-1}\\
&\qquad\text{(only }\operatorname{im} d_v^{p,q-1}\text{ is not contained in the first sum)}
\end{aligned}$$

and thereby

$$\begin{aligned}
{}^{①}E_1^{p,q} &= Z_1^{p,q} \big/ Z_0^{p+1,q-1} + d^{p+q}(Z_0^{p,q-1})\\
&= \ker d_v^{p,q} + \bigoplus_{\substack{r+s=p+q\\ r\geq p+1}} M^{r,s} \Big/ \bigoplus_{\substack{r+s=p+q\\ r\geq p+1}} M^{r,s} + \operatorname{im} d_v^{p,q-1}\\
&\cong_R \ker d_v^{p,q} \big/ \operatorname{im} d_v^{p,q-1} \qquad\qquad (\text{as } \operatorname{im} d_v^{p,q-1} \subseteq \ker d_v^{p,q} \subseteq M^{p,q})\\
&= H^{p,q}(M, d_v).
\end{aligned}$$

Since the differential ${}^{①}d_1$ is induced by the differential δ, we obtain that under this identification ${}^{①}d_1$ coincides, up to a sign, with $\bar{d}_h$. However, this sign neither affects the differential condition nor the page-turning isomorphisms (the homology does not change). Thus we can assume with out loss of generality that ${}^{①}d_1$ is given by $\bar{d}_h$.

If now M is a first-quadrant double complex the filtration ${}^{①}F$ is obviously canonically bounded, thus the construction of Theorem 1.3.4 gives that the spectral sequence $({}^{①}E_*, {}^{①}d_*)$ indeed converges to the cohomology of $\operatorname{Tot}(M)$.

The second spectral sequence $({}^{②}E_*, {}^{②}d_*)$ arises similar, by considering the row-wise filtration ${}^{②}F$ of $\operatorname{Tot}(M)$. □

Remark 1.3.11 (naturality of the construction)**.** Using the naturality of the spectral sequence of a filtered complex we also obtain the following naturality of the two spectral sequences of a double complex:

Let R be a ring and let $f\colon M \to \widetilde{M}$ be a morphism of R-double complexes. Then f obviously induces morphisms of filtered complexes between the column-wise and row-wise filtrations of M and $\widetilde{M}$. Theses morphism of filtered complexes in turn induces morphisms of the first and second spectral sequences, which under the identifications of the first pages are simply given by

$$H^{p,q}(f)\colon H^{p,q}(M,d_v) \longrightarrow H^{p,q}(\widetilde{M},\widetilde{d}_v)$$

in the case of the first spectral sequence, and by

$$H^{q,p}(f)\colon H^{q,p}(M,d_h) \longrightarrow H^{q,p}(\widetilde{M},\widetilde{d}_h)$$

in the case of the second spectral sequence.

If moreover both M and $\widetilde{M}$ are first-quadrant double complexes these morphism of spectral sequences are compatible with the induced map in cohomology of the double complex, i.e., we have that both

$$\begin{tikzcd} {}^{①}E_1^{p,q} \cong_R H^{p,q}(M,d_v) \arrow[r, Rightarrow] \arrow[d, "H^{p,q}(f)"'] & H^{p+q}\big(\mathrm{Tot}(M)\big) \arrow[d, "H^{p+q}\left(\mathrm{Tot}(f)\right)"] \\ {}^{①}\widetilde{E}_1^{p,q} \cong_R H^{p,q}(\widetilde{M},\widetilde{d}_v) \arrow[r, Rightarrow] & H^{p+q}\big(\mathrm{Tot}(\widetilde{M})\big) \end{tikzcd}$$

and

$$\begin{tikzcd} {}^{②}E_1^{p,q} \cong_R H^{q,p}(M,d_h) \arrow[r, Rightarrow] \arrow[d, "H^{q,p}(f)"'] & H^{p+q}\big(\mathrm{Tot}(M)\big) \arrow[d, "H^{p+q}\left(\mathrm{Tot}(f)\right)"] \\ {}^{②}\widetilde{E}_1^{p,q} \cong_R H^{q,p}(\widetilde{M},\widetilde{d}_v) \arrow[r, Rightarrow] & H^{p+q}\big(\mathrm{Tot}(\widetilde{M})\big) \end{tikzcd}$$

commute, where $\mathrm{Tot}(f)\colon \mathrm{Tot}(M) \to \mathrm{Tot}(\widetilde{M})$ is the map induced by f on the total complexes.

Chapter 2

Spectral sequences in bounded cohomology

Now that we have seen some examples of spectral sequences we now come to the spectral sequences in bounded cohomology.

We begin by constructing an analogue of the Hochschild-Serre spectral sequence for bounded cohomology. As one application of this spectral sequence we will prove a characterisation of *amenable* and *boundedly n-acyclic* morphisms.

Another application of spectral sequences to bounded cohomology we will discuss *cohomological Leray theorem*, with a special focus on the Leray theorem in bounded cohomology.

If the reader is *not* already familiar with the computation of bounded cohomology via resolutions and the relation of bounded cohomology with amenability we highly recommend reading the corresponding chapters in the book of Frigerio [7] before proceeding with this chapter.

Overview of this chapter

2.1 The Hochschild-Serre spectral sequence

The goal of this section is to derive a bounded cohomology analogue of the Hochschild-Serre spectral sequence and to discuss some of its applications. In the ideal case we could associate to each short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0$$

of groups and each semi-normed $R[\Gamma]$-module V a convergent spectral sequence

$$E_2^{p,q} \cong_R H_b^p\big(\Delta; H_b^q(\Lambda; V)\big) \Longrightarrow H_b^{p+q}(\Gamma; V).$$

Although it is always possible to construct a spectral sequence converging to the bounded cohomology of Γ with coefficient in V it is *not* always possible to identify the second page as above.

The Hochschild-Serre spectral sequence in bounded cohomology was first constructed by Noskov [21] and later generalised by Burger and Monod [3] to continuous bounded cohomology.

Remark 2.1.1. We note that there also is a paper of Bouarich on the Hochschild-Serre spectral sequence for bounded cohomology with semi-normed vector spaces as coefficients [2]. In this paper it is claimed that the above identification of the second page can always be made. However, a fundamental step in this proof is flawed as we will discuss in Remark 2.1.20.

2.1.1 Construction of the Hochschild-Serre spectral sequence

For simplicity we will assume in the following that Λ is a normal subgroup of Γ and that Δ is given by the quotient Γ/Λ. Moreover, in order to avoid confusion between the coboundary operators of the bounded cochain complexes we will indicate the corresponding group with a lower index, e.g., δ_Λ^* will denote the usual differential of $C_b^*(\Lambda; V)$ for some semi-normed $R[\Lambda]$-module V.

We first construct a double complex which will then give us two spectral sequences. We define the first-quadrant bigraded R-module M for $p, q \in \mathbb{N}$ by

$$M^{p,q} := C_b^q\big(\Delta; C_b^p(\Gamma; V)^\Lambda\big)^\Delta,$$

where $C_b^p(\Gamma; V)^\Lambda$ carries the Δ-action induced by the Γ-action on $C_b^p(\Gamma; V)$. On this double complex we define two differentials. The differential $d_v \colon M \to M$ of bidegree $(0, 1)$ is simply given by the standard differential

$$\delta_\Delta^q \colon C_b^q\big(\Delta; C_b^p(\Gamma; V)^\Lambda\big)^\Delta \longrightarrow C_b^{p+1}\big(\Delta; C_b^p(\Gamma; V)^\Lambda\big)^\Delta,$$

and the differential $d_h \colon M \to M$ of bidegree $(1, 0)$ induced by the standard differential of $C_b^*(\Gamma; V)$:

$$\begin{aligned} C_b^q(\Delta; \delta_\Gamma^p)^\Delta \colon C_b^q\big(\Delta; C_b^p(\Gamma; V)^\Lambda\big)^\Delta &\longrightarrow C_b^q\big(\Delta; C_b^{p+1}(\Gamma; V)^\Lambda\big)^\Delta \\ f &\longmapsto \delta_\Gamma^p \circ f. \end{aligned}$$

By definition it is easy to see that these two differentials indeed turn M into a double complex.

Now that we have a double complex we can apply Theorem 1.3.10 to obtain two spectral sequences, $({}^{①}E_*, {}^{①}d_*)$ and $({}^{②}E_*, {}^{②}d_*)$, both converging to the cohomology of the total complex $\operatorname{Tot}(M)$. The first spectral sequence $({}^{①}E_*, {}^{①}d_*)$ already collapses on the second page and thus can be used to determine the cohomology of the total complex $\operatorname{Tot}(M)$. But first we briefly mention the naturality of this construction.

