Introduction to Spectral Sequences and the Hurewicz Theorem

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Part I

Spectral Sequences

What is a spectral sequence?

A spectral sequence consists of:

- For each $n \in \mathbb{N}$ a differential bigraded module

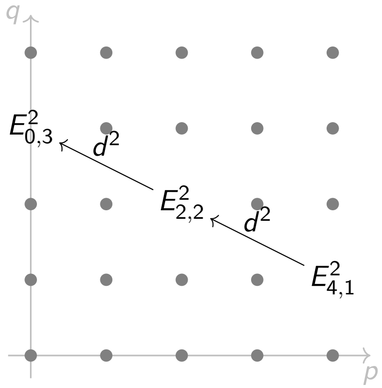
$$(E^n, d^n) = \left((E^n_{p,q})_{p,q\in\mathbb{N}}, (d^n_{p,q})_{p,q\in\mathbb{N}}\right),$$

  called the <u>n-th page</u> of the spectral sequence.

- Isomorphisms

$$E^{n+1}_{p,q} \cong H_{p,q}(E^n, d^n) = \frac{\ker d^n_{p,q}}{\operatorname{im} d^n_{p+n,q-n+1}},$$

  called <u>page turning isomorphisms</u>.



E^2

Example

Let $(C^q, \partial^q)_{q\in\mathbb{N}}$ be a family of chain complexes. We start with

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
C_0^3 & C_1^3 \arrow[l] & C_2^3 \arrow[l] & C_3^3 \arrow[l] & \cdots \arrow[l] \\
C_0^2 & C_1^2 \arrow[l] & C_2^2 \arrow[l] & C_3^2 \arrow[l] & \cdots \arrow[l] \\
C_0^1 & C_1^1 \arrow[l] & C_2^1 \arrow[l] & C_3^1 \arrow[l] & \cdots \arrow[l] \\
C_0^0 & C_1^0 \arrow[l] & C_2^0 \arrow[l] & C_3^0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}
$$

q p

E^1

where the differential d^1 is given by the chain complex differentials ∂^q.

Example

Because this is supposed to be a spectral sequence, the existence of the page turning isomorphisms force us to have

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
H_0(C^3) & H_1(C^3) & H_2(C^3) & H_3(C^3) & \cdots \\
H_0(C^2) & H_1(C^2) & H_2(C^2) & H_3(C^2) & \cdots \\
H_0(C^1) & H_1(C^1) & H_2(C^1) & H_3(C^1) & \cdots \\
H_0(C^0) & H_1(C^0) & H_2(C^0) & H_3(C^0) & \cdots
\end{array}$$

$$E^2 = E^3 = \dots$$

and we set the differentials d^n, $n \geq 2$, to be trivial.

The ∞-page of a spectral sequence

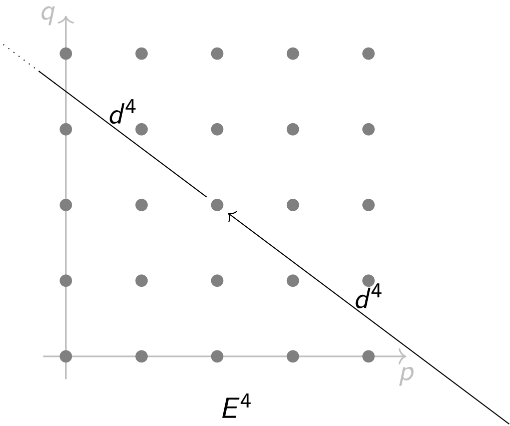
Fixing $p, q \in \mathbb{N}$ there is always a $N \in \mathbb{N}$ such that all differentials

$$E^n_{p+n,q-n+1} \xrightarrow{d^n_{p+n,q-n+1}} E^n_{p,q} \xrightarrow{d^n_{p,q}} E^n_{p-n,q+n-1},$$

with $n \geq N$, are trivial.

Thus the page turning isomorphisms give

$$E^\infty_{p,q} := E^N_{p,q} \cong E^{N+1}_{p,q} \cong \dots .$$



This leads to the definition of the ∞-page of a spectral sequence.

