Characteristic Classes

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People like vector bundles. There are multiple reasons for this. Vector bundles consist of a 'base space' X and 'fibers' $\mathbb{K}^n$ varying in a geometric way over X. This way, they can be understood as 'partially linear spaces'. The easiest example is the product $X \times \mathbb{K}^n$, but alongside with it there are many naturally occurring vector bundles of interest like tangent bundles of manifolds.

Picture a cylinder $S^1 \times \mathbb{R}$ and a Möbius strip (the line $\mathbb{R}$ making a half turn while revolving around the circle). Locally, on a small section of the circle, both look like a product, but globally their geometry looks different. This is generally true for vector bundles. They are locally given as a product, therefore it is the global structure which is of interest. One first question might be whether a given vector bundle is globally given as a product. This question might be understood using characteristic classes.

Given a vector bundle over a space X, a characteristic class is a way to functorially associate to it a cohomology class in the singular cohomology $\mathrm{H}^*(X;G)$. Those classes measure something about the vector bundle and together with geometric descriptions allow interesting insight in their geometry. Depending on the setup and spaces considered, other choices of cohomology theories work. In this workshop we will restrict to vector bundles over topological spaces and singular cohomology with coefficients in an abelian group G.

The goal of the workshop is to first understand two examples of characteristic classes (Stiefel-Whitney classes and Chern classes) together with some of their applications. Afterwards we will see the general definition of characteristic classes and end with another example.

1 Preliminaries

Talk 1 (Cohomology)**.** This first talk should provide the necessary background knowledge of singular cohomology required throughout this workshop. The references given here all refere to the book of tom Dieck [Die10], but one might also use other references such as [Hat01; May99].

- Define the singular homology and cohomology groups $\mathrm{H}_n(X, A; G)$ and $\mathrm{H}^n(X, A; G)$ for pairs of spaces (X, A) with coefficients in some abelian group G, by using the singular (co)chain complex [Die10, Sections 9.1, 9.6, 17.4].

- Without going into details talk about some properties of (co)homology:
  - Functoriality,
  - the long exact sequence [Die10, Theorem 9.1.3 and p. 416],
  - homotopy invariance [Die10, Theorem 9.3.4], and
  - the Kronecker pairing [Die10, Proposition 17.4.2].
- Give the construction of the cup product $\cup\colon \mathrm{H}^m(X;R) \times \mathrm{H}^n(X;R) \to \mathrm{H}^{m+n}(X;R)$ [Die10, Section 17.6].
- Discuss how the cup product gives a commutative graded ring $\mathrm{H}^*(X;R)$ [Die10, p. 410].
- Give the examples of the cohomology rings $\mathrm{H}^*(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$ and $\mathrm{H}^*(\mathbb{C}P^n;\mathbb{Z})$ [Hat01, Theorem 3.19].

Talk 2 (Vector bundles)**.** This talk gives a first introduction to the theory of vector bundles. These will be essential in the theory of characteristic classes. Moreover, we will see that the cylinder and Möbius strip, mentioned in the introduction, are essentially the only two real line bundles over S^1.

- Give the definition of (real) vector bundles [MS74, §2, p. 13].
- Disscuss first examples of vector bundles [MS74, §2, Examples 1–4].
- Introduce the notion of bundle maps and isomorphsims of vector bundles [MS74, §3, p. 26, 14].
- State that a vector bundle is trivial if and only if there are enough linearly independent sections [MS74, Theorem 2.2].
- Talk about constructions of vector bundles. In particular you should introduce induced bundles, products and Whitney sums [MS74, §3].
- Prove that over S^1 there are exactly two real line bundles up to isomorphism (the trivial bundle and the Möbius bundle). Pick a concrete cover of S^1, use triviality of vector bundles over contractible base spaces and that there are essentially only two possible ways to glue.

Talk 3 (Fibre bundles)**.** The goal of this talk is to generalize vector bundles to bundles with general fiber and transformation group, e.g. allowing the notion of oriented bundles.

- Introduce principal G-bundles and mention that it is a bundle with fiber G [Hus94, Definition 4.2.2, Proposition 4.2.6].
- Define fiber bundles [Hus94, Definition 4.5.1].

- Mention vector bundles as examples of fiber bundles with structure group $\mathrm{GL}_n(\mathbb{K})$ or $\mathrm{O}(n)$ in the real case, $\mathrm{U}(n)$ in the complex case, and oriented bundles for $\mathrm{SO}(n)$. Provide details as you seem fit [Hus94, Remark 5.3.3, Theorem 5.7.4, Remark 5.7.5].

- Describe the functor $\mathrm{Vect}_k \colon \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$. Feel free to restrict to a category of nice spaces **Top** [Hus94, Proposition 3.7.1]. Alternatively you can do the fiber bundle version [Die10, Section 14.4] [Hus94, Sections 4.3, 4.6, 4.10].

Talk 4 (Homotopy theory of vector bundles)**.** The goal of this talk is to understand how vector bundles are in one-to-one correspondence to certain homotopy classes, providing a topological view on the set of all vector bundles over a given space. Provide details for the statements used in the proofs as you seem fit (e.g. [Hus94, Theorems 3.4.7, 3.5.5, 3.6.2]).

- Introduce the notion of a classifying space [Die10, Section 14.4] [May99, Section 23.1].

- Introduce the (infinite) Grassmanian $\mathrm{Gr}_n(\mathbb{K}^k)$ for $k \in \mathbb{N} \cup \{\infty\}$ and the universal bundle γ_n^k [MS74, §5] [May99, Section 23.1].