Remark 2.1.2 (naturality of the two spectral sequences)**.** Let R be a normed ring, let

$$\begin{tikzcd} 0 \arrow[r] & \Lambda \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d, "\varphi"'] & \Delta \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & \widetilde{\Lambda} \arrow[r] & \widetilde{\Gamma} \arrow[r] & \widetilde{\Delta} \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of groups with exact rows, let V be a semi-normed $R[\Gamma]$-module, let $\widetilde{V}$ be a semi-normed $R[\widetilde{\Gamma}]$-module and let $f\colon \widetilde{V} \to V$ be a bounded R-morphism that is compatible with φ in the sense that

$$\forall_{g\in\Gamma}\forall_{\widetilde{v}\in\widetilde{V}}\colon f\bigl(\varphi(g)\cdot\widetilde{v}\bigr) = g\cdot f(\widetilde{v}).$$

Then it is easy to see that φ and f first induce a bounded R-cochain map

$$C_b^*(\varphi; f)\colon C_b^*(\widetilde{\Gamma}, \widetilde{V})^{\widetilde{\Lambda}} \longrightarrow C_b^*(\Gamma, V)^{\Lambda}.$$

Now by the commutativity of the above diagram these morphisms are compatible with ψ and thus induce for $p \in \mathbb{N}$ a bounded R-cochain map

$$C_b^*\bigl(\psi; C_b^p(\varphi; f)\bigr)\colon C_b^*\bigl(\widetilde{\Delta}; C_b^p(\widetilde{\Gamma}; \widetilde{V})^{\widetilde{\Lambda}}\bigr)^{\widetilde{\Delta}} \longrightarrow C_b^*\bigl(\Delta; C_b^p(\Gamma; V)^{\Lambda}\bigr)^{\Delta},$$

i.e., a morphism of double complexes $\widetilde{M} \to M$, where M is the double complex associated to the first row and the $R[\Gamma]$-module V and $\widetilde{M}$ is the double complex associated to the second row and the $R[\widetilde{\Gamma}]$-module $\widetilde{V}$.

Now the naturality of the spectral sequences of a double complex (Remark 1.3.11) give that this morphism of double complexes induces morphisms of the associated first and second second spectral sequences that are compatible with the induced morphism in cohomology of the total complexes.

Proposition 2.1.3 (cohomology of the total complex)**.** *Let R be a normed ring, let*

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0$$

be a short exact sequence of groups and let V be a semi-normed $R[\Gamma]$-module. Then the cohomology of the total complex $H^*(\mathrm{Tot}(M))$ is isomorphic to the bounded cohomology $H_b^*(\Gamma; V)$ of Γ with values in V.

Moreover, assume we are given a commutative diagram of groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda & \longrightarrow & \Gamma & \longrightarrow & \Delta & \longrightarrow & 0 \\
 & & \big\downarrow & & {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} & & \\
0 & \longrightarrow & \widetilde{\Lambda} & \longrightarrow & \widetilde{\Gamma} & \longrightarrow & \widetilde{\Delta} & \longrightarrow & 0
\end{array}$$

with exact rows, a semi-normed $R[\Gamma]$-module V, a semi-normed $R[\widetilde{\Gamma}]$-module $\widetilde{V}$ and a bounded R-morphism $f\colon \widetilde{V} \to V$ that is compatible with φ. Then under the above isomorphisms the induced map in cohomology of the total complexes of the rows is simply given by

$$H_b^*(\varphi; f)\colon H_b^*(\widetilde{\Gamma}; \widetilde{V}) \longrightarrow H_b^*(\Gamma; V).$$

Proof. We consider the first spectral sequence $({}^{\textcircled{1}}E_*, {}^{\textcircled{1}}d_*)$ of the double complex M and let $p, q \in \mathbb{N}$. For this spectral sequence we have

$${}^{\textcircled{1}}E_1^{p,q} \cong_R H^{p,q}(M, d_v) = H^q(M^{p,*}, d_v^{p,*}),$$

where by construction the cochain complex $(M^{p,*}, d_v^{p,*})$ is just the cochain complex

$$\Big(C_b^*\big(\Delta; C_b^*(\Gamma; V)^\Lambda\big)^\Delta, \delta_\Delta^*\Big).$$

Hence we have

$${}^{\textcircled{1}}E_1^{p,q} \cong_R H_b^q\big(\Delta; C_b^p(\Gamma; V)^\Lambda\big).$$

Now we use that bounded cohomology with relatively injective coefficients vanishes: The $R[\Gamma]$-module $C_b^p(\Gamma; V)$ is relatively injective, using this one can show that $C_b^p(\Gamma; V)^\Lambda$ is a relatively injective $R[\Delta]$-module [6, Proposition A.1.7]. Thus we have that ${}^{\textcircled{1}}E_1^{p,q}$ vanishes whenever $q \neq 0$, and for $q = 0$ we have

$${}^{\textcircled{1}}E_1^{p,0} \cong_R \big(C_b^p(\Gamma; V)^\Lambda\big)^\Delta = C_b^p(\Gamma; V)^\Gamma.$$

Moreover, since we know that the differential ${}^{\textcircled{1}}d_1\colon {}^{\textcircled{1}}E_1 \to {}^{\textcircled{1}}E_1$ is induced by the differential d_h, which in turn is induced by the standard differential of $C_b^*(\Gamma; V)$, we obtain that

$${}^{\textcircled{1}}E_2^{p,0} \cong_R H_b^p(\Gamma; V)$$

is the only non-zero row of ${}^{\textcircled{1}}E_2$. Thus for bidegree reasons ${}^{\textcircled{1}}E_2 = {}^{\textcircled{1}}E_\infty$, and the convergence towards $H^*(\mathrm{Tot}(M))$ gives the claim

$$H_b^p(\Gamma; V) \cong_R {}^{\textcircled{1}}E_2^{p,0} = {}^{\textcircled{1}}E_\infty^{p,0} \cong_R H^p(\mathrm{Tot}(M)).$$

Finally, for the identification of the map induced in cohomology we first see that under the above identifications the induced morphism of spectral sequences is on the first page simply given by

$$C_b^p(\varphi; f)\colon C_b^p(\widetilde{\Gamma}; \widetilde{V})^{\widetilde{\Gamma}} \longrightarrow C_b^p(\Gamma; V)^{\Gamma}.$$

Thus on the second page, and thereby ∞-page, the morphism is given by

$$H_b^p(\varphi; f)\colon H_b^p(\widetilde{\Gamma}; \widetilde{V}) \longrightarrow H_b^p(\Gamma; V).$$

Now using the compatibility of the induced morphism of spectral sequences with the morphism induced in cohomology this gives the claimed identification. $\square$

Next let us consider the second spectral sequence $({}^{②}E_*, {}^{②}d_*)$. By construction the first page of this spectral sequence is given by

$$\begin{aligned}
{}^{②}E_1^{p,q} &\cong_R H^{q,p}(M, d_h) \\
&= H^q\big(M^{*,p}, d_h^{*,p}\big) \\
&= H^q\Big(C_b^p\big(\Delta; C_b^*(\Gamma; V)^{\Lambda}\big)^{\Delta}, C_b^p(\Delta; \delta_\Gamma^*)\Big).
\end{aligned}$$

For our wished-for identification

$${}^{②}E_2^{p,q} \cong_R H_b^p\big(\Delta; H_b^q(\Lambda; V)\big)$$

we would like this homology to be simply given by

$$C_b^p\big(\Delta; H_b^q(\Lambda; V)\big)^{\Delta}.$$

But therefor we need two things: First that $C_b^p(\Delta; \,\cdot\,)^{\Delta}$ "commutes" with taking homology, and second that the homology of $C_b^*(\Gamma; V)^{\Lambda}$ indeed gives the bounded cohomology of Λ.

Let us begin with the second point.

Lemma 2.1.4. *Let R be a normed ring, let Γ be a group, let V be a semi-normed $R[\Gamma]$-module and let $\Lambda \subseteq \Gamma$ be a normal subgroup. Then we have for each $n \in \mathbb{N}$ a canonical bilipschitz isomorphism*

$$H^n\big(C_b^*(\Gamma; V)^{\Lambda}, \delta_\Gamma^*\big) \cong H_b^n(\Lambda, V).$$

Proof. We first prove that $\big(C_b^*(\Gamma; V), \delta_\Gamma^*\big)$ is a strong relatively injective resolution of the $R[\Lambda]$-module V:

The well-known contracting homotopy of $\big(C_b^*(\Gamma; V), \delta_\Gamma^*\big)$ as resolution of V as $R[\Gamma]$-module [7, Proposition 4.3] also gives a contracting homotopy as resolution of V as $R[\Lambda]$-module. Moreover, since the diagonal action $\Lambda \curvearrowright \Gamma^{n+1}$ is clearly free for each $n \in \mathbb{N}$ we obtain that the $R[\Lambda]$-modules

$$C_b^n(\Gamma; V) = \ell^\infty(\Gamma^{n+1}, V)$$

are relatively injective [6, Proposition A.1.3].

So $\big(C_b^*(\Gamma; V), \delta_\Gamma^*\big)$ is also a strong relatively injective resolution of the $R[\Lambda]$-module V. Thus we obtain the claimed bilipschitz isomorphism [7, Corollary 4.5]

$$H^n\big(C_b^*(\Gamma; V)^\Lambda, \delta_\Gamma^*\big) \cong H_b^n(\Lambda; V). \qquad \square$$

For the "commutativity" of $C_b^p(\Delta; \,\cdot\,)^\Delta$ and taking homology we first prove the following lemma.