Example

In our previous example

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
H_0(C^3) & H_1(C^3) & H_2(C^3) & H_3(C^3) & \cdots \\
H_0(C^2) & H_1(C^2) & H_2(C^2) & H_3(C^2) & \cdots \\
H_0(C^1) & H_1(C^1) & H_2(C^1) & H_3(C^1) & \cdots \\
H_0(C^0) & H_1(C^0) & H_2(C^0) & H_3(C^0) & \cdots
\end{array}
$$

$$E^\infty = E^2 = E^3 = \dots$$

all differentials d^n with $n \geq 2$ are trivial. Thus the second page is already the ∞-page.

Convergence of a spectral sequence

A spectral sequence is said to converge towards a graded module $(H_n)_{n\in\mathbb{N}}$ if there are short exact sequences

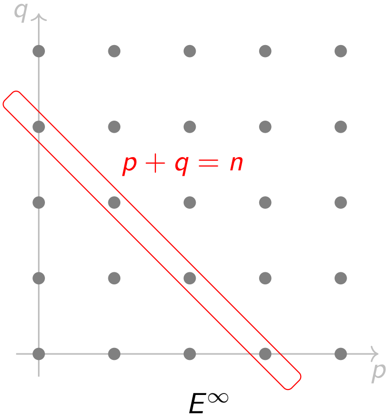
$$0 \longrightarrow 0 \longrightarrow F_0 H_n \longrightarrow E^{\infty}_{0,n} \longrightarrow 0$$

$$0 \longrightarrow F_0 H_n \longrightarrow F_1 H_n \longrightarrow E^{\infty}_{1,n-1} \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F_{n-1} H_n \longrightarrow H_n \longrightarrow E^{\infty}_{n,0} \longrightarrow 0.$$

In this case we write $E^r_{p,q} \Longrightarrow H_{p+q}$.



Example

Take again the previous example, but this time with just $C^n = C$ non-trivial.

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \cdots \\
H_0(C) & H_1(C) & H_2(C) & H_3(C) & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \cdots
\end{array}
$$

$$E^\infty = E^2 = E^3 = \dots$$

Then the only non-trivial entrys on the diagonals are the ones in the n-th row. Thus

$$E^2_{p+q} \Longrightarrow H_{p+q}(C[n])$$

Part II

The Hurewicz theorem

The theorem

For a topological space X denote by $H_*(X)$ the singular homology with integer coefficients.

Theorem (Hurewicz)

Let X be a path-connected topological space. If X is n-connected, i.e., $\pi_i(X) \cong 0$ for $i \leq n$, we have

$$H_i(X) \cong 0 \quad \textit{for} \quad 1 \leq i \leq n$$

and

$$H_{n+1}(X) \cong \begin{cases} \pi_1(X)_{ab} & \textit{if } n = 0 \\ \pi_{n+1}(X) & \textit{otherwise.} \end{cases}$$

The path space fibration

Definition (path space, loop space)

Let (X, x_0) be a pointed topological space. Then

$$PX = \{\gamma\colon [0,1] \to X \mid \gamma \text{ continuous with } \gamma(0) = x_0\},$$

equipped with the compact-open topology, is called the associated <u>path space</u>. The subspace

$$\Omega X = \{\gamma\colon [0,1] \to X \mid \gamma \text{ continuous with } \gamma(0) = \gamma(1) = x_0\}.$$

is called the <u>loop space</u>.

The map $p\colon PX \to X, \gamma \mapsto \gamma(1)$ is an example of a <u>fibration</u>. Its <u>fiber</u> $p^{-1}(x_0)$ is ΩX.

Fibrations and homotopy groups

Theorem

Let $p\colon E \to B$ be a fibration, let $b_0 \in B$ and $x_0 \in F := p^{-1}(b_0)$. The maps

$$(F, x_0) \hookrightarrow (E, x_0) \xrightarrow{p} (B, b_0)$$

induce a long exact sequence

$$\ldots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \ldots \longrightarrow \pi_0(F) \longrightarrow \pi_0(E).$$

Since the path space PX is contractible, and thus $\pi_*(PX) \cong 0$, we immediately get:

Corollary

$$\pi_{n+1}(X) \cong \pi_n(\Omega X) \qquad \textit{for all } n \in \mathbb{N}.$$

In particular, if X is n-connected the loop space ΩX is $(n-1)$-connected.