- Describe the natural transformation $[-, \mathrm{Gr}_k(\mathbb{K}^\infty)] \to \mathrm{Vect}_k$ and prove that it is an isomorphism [Hus94, Theorem 3.7.2] [May99, Section 23.1 and 23.8] [MS74, Corollary 5.10].

2 Stiefel-Whitney and Chern classes

Talk 5 (Axioms and properties)**.** In this talk we introduce the Stiefel-Whitney and Chern classes axiomatically, and prove some first properties using just these axioms.

- Introduce the axioms of the Stiefel-Whitney and Chern classes [Hus94, Properties 17.3.1, 17.3.2].

- Talk about some first properties following from these axioms [Hus94, Section 17.4].

- Prove that the axioms uniquely determine the Stiefel-Whitney and Chern classes [Hus94, Theorem 17.3.4, Section 17.5].

Talk 6 (Existence)**.** After introducing Stiefel-Whitney and Chern classes axiomatically in the last talk, this talk will prove the existence of classes satisfying these axioms.

- As preperation prove the Leray-Hirsch Theorem. For simplicity restrict to the case of a single space instead of pairs of spaces [Hus94, Section 17.1] [Hat01, Theorem 4D.1].

- Give the construction of the Stiefel-Whitney and Chern classes [Hus94, Section 17.2].

- Verify that these classes satisfy the axioms of the previous talk [Hus94, Proposition 17.3.3, Section 17.6].

Talk 7 (Application I: Manifolds as boundaries)**.** Since we have now proven the uniqueness and existence of the Stiefel-Whitney classes this talk discusses some applications of these characteristic classes. This talk will focus on results by Pontrjagin and Thom, giving a complete characterization of which manifolds can occur as boundary of another manifold.

- Briefly talk abount (homological) orientability of manifolds (with boundary) and that every manifold is $\mathbb{Z}/2\mathbb{Z}$-orientable [Hus94, Section 18.4] [May99, Section 20.3].
- Define the Stiefel-Whitney classes and numbers of a manifold [Hus94, Definitions 18.8.1, 18.9.1].
- Prove the theorem of Pontrjagin and give some easy examples [Hus94, Theorem 18.9.2, Corollaries 18.9.4, 18.9.5].
- Mention the result of Thom, giving the convers of Pontrjagins theorem [Hus94, Remark 18.9.7].
- If time permits: Introduce cobordism classes and give a characterization when two manifolds are in the same cobordism class [Hus94, Definition 18.9.6, Theorem 18.9.7, Remark 18.9.8].

Talk 8 (Application II: Immersions)**.** This second talk on applications of characteristic classes will give some obstructions which manifolds can be embedded into $\mathbb{R}^n$.

- Recall the tangent bundle of a manifold [Hus94, Definition 18.2.1] [MS74, §2 Example 2].
- Give the definitions of immersions and embeddings of manifolds [Hus94, Definition 18.2.5].
- Define the normal bundle of an immersion and briefly mention its relation to the normal bundle of a submanifold $M \subseteq \mathbb{R}^n$ introduced in the second talk [Hus94, Definition 18.2.6] [MS74, §2 Example 3].
- Define the dual Stiefel-Whitney class for both vector bundles and manifolds [Hus94, Definition 18.10.1].
- Prove the condition on the dual Stiefel-Whitney class of a manifold M^n immersed in $\mathbb{R}^{n+k}$ [Hus94, Theorem 18.10.2].
- State the similar result for embeddings (while ignoring the part on the Euler class) [Hus94, Theorem 18.10.2].
- Discuss the specialized results on immersions/embeddings of $\mathbb{R}P^n$ into $\mathbb{R}^k$ [Hus94, Theorem 18.10.3, Corollary 18.10.4].

3 Characteristic classes

Talk 9 (General Characteristic classes)**.** The goal of this talk is to introduce the general notion of characteristic classes and explain the special role of Stiefel-Whitney and Chern classes.

- Recall the Yoneda lemma [Hus94, Theorem 20.1.2] [May99, Section 22.5].
- Define characteristic classes. Feel free to restrict to $\mathrm{H}^*(-;R)$ [Hus94, Definition 20.2.1] [May99, Section 23.2].
- Prove that characteristic classes with values in $\mathrm{H}^*(-;\mathbb{Z})$ are polynomials in the Chern classes [Hus94, Theorem 20.3.4] [May99, Section 23.7, p. 199].
- State the analogue for $\mathrm{H}^*(-;\mathbb{Z}/2\mathbb{Z})$ and Stiefel-Whitney classes [Hus94, Theorem 20.5.4] [May99, Section 23.2, p. 191].

Talk 10 (Euler class)**.** The goal of this talk is to introduce the Euler class for real oriented vector bundles, show the connection to the Euler characteristic and deduce the hairy ball theorem.

- Recall oriented real vector bundles [MS74, §9].
- Define Euler classes of oriented vector bundles [Hus94, Definition 17.7.4] [MS74, §9].
- Prove that an oriented vector bundle with nowhere vanishing section has trivial Euler class [Hus94, Theorem 17.8.3] [MS74, Property 9.7].
- Show the relation between Euler class and Euler characteristic [Hus94, Theorem 18.7.2] [MS74, Corollary 11.12].
- Deduce the hairy ball theorem [Hus94, Remark 18.7.4].

References

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