Lemma 2.1.5. *Let R be a normed ring, let Γ be a group, let $\Gamma \curvearrowright S$ be a group action of Γ on a set S, let V be a semi-normed $R[\Gamma]$-module and let $W \subseteq V$ be a $R[\Gamma]$-submodule of V. Then we have a canonical R-isomorphism*

$$\frac{\ell^\infty(S, V)^\Gamma}{\ell^\infty(S, W)^\Gamma} \longrightarrow \ell^\infty(S, V/W)^\Gamma.$$

Proof. We consider the R-linear morphism

$$\begin{aligned} \varphi\colon \ell^\infty(S, V)^\Gamma &\longrightarrow \ell^\infty(S, V/W)^\Gamma \\ f &\longmapsto \big(s \mapsto [f(s)]\big) \end{aligned}$$

and first show the surjectivity of this map. Let $g \in \ell^\infty(S, V/W)^\Gamma$ and let $\varepsilon > 0$. Then we have for each $s \in S$ that $g(s) \in V/W$, and by construction of the semi-norm on V/W there exists some $f(s) \in V$ with $[f(s)] = g(s)$ and

$$\big\|f(s)\big\|_V \le \big\|g(s)\big\|_{V/W} + \varepsilon \le \|g\|_\infty + \varepsilon.$$

Thus f is a map $f\colon S \to V$ with $\|f\|_\infty \le \|g\|_\infty + \varepsilon$. Since g was by assumption Γ-invariant we can also choose f to be Γ-invariant as well, i.e., such that we have $f \in \ell^\infty(S, V)$ with

$$\varphi(f)(s) = [f(s)] = g(s)$$

for each $s \in S$. This shows the surjectivity of φ. Moreover, we have that the kernel of φ is given by $\ker \varphi = \ell^\infty(S, W)^\Gamma$: For $f \in \ell^\infty(S, W)^\Gamma$ and $s \in S$ we have

$$\varphi(f)(s) = [\underbrace{f(s)}_{\in W}] = 0,$$

i.e., $\varphi(f) = 0$ which gives $\ell^\infty(S, W)^\Delta \subseteq \ker\varphi$. Conversely, if we have $f \in \ell^\infty(S, V)^\Gamma$ with $\varphi(f) = 0$ we obtain for $s \in S$ that

$$0 = \varphi(f)(s) = [f(s)]$$

and thus $f(s) \in W$ which gives the inclusion $\ker\varphi \subseteq \ell^\infty(S, W)^\Gamma$.

Hence φ induces an isomorphism

$$\frac{\ell^\infty(S, V)^\Gamma}{\ell^\infty(S, W)^\Gamma} \longrightarrow \ell^\infty(S, V/W)^\Gamma. \qquad \square$$

Now in order to be able to apply the above lemma to identify the first page of the spectral sequence $({}^{②}E_*, {}^{②}d_*)$ we still need

$$\ker\bigl(C_b^p(\Delta; \delta_\Gamma^q)^\Delta\bigr) = C_b^p(\Delta; \ker\delta_\Gamma^q)^\Delta$$

as well as

$$\operatorname{im}\bigl(C_b^p(\Delta; \delta_\Gamma^{q-1})^\Delta\bigr) = C_b^p(\Delta; \operatorname{im}\delta_\Gamma^{q-1})^\Delta.$$

For the kernel it is fairly easy to see that this equality always holds without any further assumptions. For the image, however, only one inclusion, namely the one from left to right, is easy to show. But the other inclusion poses a problem: In general it *does not* hold.

Example 2.1.6 (a positve example)**.** Let R be a normed ring, let

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0$$

be a short exact sequence of groups where Δ is finite, and let V be a semi-normed $R[\Gamma]$-module. Since *any* map of a finite set is already finite, the condition that Δ is finite gives that

$$C_b^*(\Delta; W) = \ell^\infty(\Delta^{*+1}, W) = \operatorname{Hom}_{\mathrm{Set}}(\Delta^{*+1}, W) = C^*(\Delta; W)$$

for each $R[\Delta]$-module W. Hence, the desired equality

$$\operatorname{im}\bigl(C_b^p(\Delta; \delta_\Gamma^{q-1})^\Delta\bigr) = C_b^p(\Delta; \operatorname{im}\delta_\Gamma^{q-1})^\Delta$$

reduces to

$$\operatorname{im}\bigl(\operatorname{Hom}_{\mathrm{Set}}(\Delta^{p+1}, \delta_\Gamma^{q-1})^\Delta\bigr) = \operatorname{Hom}_{\mathrm{Set}}(\Delta^{p+1}, \operatorname{im}\delta_\Gamma^{q-1})^\Delta$$

which is easily seen to be true for all $p, q \in \mathbb{N}$.

In the case that our coefficient module V is a *Banach* Γ*-module*, i.e., a Banach space with isometric Γ-action, we have the following:

Proposition 2.1.7 (characterization of equality for Banach coefficients)**.** *Let Γ be a group, let $\Gamma \curvearrowright S$ be a group action of Γ on a set S and let $f\colon V \to W$ be a morphism of Banach Γ-modules. Then we have the following:*

1. *If the image of f is closed in W we have*
$$\operatorname{im}\bigl(\ell^\infty(S,f)^\Gamma\bigr) = \ell^\infty(S, \operatorname{im} f)^\Gamma.$$

2. *If Γ is infinite and the image of f is* not *closed then we have for all $n \in \mathbb{N}_{\geq 1}$*
$$\operatorname{im}\bigl(\ell^\infty(\Gamma^{n+1},f)^\Gamma\bigr) = \ell^\infty(\Gamma^{n+1}, \operatorname{im} f)^\Gamma.$$

In particular we obtain for a short exact sequence
$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0$$
groups and a Banach Γ-module V the following:

3. *If for $q \in \mathbb{N}$ the semi-norm on $H_b^q(\Lambda;V)$ is a norm we have*
$$\operatorname{im}\bigl(C_b^p(\Delta;\delta_\Gamma^{q-1})^\Delta\bigr) = C_b^p(\Delta; \operatorname{im}\delta_\Gamma^{q-1})^\Delta$$
for all $p \in \mathbb{N}$.

4. *If Δ is infinite we have for $q \in \mathbb{N}$ that*
$$\operatorname{im}\bigl(C_b^p(\Delta;\delta_\Gamma^{q-1})^\Delta\bigr) = C_b^p(\Delta; \operatorname{im}\delta_\Gamma^{q-1})^\Delta$$
holds for all $p \in \mathbb{N}_{\geq 1}$ if and only if the semi-norm on $H_b^q(\Lambda;V)$ is a norm.

For the proof of the above proposition we will use the following functional analytic fact.

Proposition 2.1.8 (characterisation of closed images [1, Corollary 2.15])**.** *For a bounded morphism $f\colon V \to W$ between Banach spaces the following are equivalent:*

1. *The image of f is closed in W.*

2. *There exists a $c \in \mathbb{R}_{>0}$ such that for each $w \in \operatorname{im} f$ there exists a $v \in V$ with $f(v) = w$ and $\|v\|_V \leq c \cdot \|w\|_W$.*

Proof of Proposition 2.1.7.

1. We prove the two inclusions:

"$\subseteq$": First let $g \in \operatorname{im}\big(\ell^\infty(S, f)^\Gamma\big)$. Then there is some $h \in \ell^\infty(S, V)^\Gamma$ such that $g = f \circ h$. But this gives in particular that $\operatorname{im} g \subseteq \operatorname{im} f$, i.e., we can consider g as element of $\ell^\infty(S, \operatorname{im} f)^\Gamma$.

As we already noted above this inclusion uses neither the completeness of V and W nor the closedness of $\operatorname{im} f$ and in fact holds for any bounded morphism $f\colon V \to W$ of semi-normed $R[\Gamma]$-module over some normed ring R.

"$\supseteq$": Now let $g \in \ell^\infty(S, \operatorname{im} f)^\Gamma$. Since g is a map $S \to \operatorname{im} f$ we can choose for each $s \in S$ a preimage of $g(s)$ under f to obtain a map $h\colon S \to V$ with $f \circ h = g$. Moreover, as both f and g are Γ-invariant we can also choose h to be Γ-invariant.

Now at first glance it is not clear why h should be bounded, i.e., a element in $\ell^\infty(S, V)^\Gamma$, but here we can use the closedness of $\operatorname{im} f$: From Proposition 2.1.8 we obtain that there is a $c \in \mathbb{R}_{>0}$ such that we have

$$\big\|h(s)\big\|_V \leq c \cdot \big\|g(s)\big\|_W \leq c \cdot \|g\|_\infty .$$

So h is indeed bounded.

2. We will construct an element in $\ell^\infty(\Gamma^{n+1}, \operatorname{im} f)^\Gamma$ that does not have a preimage under $\ell^\infty(\Gamma^{n+1}, f)^\Gamma$:

   Since the image of f is not closed we can apply Proposition 2.1.8 to obtain for each $m \in \mathbb{N}$ an element $w_m \in \operatorname{im} f$ with

   $$\forall_{v \in f^{-1}(\{w_m\})}\colon \|v\|_V > m \cdot \|w_m\| . \qquad (*)$$

   In particular each w_m in non-zero as zero can't be a preimage. So we can multiply each w_m with $1/\|w_m\|_W$ and thus assume without loss of generality that we have $\|w_m\| = 1$ for each $m \in \mathbb{N}$.

   Moreover, as Γ was assumed to be infinite there is a surjective map $\varphi\colon \Gamma \to \mathbb{N}$ and we define

   $$\begin{aligned} h\colon \Gamma^{n+1} &\longrightarrow \operatorname{im} f \\ (g_0, \ldots, g_n) &\longmapsto g_1 \cdot w_{\varphi(g_1^{-1} \cdot g_0)} . \end{aligned}$$

   Since each w_m lies in the image of f and f is Γ-equivariant this map is indeed well-defined. Moreover, since $\|w_m\|_W = 1$ for all $m \in \mathbb{N}$ and the action $\Gamma \curvearrowright W$ is isometric we have that $\|h\|_\infty = 1$. Finally h is also Γ-equivariant, since we have for $g, g_0, \ldots, g_n \in \Gamma$ that

   $$g \cdot \big(h(g_0, \ldots, g_n)\big) = g \cdot g_1 \cdot w_\varphi(g_1^{-1} \cdot g_0)$$

$$\begin{aligned} &= g \cdot g_1 \cdot w_\varphi(g_1^{-1} \cdot g \cdot g^{-1} \cdot g_0) \\ &= h(g \cdot g_0, \dots, g \cdot g_n). \end{aligned}$$

Now *assume for a contradiction* that there exists some $k \in \ell^\infty(\Gamma^{n+1}, V)^\Gamma$ with $f \circ k = h$. Then we have in particular for $g \in \Gamma$ that

$$f\big(k(g, e, \dots, e)\big) = h(g, e, \dots, e) = w_{\varphi(g)}.$$

Hence we obtain

$$\begin{aligned} \|k\|_\infty &= \sup_{g_0,\dots,g_n \in \Gamma} \big\|k(g_0, \dots, g_n)\big\|_V \\ &\geq \sup_{g \in \Gamma} \big\|k(g, e, \dots, e)\big\|_V \\ &> \sup_{g \in \Gamma} \varphi(g) \cdot \big\|w_{\varphi(g)}\big\|_W && \text{(by } (*)) \\ &= \sup_{g \in \Gamma} \varphi(g) && (\|w_m\| = 1 \text{ for all } m \in \mathbb{N}) \\ &= \sup_{n \in \mathbb{N}} n && \text{(as } \varphi \text{ is surjective)} \\ &= \infty, \end{aligned}$$

which is a contradiction. So $h \in \ell^\infty(\Gamma^{n+1}, \operatorname{im} f)^\Gamma$ has no preimage under $\ell^\infty(\Gamma^{n+1}, f)$.