Fibrations and homology

Theorem (Serre spectral sequence)

Let $p\colon E \to B$ be a fibration where B is simply connected, i.e., $\pi_1(B) \cong 0$, and the fiber F is connected. Then there exists a spectral sequence

$$E^2_{p,q} \cong H_p\big(B; H_q(F)\big) \Longrightarrow H_{p+q}(E).$$

We will later proof:

Corollary

Let X be a simply-connected space with $H_i(X) = 0$ for $1 \leq i \leq n$. Then

$$H_{n+1}(X) \cong H_n(\Omega X).$$

Proof of the Hurewicz Theorem

Proof by induction.

Base case: Considering loops $\mathbb{S}^1 \to X$ as singular simplices induces a well-defined isomorphism $\pi_1(X)_{ab} \to H_1(X)$.

Induction step: If X is now n-connected, it is in particular $(n-1)$-connected. Thus induction gives $H_i(X) \cong 0$ for $1 \leq i \leq n-1$ and

$$H_n(X) \cong \pi_n(X) \cong 0.$$

Then by the corollary and induction we have

$$H_{n+1}(X) \cong H_n(\Omega X) \cong \pi_n(\Omega X) \cong \pi_{n+1}(X). \qquad \square$$

Proof of the corollary

Corollary

Let X be a simply-connected space with $H_i(X) = 0$ for $1 \leq i \leq n$. Then

$$H_{n+1}(X) \cong H_n(\Omega X).$$

Proof.

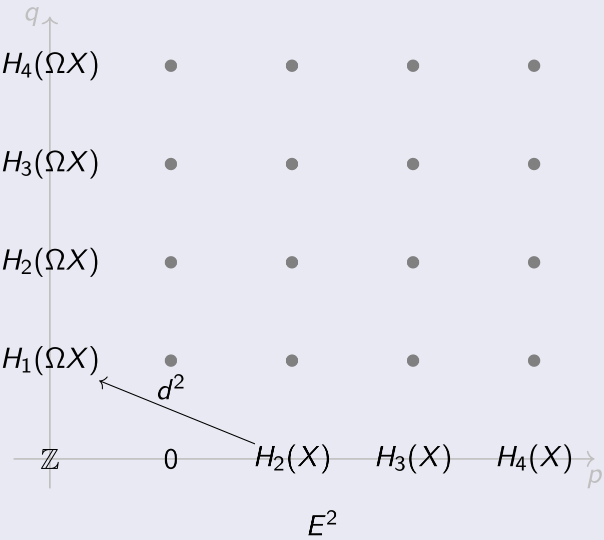
Consider the Serre spectral sequence

$$E^2_{p,q} \cong H_p\big(X; H_q(\Omega X)\big) \Longrightarrow H_{p+q}(PX) \cong \begin{cases} \mathbb{Z} & \text{if } p+q=0 \\ 0 & \text{else.} \end{cases}$$

Since X is simply-connected ΩX is path-connected. Thus

$$E^2_{p,0} \cong H_p\big(X; H_0(\Omega X)\big) \cong H_p(X) \qquad \text{and} \qquad E^2_{0,q} \cong H_0\big(X; H_q(\Omega X)\big) \cong H_q(\Omega X).$$

Proof (base case $n = 1$).



Since PX is contractible all E^∞-terms (except for $E^\infty_{0,0}$) are trivial. In particular

$$0 \cong E^\infty_{2,0} \cong \ker d^2_{2,0}$$

and

$$0 \cong E^\infty_{0,1} \cong H_1(\Omega X) / \operatorname{im} d^2_{2,0}.$$

So $d^2_{2,0} \colon H_2(X) \to H_1(\Omega X)$ has to be an isomorphism.

Proof (induction step).

$$
\begin{tikzcd}
H_4(\Omega X) & \bullet & \bullet & \bullet & \bullet \\
H_3(\Omega X) & \bullet & \bullet & \bullet & \bullet \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & 0 & H_4(X) \arrow[uuullll, "d^4"']
\end{tikzcd}
$$

E^4

Assume $H_i(X) \cong 0$ for $1 \leq i \leq n$. As before we have

$$0 \cong E^{\infty}_{n+1,0} \cong \ker d^{n+1}_{n+1,0}$$

and

$$0 \cong E^{\infty}_{0,n} \cong H_n(\Omega X) / \operatorname{im} d^{n+1}_{n+1,0}.$$

So, as before, the differential

$$d^{n+1}_{n+1,0} \colon H_{n+1}(X) \to H_n(\Omega X)$$

is an isomorphism. $\square$