Finally, since we have by Lemma 2.1.4 for each $q \in \mathbb{N}$ an isometric isomorphism

$$H_b^q(\Lambda; V) \cong_R H^q\big(C_b^*(\Gamma; V)^\Lambda, \delta_\Gamma^*\big) = {\ker \delta_\Gamma^q} \big/ {\operatorname{im} \delta_\Gamma^{q-1}}$$

we get that the $\operatorname{im} \delta_\Gamma^{q-1}$ is closed if and only if $H_b^q(\Lambda; V)$ is a normed space. Hence both 3. and 4. follow from the first two parts. □

Remark 2.1.9. Both Ivanov [11] and Matsumoto and Morita [16] showed independently that for every group Γ the second bounded cohomology $H_b^2(\Gamma; \mathbb{R})$ with real coefficients is a normed space. However, this result does not extend to higher degrees: Soma [24] proved that the semi-norm on the third bounded cohomology group $H_b^3(F_2; \mathbb{R})$ of the free group of rank 2 is *not* a norm.

As we have seen we can not prove that we always have

$$\operatorname{im}\big(C_b^p(\Delta; \delta_\Gamma^{q-1})^\Delta\big) = C_b^p(\Delta; \operatorname{im} \delta_\Gamma^{q-1})^\Delta$$

we will simply take it as an assumption in the following proposition.

Proposition 2.1.10 (identification of ${}^{②}E_1$ and ${}^{②}E_2$ terms)**.** *Let R be a normed ring, let*

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0$$

be a short exact sequence of groups and let V be a semi-normed $R[\Gamma]$-module.

1. *Let $p, q \in \mathbb{N}$ such that we have*

   $$\operatorname{im}\bigl(C_b^p(\Delta; \delta_\Gamma^{q-1})^\Delta\bigr) = C_b^p(\Delta; \operatorname{im} \delta_\Gamma^{q-1})^\Delta.$$

   Then we have a canonical isomorphism

   $${}^{②}E_1^{p,q} \cong_R C_b^p\bigl(\Delta; H_b^q(\Lambda; V)\bigr)^\Delta.$$

2. *Let $p, q \in \mathbb{N}$ such that we have the above canonical identification possible for both ${}^{②}E_1^{p,q}$ and ${}^{②}E_1^{p+1,q}$. Then the differential ${}^{②}E_1^{p,q} \colon {}^{②}E_1^{p,q} \to {}^{②}E_1^{p+1,q}$ corresponds under these isomorphisms to the usual differential*

   $$\delta_\Delta^p \colon C_b^p\bigl(\Delta; H_b^q(\Lambda; V)\bigr)^\Delta \longrightarrow C_b^{p+1}\bigl(\Delta; H_b^q(\Lambda; V)\bigr)^\Delta.$$

3. *In particular: If $p, q \in \mathbb{N}$ such that we can apply the identification of 1. to ${}^{②}E_1^{p-1,q}, {}^{②}E_1^{p,q}$ and ${}^{②}E_1^{p+1,q}$ then we have a canonical isomorphism*

   $${}^{②}E_2^{p,q} \cong_R H_b^p\bigl(\Delta; H_b^q(\Lambda; V)\bigr).$$

Proof.

1. We know that ${}^{②}E_1^{p,q}$ is given by

   $$\begin{aligned} {}^{②}E_1^{p,q} &\cong_R H^{q,p}(M, d_h) \\ &= H^q\bigl(M^{*,p}, d_h^{*,p}\bigr) \\ &= H^q\Bigl(C_b^p\bigl(\Delta; C_b^*(\Gamma; V)^\Lambda\bigr)^\Delta, C_b^p(\Delta; \delta_\Gamma^*)^\Delta\Bigr) \end{aligned}$$

   and that the differential ${}^{②}d_1$ is induced by the standard differential of

   $$\Bigl(C_b^*\bigl(\Delta; C_b^q(\Gamma, V)^\Lambda\bigr), \delta_\Delta^*\Bigr).$$

   Since it is easy to see that we have

   $$\ker\bigl(C_b^p(\Delta; \delta_\Gamma^q)^\Delta\bigr) = C_b^p(\Delta; \ker \delta_\Gamma^q)^\Delta$$

   our assumption on the image gives canonical isomorphisms

   $${}^{②}E_1^{p,q} \cong_R H^q\Bigl(C_b^p\bigl(\Delta; C_b^*(\Gamma; V)^\Lambda\bigr)^\Delta, C_b^p(\Delta; \delta_\Gamma^*)\Bigr)$$

$$
\begin{aligned}
&= \frac{\ker\bigl(C_b^p(\Delta;\delta_\Gamma^q)^\Delta\bigr)}{\operatorname{im}\bigl(C_b^p(\Delta;\delta_\Gamma^{q-1})^\Delta\bigr)} \\
&= \frac{C_b^p(\Delta;\ker\delta_\Gamma^q)^\Delta}{C_b^p(\Delta;\operatorname{im}\delta_\Gamma^{q-1})^\Delta} \\
&\cong_R C_b^p\bigl(\Delta;\ker\delta_\Gamma^q/\operatorname{im}\delta_\Gamma^{q-1}\bigr)^\Delta && \text{(by Lemma 2.1.5)} \\
&= C_b^p\Bigl(\Delta;H^q\bigl(C_b^*(\Gamma;V)^\Lambda,\delta_\Gamma^*\bigr)\Bigr)^\Delta \\
&\cong_R C_b^p\bigl(\Delta;H_b^q(\Lambda;V)\bigr)^\Delta. && \text{(by Lemma 2.1.4)}
\end{aligned}
$$

2. By the construction of the above isomorphism it is clear that the differential

$$
{}^{②}d_1^{p,q}\colon {}^{②}E_1^{p,q} \longrightarrow {}^{②}E_1^{p+1,q}
$$

corresponds to the standard differential

$$
\delta_\Delta^p\colon C_b^p\bigl(\Delta;H_b^q(\Lambda;V)\bigr)^\Delta \longrightarrow C_b^{p+1}\bigl(\Delta;H_b^q(\Lambda;V)\bigr)^\Delta.
$$

3. Using the identification of the entries on the first page and the identifications of the differentials, the page-turning isomorphism gives

$$
\begin{aligned}
{}^{②}E_2^{p,q} &\cong_R H^{p,q}({}^{②}E_1,{}^{②}d_1) \\
&= \frac{\ker d_1^{p,q}}{\operatorname{im} d_1^{p-1,q}} \\
&\cong_R \frac{\ker\delta_\Delta^p}{\operatorname{im}\delta_\Delta^{p-1}} \\
&= H^p\Bigl(C_b^*\bigl(\Delta;H_b^q(\Lambda;V)\bigr)^\Delta;\delta_\Delta^*\Bigr) \\
&= H_b^p\bigl(\Delta;H_b^q(\Lambda;V)\bigr).
\end{aligned}
$$

□

Corollary 2.1.11 (identification of ${}^{②}E_1^{p,0}$ and ${}^{②}E_2^{p,0}$)**.** *Let R be a normed ring, let*

$$
0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0
$$

be a short exact sequence of groups and let V be a semi-normed $R[\Gamma]$-module. Then we have for each $p\in\mathbb{N}$ canonical isomorphisms

$$
{}^{②}E_1^{p,0} \cong_R C_b^p(\Delta;V^\Lambda)^\Delta \qquad \textit{and} \qquad {}^{②}E_2^{p,0} \cong_R H_b^p(\Delta;V^\Lambda).
$$

Proof. Since we have that δ_Γ^{-1} is trivial we obviously have for each $p\in\mathbb{N}$

$$
\operatorname{im}\bigl(C_b^p(\Delta;\delta_\Gamma^{-1})^\Delta\bigr) = 0 = C_b^p(\Delta;\operatorname{im}\delta_\Gamma^{-1}).
$$

Hence we can apply Proposition 2.1.10 to obtain canonical isomorphisms

$$ {}^{②}E_1^{p,0} \cong_R C_b^p\big(\Delta; H_b^0(\Lambda; V)\big)^\Delta \qquad \text{and} \qquad {}^{②}E_2^{p,0} \cong_R H_b^p\big(\Delta; H_b^0(\Lambda; V)\big). $$

Finally the identification $H_b^0(\Lambda; V) \cong_R V^\Lambda$ gives the claim. □

Corollary 2.1.12. *Let R be a normed ring, let*

$$ 0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0 $$

be a short exact sequence of groups with finite Δ and let V be a semi-normed $R[\Gamma]$-module. Then there are for all $p, q \in \mathbb{N}$ canonical isomorphisms

$$ {}^{②}E_1^{p,q} \cong_R C_b^p\big(\Delta; H_b^q(\Lambda; V)\big)^\Delta \qquad \textit{and} \qquad {}^{②}E_2^{p,q} \cong_R H_b^p\big(\Delta; H_b^q(\Lambda; V)\big). $$

Proof. Due to Example 2.1.6 this follows directly from Proposition 2.1.10. □

Corollary 2.1.13 (identification for Banach coefficients)**.** *Let*

$$ 0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0 $$

be a short exact sequence of groups and let V be a Banach Γ-module. If for $q \in \mathbb{N}$ the semi-norm on $H_b^q(\Lambda; V)$ is a norm, we have for each $p \in \mathbb{N}$ canonical isomorphisms

$$ {}^{②}E_1^{p,q} \cong_R C_b^p\big(\Delta; H_b^q(\Lambda; V)\big)^\Delta $$

and

$$ {}^{②}E_2^{p,q} \cong_R H_b^p\big(\Delta; H_b^q(\Lambda; V)\big). $$

Proof. Since $H_b^q(\Lambda; V)$ is a normed vector space we can apply Proposition 2.1.7 to obtain for each $p \in \mathbb{N}$ the equality

$$ \operatorname{im}\big(C_b^p(\Delta; \delta_\Gamma^{q-1})^\Delta\big) = C_b^p(\Delta; \operatorname{im} \delta_\Gamma^{q-1})^\Delta $$

and an application of Proposition 2.1.10 finishes the proof. □

In addition to the condition in Proposition 2.1.10 we can use another argument to always identify the terms ${}^{②}E_1^{0,q}$ and ${}^{②}E_2^{0,q}$.

Lemma 2.1.14. *Let R be a normed ring, let Γ be a group and let V be a semi-normed $R[\Gamma]$-module. Then there are isometric R-isomorphisms*

$$ C_b^0(\Gamma; V)^\Gamma \cong_R V \qquad \textit{and} \qquad C_b^1(\Gamma; V)^\Gamma \cong_R \ell^\infty(\Gamma; V), $$

which are natural in the coefficient modules, i.e., each bounded morphism $f\colon V \to W$ of semi-normed $R[\Gamma]$-modules induces a commutative diagram

$$\begin{tikzcd} C_b^0(\Gamma;V)^\Gamma \arrow[r, "\cong_R"] \arrow[d, "C_b^0(\Gamma;f)"'] & V \arrow[d, "f"] \\ C_b^0(\Gamma;W)^\Gamma \arrow[r, "\cong_R"'] & W, \end{tikzcd}$$

and similar for $C_b^1(\Gamma;\,\cdot\,)^\Gamma$.

Proof. First we consider the maps

$$\begin{aligned} \varphi_0\colon \ell^\infty(\Gamma,V)^\Gamma &\longrightarrow V \\ f &\longmapsto f(e) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \psi_0\colon V &\longrightarrow \ell^\infty(\Gamma,V) \\ v &\longmapsto \big(g \mapsto g\cdot v\big). \end{aligned}$$

These maps are obviously well-defined and R-linear with $\|\varphi_0\| \leq 1$ and $\|\psi_0\| \leq 1$. Moreover the image of ψ_0 is in fact already Γ-invariant: For $v \in V$ and $h, g \in \Gamma$ we have

$$\begin{aligned} \big(h\cdot\psi_0(v)\big)(g) &= h\cdot\psi_0(v)(h^{-1}\cdot g) \\ &= h\cdot h^{-1}\cdot g\cdot v \\ &= g\cdot v \\ &= \psi_0(v)(g). \end{aligned}$$

Now when restricting ψ_0 to a map $V \to \ell^\infty(\Gamma,V)^\Gamma$ it is obvious that φ_0 and ψ_0 are mutually inverse, and thus are isometric isomorphisms

$$C_b^0(\Gamma;V)^\Gamma = \ell^\infty(\Gamma,V)^\Gamma \cong_R V.$$

Next let us consider the maps

$$\begin{aligned} \varphi_1\colon \ell^\infty(\Gamma^2,V)^\Gamma &\longrightarrow \ell^\infty(\Gamma,V) \\ f &\longmapsto \big(g \mapsto f(e,g)\big) \end{aligned}$$

and

$$\begin{aligned} \psi_1\colon \ell^\infty(\Gamma,V) &\longrightarrow \ell^\infty(\Gamma^2,V)^\Gamma \\ f &\longmapsto \big((g_0,g_1) \mapsto g_0\cdot f(g_0^{-1}\cdot g_1)\big). \end{aligned}$$

Again it is clear that these two maps are well-defined and R-linear with $\|\varphi_1\| \leq 1$ and $\|\psi_1\| \leq 1$. Similar as in the first part the image of ψ_1 is Γ-invariant:

Let $f \in \ell^\infty(\Gamma, V)$ and $g, g_0, g_1 \in \Gamma$. Then

$$\begin{aligned}\big(g \cdot \psi_1(f)\big)(g_0, g_1) &= g \cdot \psi_1(f)(g^{-1} \cdot g_0, g^{-1} \cdot g_1)\\ &= g \cdot g^{-1} \cdot g_0 \cdot f(g_0^{-1} \cdot g \cdot g^{-1} \cdot g_1)\\ &= g_0 \cdot f(g_0^{-1} \cdot g_1)\\ &= f(g_0, g_1).\end{aligned}$$

As before by restricting the codomain of ψ_1 we obtain a inverse of φ_1 which provides the isometric isomorphism

$$C_b^1(\Gamma; V)^\Gamma = \ell^\infty(\Gamma^2, V)^\Gamma \cong_R \ell^\infty(\Gamma; V).$$

In both cases the naturality is clear by construction. □

Remark 2.1.15 (bar resolution)**.** By generalising the above proof one can obtain for each $n \in \mathbb{N}$ an isometric isomorphism

$$C_b^n(\Gamma; V)^\Gamma = \ell^\infty(\Gamma^{n+1}, V)^\Gamma \cong_R \ell^\infty(\Gamma^n, V).$$

Under these isomorphisms we can identify the standard differential δ_Γ^* with a differential $\bar{\delta}_\Gamma^*$ of $\ell^\infty(\Gamma^*, V)$, such that the cochain complex

$$\big(\ell^\infty(\Gamma^*, V), \bar{\delta}^*\big),$$

the so-called *bar cochain complex*, can be used to isometrically compute the bounded cohomology $H_b^*(\Gamma; V)$.

As an example let us compute the differential $\bar{\delta}_\Gamma^0$: Let $v \in V$ and let $g \in \Gamma$. With ψ_0 and φ_1 as in the above proof we have

$$\begin{aligned}\bar{\delta}_\Gamma^0(v)(g) &= \varphi_1 \circ \delta_\Gamma^0 \circ \psi_0(v)(g)\\ &= \delta_\Gamma^0 \circ \psi_0(v)(e, g)\\ &= \psi_0(v)(e) - \psi_0(v)(g)\\ &= v - g \cdot v,\end{aligned}$$

which immediately gives another proof of

$$H_b^0(\Gamma; V) \cong_R \ker \bar{\delta}_\Gamma^0 = V^\Gamma.$$

Proposition 2.1.16 (identification of ${}^{②}E_1^{0,q}$ and ${}^{②}E_2^{0,q}$)**.** *Let R be a normed ring, let*

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0$$

be a short exact sequence of groups and let V be a semi-normed $R[\Gamma]$-module. Then we have for each $q \in \mathbb{N}$ the canonical identifications

$${}^{②}E_1^{0,q} \cong_R H_b^q(\Lambda; V) \qquad \textit{and} \qquad {}^{②}E_2^{0,q} \cong_R H_b^q(\Lambda; V)^\Delta.$$

Proof. Again we have for ${}^{②}E_1^{0,q}$ the identification

$$\begin{aligned}
{}^{②}E_1^{0,q} &\cong_R H^{q,0}(M, d_h) \\
&= H^q\big(M^{*,0}, d_h^{*,0}\big) \\
&= H^q\Big(C_b^0\big(\Delta; C_b^*(\Gamma; V)^\Lambda\big)^\Delta, C_b^0(\Delta; \delta_\Gamma^*)^\Delta\Big).
\end{aligned}$$

Now using the natural isomorphism of Lemma 2.1.14 this cohomology is canonically isomorphic to

$$\begin{aligned}
{}^{②}E_1^{0,q} &= H^q\Big(C_b^0\big(\Delta; C_b^*(\Gamma; V)^\Lambda\big)^\Delta, C_b^0(\Delta; \delta_\Gamma^*)^\Delta\Big) \\
&\cong_R H^q\big(C_b^*(\Gamma; V)^\Lambda, \delta_\Gamma^*)\big) \\
&\cong_R H_b^q(\Lambda; V). && \text{(by Lemma 2.1.4)}
\end{aligned}$$

Moreover, we can also apply the second natural isomorphism of Lemma 2.1.14 to see that

$$\begin{aligned}
{}^{②}E_1^{1,q} &\cong_R H^{q,1}(M, d_h) \\
&= H^q\big(M^{*,1}, d_h^{*,1}\big) \\
&= H^q\Big(C_b^1\big(\Delta; C_b^*(\Gamma; V)^\Lambda\big)^\Delta, C_b^1(\Delta; \delta_\Gamma^*)^\Delta\Big) \\
&\cong_R H^q\Big(\ell^\infty\big(\Delta, C_b^*(\Gamma; V)^\Lambda\big), \ell^\infty(\Delta, \delta_\Gamma^*)\Big),
\end{aligned}$$

where the differential ${}^{②}d_1^{0,q} \colon {}^{②}E_1^{0,q} \to {}^{②}E_1^{1,q}$ is induced by the differential $\bar{\delta}_\Delta^0$. Hence the description of $\bar{\delta}_\Delta^0$ in Remark 2.1.15 gives that the image of $[f] \in H_b^q(\Lambda; V)$ under ${}^{②}d_1^{0,1}$ in ${}^{②}E_1^{1,q}$ is represented by the map $g \mapsto f - g \cdot f$. So, using the page-turning isomorphism, we obtain

$$\begin{aligned}
{}^{②}E_2^{0,1} &\cong_R H^{p,q}({}^{②}E_1, {}^{②}d_1) \\
&\cong_R \ker {}^{②}d_1^{0,1} && ({}^{②}d_1^{-1,0} \text{ is trivial}) \\
&= H_b^q(\Lambda; V)^\Delta. && \square
\end{aligned}$$

Now let us collect all the above results of concrete identifications, and give the spectral sequence a proper name.

Theorem 2.1.17 (the Hochschild-Serre spectral sequence for bounded cohomology)**.** *Let R be a normed ring, let*

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \Delta \longrightarrow 0$$

be a short exact sequence of groups and let V be a semi-normed $R[\Gamma]$-module. Then there exist a first-quadrant spectral sequence (E_*, d_*), called the Hochschild-Serre spectral sequence*, with*

$$E_2^{p,0} \cong_R H_b^p(\Delta; V^\Lambda) \qquad \textit{and} \qquad E_2^{0,q} \cong_R H_b^q(\Lambda; V)^\Delta$$

for all $p, q \in \mathbb{N}$, converging to the bounded cohomology of Γ with coefficients in V

$$E_2^{p,q} \Longrightarrow H^{p+q}(\Gamma; V).$$

Moreover, in the following cases we can further identify the second page:

1. *If the group Δ is finite we have for all $p, q \in \mathbb{N}$*

$$E_2^{p,q} \cong_R H_b^p\big(\Delta; H_b^q(\Lambda; V)\big).$$

2. *If V is a Banach Γ-module and for $q \in \mathbb{N}$ the semi-norm on $H_b^q(\Lambda; V)$ is a norm we have for each $p \in \mathbb{N}$ the identification*

$$E_2^{p,q} \cong_R H_b^p\big(\Delta; H_b^q(\Lambda; V)\big).$$

Proof. We take (E_*, d_*) to be the spectral sequence $({}^{②}E_*, {}^{②}d_*)$ constructed above. By the construction of the spectral sequence and Proposition 2.1.3 we know that this spectral sequence converges to $H_b^*(\Gamma; V)$. Moreover, the identifications of $E_2^{p,0}$ and $E_2^{0,q}$ are just the results of Corollary 2.1.11 and Proposition 2.1.16.

The statements about the further identification of the second page are nothing but Corollary 2.1.12 and Corollary 2.1.13. □

Of course we could always use Proposition 2.1.10 to further identify entries in the Hochschild-Serre spectral sequence.

Remark 2.1.18 (naturality of the Hochschild-Serre spectral sequence)**.** Let R be a normed ring, let

$$\begin{tikzcd} 0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "\tau"'] & \Gamma \arrow[r] \arrow[d, "\varphi"'] & \Delta \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & \widetilde{\Lambda} \arrow[r] & \widetilde{\Gamma} \arrow[r] & \widetilde{\Delta} \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of groups with exact rows, let V be a semi-normed $R[\Gamma]$-module, let $\widetilde{V}$ be a semi-normed $R[\widetilde{\Gamma}]$-module and let $f \colon \widetilde{V} \to V$ be a bounded R-morphism that is compatible with φ. Moreover, denote by (E_*, d_*) the Hochschild-Serre spectral sequence associated to the first row with coefficient module V and denote by $(\widetilde{E}_*, \widetilde{d}_*)$ the Hochschild-Serre spectral sequence of the second row with coefficient module $\widetilde{V}$.

As we have seen in Remark 2.1.2 the morphism

$$C_b^q\big(\psi; C_b^p(\varphi; f)\big) \colon C_b^q\big(\widetilde{\Delta}; C_b^p(\widetilde{\Gamma}; \widetilde{V})^{\widetilde{\Lambda}}\big)^{\widetilde{\Delta}} \longrightarrow C_b^q\big(\Delta; C_b^p(\Gamma; V)^{\Lambda}\big)^{\Delta}$$

of double complexes induces a morphism of spectral sequences $(\widetilde{E}_*, \widetilde{d}_*) \to (E_*, d_*)$, which is compatible with the induced morphism in cohomology of the total complex. In Proposition 2.1.3 we have identified this morphism in cohomology to be

$$H_b^*(\varphi; f) \colon H_b^*(\widetilde{\Gamma}; \widetilde{V}) \to H_b^*(\Gamma; V).$$

Hence we have a commutative diagram

$$\begin{tikzcd}
\widetilde{E}_2^{p,q} \arrow[r, Rightarrow] \arrow[d] & H_b^{p+q}(\widetilde{\Gamma}; \widetilde{V}) \arrow[d, "H_b^{p+q}(\varphi; f)"] \\
E_2^{p,q} \arrow[r, Rightarrow] & H_b^{p+q}(\Gamma; V).
\end{tikzcd}$$

Now whenever we can identify for $p, q \in \mathbb{N}$ both

$$\widetilde{E}_2^{p,q} \cong_R H_b^p\big(\widetilde{\Delta}; H_b^q(\widetilde{\Lambda}; \widetilde{V})\big) \qquad \text{and} \qquad E_2^{p,q} \cong_R H_b^p\big(\Delta; H_b^q(\Lambda; V)\big)$$

with one of the identifications in Theorem 2.1.17, we get that the map between the second pages corresponds to

$$H_b^p\big(\psi; H_b^q(\tau; f)\big) \colon H_b^p\big(\widetilde{\Delta}; H_b^q(\widetilde{\Lambda}; \widetilde{V})\big) \longrightarrow H_b^p\big(\Delta; H_b^q(\Lambda; V)\big).$$

Since the Hochschild-Serre spectral sequence is a convergent first-quadrant spectral sequence we obtain in particular a five-term exact sequence.

Corollary 2.1.19 (five-term exact sequence in bounded cohomology)**.** *Let R be a normed ring, let*

$$0 \longrightarrow \Lambda \xrightarrow{\ i\ } \Gamma \xrightarrow{\ \varphi\ } \Delta \longrightarrow 0$$

be a short exact sequence of groups and let V be a semi-normed $R[\Gamma]$-module. Then we have an exact sequence

$$0 \longrightarrow H_b^1(\Delta; V^{\Lambda}) \longrightarrow H_b^1(\Gamma; V) \longrightarrow H_b^1(\Lambda; V)^{\Delta} \longrightarrow H_b^2(\Delta; V^{\Lambda}) \longrightarrow H_b^2(\Gamma; V),$$

where for $i \in \{1, 2\}$ the map $H_b^i(\Delta; V^{\Lambda}) \to H_b^i(\Gamma; V)$ is given by

$$H_b^i(\varphi; I_V) \colon H_b^i(\Delta; V^{\Lambda}) \to H_b^i(\Gamma; V),$$

with $I_V \colon V^{\Lambda} \to V$ as the inclusion, and $H_b^1(\Gamma; V) \to H_b^1(\Lambda; V)^{\Delta}$ is given by

$$H_b^1(i; \mathrm{id}_V) \colon H_b^1(\Gamma; V) \to H_b^1(\Lambda; V)^{\Delta}.$$

Proof. By applying the usual five-term exact sequence of a convergent first-quadrant spectral sequence Proposition 1.1.17 to the Hochschild-Serre spectral sequence we obtain the exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow H_b^1(\Gamma;V) \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow E_\infty^{2,0} \longrightarrow 0$$

and that $E_\infty^{2,0}$ is isomorphic to a submodule of $H_b^2(\Gamma;V)$. Hence using our identifications for $E_2^{p,0}$ and $E_2^{0,q}$ we obtain an exact sequence

$$0 \longrightarrow H_b^1(\Delta;V^\Lambda) \longrightarrow H_b^1(\Gamma;V) \longrightarrow H_b^1(\Lambda;V)^\Delta \longrightarrow H_b^2(\Delta;V^\Lambda) \longrightarrow H_b^2(\Gamma;V).$$

For the identification of the maps we use the naturality of the Hochschild-Serre spectral sequence Remark 2.1.18 and the naturality of the five term exact sequence Proposition 1.1.17: Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r, "\mathrm{id}_\Lambda"] \arrow[d, "\mathrm{id}_\Lambda"'] & \Lambda \arrow[r] \arrow[d, "i"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda \arrow[r, "i"] \arrow[d] & \Gamma \arrow[r, "\varphi"] \arrow[d, "\varphi"] & \Delta \arrow[r] \arrow[d, "\mathrm{id}_\Delta"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \Delta \arrow[r, "\mathrm{id}_\Delta"'] & \Delta \arrow[r] & 0
\end{tikzcd}$$

of groups with exact rows with the coefficient modules V for the top and middle row, and V^Λ for the bottom row. The the naturality of the Hochschild-Serre spectral sequence and the five-term exact sequence gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & H_b^1(\Lambda;V) \arrow[r, "\mathrm{id}"] & H_b^1(\Lambda;V) \arrow[r] & 0 \arrow[r] & H_b^2(\Lambda;V) \\
0 \arrow[r] & H_b^1(\Delta;V^\Lambda) \arrow[r] \arrow[u] & H_b^1(\Gamma;V) \arrow[r] \arrow[u, "H_b^1(i;\mathrm{id}_V)"] & H_b^1(\Lambda;V)^\Delta \arrow[r] \arrow[u, "\mathrm{incl.}"'] & H_b^2(\Delta;V^\Lambda) \arrow[r] \arrow[u] & H_b^2(\Gamma;V) \arrow[u, "H_b^2(i;\mathrm{id}_V)"'] \\
0 \arrow[r] & H_b^1(\Delta;V^\Lambda) \arrow[r, "\mathrm{id}"'] \arrow[u, "\mathrm{id}"] & H_b^1(\Delta;V^\Lambda) \arrow[r] \arrow[u, "H_b^1(\varphi;I_V)"'] & 0 \arrow[r] \arrow[u] & H_b^2(\Delta;V^\Lambda) \arrow[r, "\mathrm{id}"'] \arrow[u, "\mathrm{id}"] & H_b^2(\Delta;V^\Lambda), \arrow[u, "H_b^2(\varphi;I_V)"']
\end{tikzcd}$$

where the identification of the vertical maps follows from Remark 2.1.18. Now the commutativity of the diagram allows us to identify the horizontal morphism of the middle row as desired. □

Remark 2.1.20 (the flaw of Bouarich's proof)**.** As we have seen above in Proposition 2.1.10 an important step in the identification of the entries of the first and second page is the equality

$$\mathrm{im}\big(C_b^p(\Delta;\delta^{q-1})\big)^\Delta = C_b^p(\Delta;\mathrm{im}\,\delta^{q-1})^\Delta.$$

However, we have also seen in Proposition 2.1.7 that this equality does not always hold, even in the “good” case of Banach modules as coefficients.

Yet, Bouarich claims, without any proof, in his paper on the Hochschild-Serre spectral sequence in bounded cohomology with semi-normed vector spaces as coefficients that it is always true [2, p. 334].

Besides this flaw the rest of his proof for the identification of the first and second page works, in fact our proof of Proposition 2.1.10 is not much different. Moreover, any corollaries he deduces from his spectral sequence only require identifications that can always be made, e.g., the identifications of $E_2^{p,0}$ and $E_2^{0,q}$, so they hold regardless.

2.1.2 Application: Amenability and bounded acyclicity

One example of an application of the Hochschild-Serre spectral sequence a proof of a characterisation of *amenable* and *boundedly n-acyclic* morphisms. This characterisation was just recently proved by Moraschini and Raptis [19], without the use of the Hochschild-Serre spectral sequence. However, they noted that a proof is also possible using the spectral sequence.

Let us start by proving a “poor man’s” version of the mapping theorem:

Theorem 2.1.21 (poor man’s mapping theorem)**.** *Let $\varphi\colon \Gamma \to \Delta$ be a surjective group homomorphism with amenable kernel Λ and let V be a semi-normed $\mathbb{R}[\Gamma]$-module. Then the canonical $\mathbb{R}$-morphism*

$$H_b^n(\varphi; \mathrm{I}_{V^\#})\colon H_b^n\big(\Delta; (V^\#)^\Lambda\big) \longrightarrow H_b^n(\Gamma; V^\#),$$

induced by the inclusion $\mathrm{I}_{V^\#}\colon (V^\#)^\Lambda \to V^\#$, is an isomorphism for each $n \in \mathbb{N}$.

Note that in contrast to the “real” mapping theorem the above theorem makes no statement about the map being an *isometric* isomorphism.

Proof of Theorem 2.1.21. Consider the following commutative diagram of groups with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda & \hookrightarrow & \Gamma & \xrightarrow{\varphi} & \Delta & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\mathrm{id}_\Delta} & & \\
0 & \longrightarrow & 0 & \longrightarrow & \Delta & \xrightarrow[\mathrm{id}_\Delta]{} & \Delta & \longrightarrow & 0.
\end{array}$$

Using the amenability of Λ we know that the bounded cohomology $H_b^{q\geq 1}\big(\Lambda; V^\#\big)$ vanishes [7, Theorem 3.6]. So we have that

$$H_b^q\big(\Lambda; V^\#\big) = \begin{cases} (V^\#)^\Lambda & \text{if } q = 0 \\ 0 & \text{else} \end{cases}$$

is for each $q \in \mathbb{N}$ a normed space, both $V^{\#}$ and $(V^{\#})^{\Lambda}$ are Banach spaces and φ is compatible with $\mathrm{I}_{V^{\#}}\colon (V^{\#})^{\Lambda} \hookrightarrow V^{\#}$ we obtain the following commutative diagram

$$\begin{array}{ccc} \widetilde{E}_2^{p,q} \cong_R H_b^p\Big(\Delta; H_b^q\big(0; (V^{\#})^{\Lambda}\big)\Big) & \Longrightarrow & H_b^{p+q}\big(\Delta; (V^{\#})^{\Lambda}\big) \\ {\scriptstyle H_b^p\left(\mathrm{id}_{\Delta}; H_b^q(0, \mathrm{I}_{V^{\#}})\right)}\Big\downarrow & & \Big\downarrow{\scriptstyle H_b^{p+q}(\varphi; \mathrm{I}_{V^{\#}})} \\ E_2^{p,q} \cong_R H_b^p\big(\Delta; H_b^q(\Lambda; V^{\#})\big) & \Longrightarrow & H_b^{p+q}(\Gamma; V^{\#}). \end{array}$$

Now both $H_b^q\big(0; (V^{\#})^{\Lambda}\big)$ and $H_b^q(\Lambda; V^{\#})$ are trivial for $q \neq 0$. Thus both of the above two spectral sequences have the second page as ∞-page. Together with the above convergence and compatibility this gives for each $p \in \mathbb{N}$ a commutative diagram

$$\begin{array}{ccc} \widetilde{E}_{\infty}^{p,0} \cong_R H_b^p\Big(\Delta; H_b^0\big(0; (V^{\#})^{\Lambda}\big)\Big) & \xrightarrow{\cong_R} & H_b^p(\Gamma; V^{\#}) \\ {\scriptstyle H_b^p\left(\mathrm{id}_{\Delta}; H_b^0(0; \mathrm{I}_{V^{\#}})\right)}\Big\downarrow & & \Big\downarrow{\scriptstyle H_b^p(\varphi; \mathrm{I}_{V^{\#}})} \\ E_{\infty}^{p,0} \cong_R H_b^p\big(\Delta; H_b^0(\Lambda; V^{\#})\big) & \xrightarrow[\cong_R]{} & H_b^p(\Gamma; V^{\#}). \end{array}$$

Under the canonical identifications

$$H_b^0\big(0; (V^{\#})^{\Lambda}\big) \cong_R (V^{\#})^{\Lambda} \cong_R H_b^0(\Lambda; V^{\#})$$

both the right hand and upper morphism are given by the identity. Hence we are given for each $p \in \mathbb{N}$ the commutative diagram

$$\begin{array}{ccc} H_b^p\big(\Delta; (V^{\#})^{\Lambda}\big) & = & H_b^p(\Gamma; V^{\#}) \\ \Big\| & & \Big\downarrow{\scriptstyle H_b^p(\varphi; \mathrm{I}_{V^{\#}})} \\ H_b^p\big(\Delta; (V^{\#})^{\Lambda}\big) & \xrightarrow[\cong_R]{} & H_b^p(\Gamma; V^{\#}), \end{array}$$

which gives that

$$H_b^p(\varphi; \mathrm{I}_{V^{\#}})\colon H_b^p\big(\Delta; (V^{\#})^{\Lambda}\big) \longrightarrow H_b^p(\Gamma; V^{\#})$$

is an isomorphism for each $p \in \mathbb{N}$. $\square$

Moreover one can also prove a converse of the mapping theorem. Therefore let us introduce the following notion.

Definition 2.1.22 (amenable group homomorphism)**.** A surjective group homomorphism $\varphi\colon \Gamma \to \Delta$ with kernel Λ is called *amenable*, if for each semi-normed $\mathbb{R}[\Gamma]$-module and each $n \in \mathbb{N}$ the induced map

$$H_b^n(\varphi; \mathrm{I}_{V^{\#}})\colon H_b^n\big(\Delta; (\mathrm{I}_{V^{\#}})^{\Lambda}\big) \longrightarrow H_b^n(\Gamma; V^{\#})$$

is an isometric isomorphism.

Theorem 2.1.23 (characterisation of amenable morphisms [19, Theorem 3.1.3])**.** *Let $\varphi\colon \Gamma \to \Delta$ be a surjective group homomorphism with kernel Λ. Then the following are equivalent:*

1. *The group homomorphism φ is amenable.*
2. *For all semi-normed $\mathbb{R}[\Gamma]$-module V the induced map*
$$H_b^1(\varphi; \mathrm{I}_{V^\#})\colon H_b^1\big(\Delta; (V^\#)^\Lambda\big) \longrightarrow H_b^1(\Gamma; V^\#)$$
is an isomorphism.
3. *The group Λ is amenable.*

Sketch of proof. Both the implications "1. $\Rightarrow$ 2." and "3. $\Rightarrow$ 1." are clear by definition and the mapping theorem . Thus we only have to show "2. $\Rightarrow$ 3.", which uses a similar argument as the characterisation of amenability using bounded cohomology [7, Theorem 3.10]:

We consider the normed $\mathbb{R}[\Gamma]$-module $V = \ell^\infty(\Gamma, \mathbb{R})/\mathbb{R}$, where we identify $\mathbb{R}$ as the subspace of constant maps. Since
$$H_b^1(\varphi; \mathrm{I}_{V^\#})\colon H_b^1\big(\Delta; (V^\#)^\Lambda\big) \longrightarrow H_b^1(\Gamma; V^\#)$$
is assumed to be an isomorphism the beginning of the five-term exact sequence in bounded cohomology Corollary 2.1.19
$$0 \longrightarrow H_b^1\big(\Delta; (V^\#)^\Lambda\big) \xrightarrow{H_b^1(\varphi; \mathrm{I}_{V^\#})} H_b^1(\Gamma; V^\#) \xrightarrow{H_b^1(i; \mathrm{id}_{V^\#})} H_b^1(\Lambda; V^\#)^\Delta$$
gives that
$$H_b^1(i; \mathrm{id}_{V^\#})\colon H_b^1(\Gamma; V^\#) \longrightarrow H_b^1(\Lambda; V^\#),$$
where $i\colon \Lambda \to \Gamma$ is the inclusion, is trivial. In particular the image of the class represented by the *Johnson cocycle* vanishes. With this triviality at hand one can now construct a non-trivial Λ-invariant functional
$$\ell^\infty(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R},$$
which suffices to give the amenability of Λ.

For greater details we refer the reader to the paper of Moraschini and Raptis [19]. □

Definition 2.1.24 ($\mathbb{R}$-generated Banach module)**.** Let Γ be a group. For a group action $\Gamma \curvearrowright S$ of Γ on a set S, the normed $\mathbb{R}[\Gamma]$-module $\ell^\infty(S, \mathbb{R})$ is called an $\mathbb{R}$-*generated Banach* Γ-*module*

Remark 2.1.25 (pullback of a R-generated Banach module)**.** Let $\varphi\colon \Gamma \to \Delta$ be group homomorphism and let $V = \ell^\infty(S, \mathbb{R})$ be an R-generated Banach Δ-module. Then it is easy to see that the pullback module $\varphi^{-1}V$ is an $\mathbb{R}$-generated Banach Γ-module.

Definition 2.1.26 (boundedly n-acyclic morphisms and groups)**.** Let $n \in \mathbb{N}$.

- A group homomorphism $\varphi\colon \Gamma \to \Delta$ is called *boundedly n-acyclic* if for every $\mathbb{R}$-generated Banach Δ-module V the restriction map

$$H_b^*(\varphi; V)\colon H_b^*(\Delta; V) \longrightarrow H_b^*(\Gamma; \varphi^{-1}V)$$

  is an isomorphism for $i \leq n$ and injective for $i = n + 1$.

- A group Γ is called *boundedly n-acyclic* if for each $i \in \{1, \ldots, n\}$ we have

$$H_b^i(\Gamma; \mathbb{R}) \cong_R 0.$$

Moreover, we call a group homomorphism or group *boundedly acyclic* if it is boundedly n-acyclic for each $n \in \mathbb{N}$.

When comparing the definition of boundedly n-acyclic morphisms and groups there is a slight discrepancy: For a boundedly n-acyclic group we only consider the coefficient module $\mathbb{R}$, where as in the case of boundedly n-acyclic morphism we consider any $\mathbb{R}$-generated Banach modules. But this is (at least to some extend) resolved by the following.

Proposition 2.1.27 ([19, Proposition 2.6.4])**.** *Let $n \in \mathbb{N}$ and let Γ be a boundedly n-acyclic group. Then for each $\mathbb{R}$-generated Banach Γ-module with trivial Γ-action and each $i \in \{1, \ldots, n\}$ we have $H_b^i(\Gamma; V) \cong_R 0$.*

Now that we have introduced all the relevant notions we come to the characterisation of boundedly n-acyclic morphisms.

Theorem 2.1.28 (characterisation of boundedly n-acyclic morphisms [19, Theorem 4.1.1])**.** *Let $n \in \mathbb{N}$ and let $\varphi\colon \Gamma \to \Delta$ be a group homomorphism with kernel Λ. Then the following are equivalent:*

1. *The morphism φ is boundedly n-acyclic.*
2. *For every $\mathbb{R}$-generated Banach Δ-module V the induced restriction map*

$$H_b^i(\varphi; V)\colon H_b^i(\Delta; V) \longrightarrow H_b^i(\Gamma; \varphi^{-1}V)$$

   is surjective for $i \in \{0, \ldots n\}$.

3. *The morphism φ is surjective and for each relatively injective $\mathbb{R}$-generated Banach Δ-module V we have*

$$H_b^i(\Lambda; \varphi^{-1}V) \cong_{\mathbb{R}} 0$$

for $i \in \{1, \ldots, n\}$.

4. *The morphism φ is surjective and the group Λ is boundedly n-acyclic.*

Proof of "4. $\Rightarrow$ 1." using the Hochschild-Serre spectral sequence. All other implications (and a proof of "4. $\Rightarrow$ 1." without using spectral sequences) can be found in the original paper of Moraschini and Raptis [19].

Let V be an $\mathbb{R}$-generated Banach Δ-module. Since Λ is the kernel of φ it is clear that the restricted Λ-action on $\varphi^{-1}V$ is trivial. Hence Proposition 2.1.27 gives that

$$H_b^i(\Lambda; \varphi^{-1}V) \cong_R 0$$

for $i \in \{1, \ldots, n\}$ and we moreover have

$$H_b^0(\Lambda; \varphi^{-1}V) = (\varphi^{-1}V)^\Lambda = V.$$

Now this gives that in the Hochschild-Serre spectral sequence of the short exact sequence

$$0 \longrightarrow \Lambda \hookrightarrow \Gamma \xrightarrow{\varphi} \Delta \longrightarrow 0$$

with coefficient module $\varphi^{-1}V$ we can identify

$$E_2^{p,q} \cong_R H_b^p\big(\Delta; H_b^q(\Lambda; \varphi^{-1}V)\big) \cong_R \begin{cases} H_b^p(\Delta; V) & \text{if } q = 0 \\ 0 & \text{if } q \in \{1, \ldots, n\} \end{cases}$$

for all $p \in \mathbb{N}$. In particular we obtain for each $r \in \mathbb{N}_{\geq 2}$ and each $p \in \{0, \ldots, n+1\}$ that the differential

$$d_r^{p-r,r-1} \colon E_r^{p-r,r-1} \longrightarrow E_r^{p,0}$$

is trivial, as either $r > p$, in which case the domain lies outside the first quadrant, or $r \leq p \leq n$, where the domain is trivial by the above identification. Hence, we get with the above identification of the second page that

$$E_\infty^{p,q} \cong_R \begin{cases} H_b^p(\Delta; V) & \text{if } q = 0 \text{ and } p \in \{0, \ldots, n+1\} \\ 0 & \text{if } q \in \{1, \ldots, n\} \text{ and } p \in \mathbb{N}. \end{cases}$$

$$
\begin{array}{c|ccccc}
q & & & & & \\
H_b^{n+1}(\Lambda;\varphi^{-1}V) & \bullet & \bullet & \bullet & \bullet & \\
0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \\
\hline
V^{\Delta} & H_b^1(\Delta;V) & H_b^2(\Delta;V) & H_b^3(\Delta;V) & H_b^4(\Delta;V) & p
\end{array}
$$

$$E_2$$

Figure 2.1: The second page of the spectral sequence used in the proof of Theorem 2.1.28

Now for $i \in \{0, \ldots, n+1\}$ the convergence towards $H_b^*(\Gamma;\varphi^{-1}V)$ gives the extension problems

$$0 \longrightarrow 0 \longrightarrow F^iH^i \longrightarrow E_\infty^{i,0} \cong_R H_b^i(\Delta;V) \longrightarrow 0$$

$$0 \longrightarrow F^iH^i \longrightarrow F^{i-1}H^i \longrightarrow E_\infty^{i-1,1} = 0 \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F^2H^i \longrightarrow F^1H^i \longrightarrow E_\infty^{1,i-1} = 0 \longrightarrow 0$$

$$0 \longrightarrow F^1H^i \longrightarrow H_b^i(\Gamma;\varphi^{-1}V) \longrightarrow E_\infty^{0,i} \longrightarrow 0.$$

Inductively solving these problems results in the short exact sequence

$$0 \longrightarrow E_\infty^{i,0} \cong_R H_b^i(\Delta; V) \longrightarrow H_b^i(\Gamma; \varphi^{-1}V) \longrightarrow E_\infty^{0,i} \longrightarrow 0.$$

Hence we have a map

$$H_b^i(\Delta; V) \longrightarrow H_b^i(\Gamma; \varphi^{-1}V)$$

which is for $i = n+1$ injective and for $i \in \{0, \ldots, n\}$ an isomorphisms as in this case $E_\infty^{0,i} = 0$.

In order to identify this map as $H_b^i(\varphi; V)$ we use naturality: By considering

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r, hook] \arrow[d] & \Gamma \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & \Delta \arrow[r] \arrow[d, "\mathrm{id}_\Delta"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \Delta \arrow[r, "\mathrm{id}_\Delta"'] & \Delta \arrow[r] & 0
\end{tikzcd}$$

the naturality of the Hochschild-Serre spectral sequence Remark 2.1.18 results in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E_\infty^{i,0} \cong_R H_b^i(\Delta; V) \arrow[r] & H_b^i(\Gamma; \varphi^{-1}V) \arrow[r] & E_\infty^{0,i} \arrow[r] & 0 \\
0 \arrow[r] & \widetilde{E}_\infty^{i,0} \cong_R H_b^i(\Delta; V) \arrow[r, "\mathrm{id}"'] \arrow[u, "\mathrm{id}"] & H_b^i(\Delta; V) \arrow[r] \arrow[u, "H_b^i(\varphi; V)"'] & 0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

of extension problems, where $(\widetilde{E}_*, \widetilde{d}_*)$ denotes the Hochschild-Serre spectral sequence of the lower exact sequence. This gives the desired identification. □

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