Centralizers and Fixed Points of Automorphisms in Finite and Locally Finite Groups

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Preface

This Habilitation thesis is a collection of former research of mine. In **Chapter 1**, Introduction, I give an outline of my research program in general.

- **Chapter 2** is the paper of mine, "Infinite Groups with an Anticentral Element", [Ers12], appeared in Comm. Alg. 2012.
- **Chapter 3** is the paper of mine, "Finite Groups with a Splitting Automorphism of Odd Order", [Ers16], appeared in Arch. Math. 2016.
- **Chapter 4** is the paper of us together with C.K. Gupta, with name "Locally Finite Groups with Centralizers of Finite Rank", [EG], appeared in Comm. Alg. 2016.
- **Chapter 5** is the paper of us together with M. Kuzucuoğlu and P. Shumyatsky, with name "Locally Finite Groups and Their Subgroups with Small Centralizers", [EKS], appeared in J. Algebra, 2017.
- **Chapter 6** is the paper of mine, "Centralizers of p-subgroups in Simple Locally Finite Groups", [Ers19], that will appear in Glasgow Mathematical Journal.
- **Chapter 7** is the paper of us, together with A. Tortora and M. Tota, with name "On groups with all subgroups subnormal or soluble of bounded derived length", [ETT], appeared in Glasgow Math. J., 2014.

To my son İnanç,

His presence always gives me inspiration.

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I would like to thank my collaborators Prof. Maria Tota, Prof. Antonio Tortora, Prof. Pavel Shumyatsky and Prof. Mahmut Kuzucuoğlu. They gave permission for our joint works to appear in this text. I will always remember Prof. Chander Kanta Gupta and our memories about our joint works with great admire.

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CHAPTER 1

INTRODUCTION

In 20th century, there were two big projects in group theory. The first of them was initiated by a question about periodic groups, raised by Burnside in 1902 (see [Burn]). Recall that a group is called *periodic* if every element has finite order.

Question 1.1. ***(Burnside)*** *Is every finitely generated periodic group necessary finite?*

A group is called *locally finite* if every finitely generated subgroup is finite. That is, Question 1.1 can be formulated as the following: Is every periodic group locally finite?

The answer of Question 1.1, which was later known as General Burnside Problem, is negative. The first examples of periodic finitely generated infinite groups were given by Golod and Shafarevich (see [GS64] and [Gol64]). Later, other examples were given by Gupta-Sidki ([GS83]), Grigorchuk ([Gri80]) and Olshanskii ([Ols81]). The groups constructed by Olshanskii were 2-generated infinite simple p-groups for primes greater than 10^{75} in which every proper subgroup has prime order and any two elements are conjugate (see [Ols81] or [Ols91]). Clearly, if an infinite group G has all its proper subgroups cyclic of order p, then G must be generated by arbitrary two elements that are not in the same cyclic subgroup. Moreover, these groups contain a unique non-trivial conjugacy class and hence they are simple.

After the construction of examples of periodic groups which are not locally finite, locally finite groups constituted an active area of research since it was interesting to find conditions for a given periodic group to be locally finite. For example, Shunkov proved in [Sun72] that an infinite periodic (residually finite) group with a finite centralizer of an involution is locally finite.

The other important project of 20th century was the Classification of Finite Simple Groups. By Jordan-Hölder Theorem, every finite group has a composition series consisting of a unique list of finite simple groups. So, finite simple groups are building blocks of all finite groups. It was F. Klein who first suggested to classify all finite simple groups. In 1950's Brauer and Fowler proved the following:

Theorem 1.2. ***(Brauer-Fowler Theorem [BF])*** *There exists a natural valued function f such that if a finite simple group contains an involution whose centralizer has order k, then the order of the group is less than $f(k)$.*

Moreover, in 1960's, Feit and Thompson proved that every group of odd order is solvable. As a consequence of this result one can easily deduce that any finite non-abelian simple group contains involutions. Since the center of a non-abelian simple group must be trivial, centralizer of involutions are large proper subgroups of finite simple groups. Hence, the centralizers of involutions became key objects to study to classify all finite simple groups.

One of the biggest achievements in finite group theory of 20th century was the following theorem of Thompson:

Theorem 1.3. ***(Thompson, [Th59])*** *Let G be a finite group with a fixed point free automorphism $\alpha \in Aut(G)$ of prime order p. Then G is nilpotent.*

Higman, Kreknin and Kostrikin proved that the nilpotency class is bounded in terms of p (see [Hig57] and [KK63]).

Thompson's result on nilpotence and Brauer-Fowler Theorem are examples of two types of results about centralizers in group theory. First of these is proving the group has a restricted structure when the centralizer of an element (or equivalently the set of fixed points of an automorphism) is "small", that is, has restrictions on it. In infinite group theory, first example of results assuming a restriction on the fixed point of an automorphism, is the important theorem of B. Neumann. Neumann proved that if G is a periodic group with a fixed-point-free automorphism of order 2, then G is abelian without elements of order 2, so locally finite (see [Neu40]). Shunkov proved in [Sun72] that a periodic group with an involutory automorphism with finite centralizer is locally finite. Because of General Burnside Problem (Question 1.1) , it was interesting to know under which conditions a periodic group has to be locally finite, so, the results of Neumann and Shunkov can be seen as first examples. Later, Hartley and Meixner proved in [HM80] that there is a natural valued function f such that if a periodic group G has an involution i with $C_G(i)$ of order less than m, then G has a nilpotent normal subgroup of nilpotency class at most 2, and index less than $f(m)$.

The second type of results related to centralizers are obtained by proving that the centralizers are "big" when the group is simple. A classical and most important example of a result of this kind is the Brauer-Fowler Theorem. In locally finite simple groups, also it has analogues. Kegel and Wehrfritz asked whether the centralizer of any element in an infinite simple locally finite group is infinite (see [KW73, Question II.4]). This was later proved by Hartley and Kuzucuoğlu (see [HK91]). Most of the theorems proven throughout my mathematical research fit into one of these two types of questions, either proving the group has a restricted structure when the centralizer is "small" or proving the centralizer (of an element, a subgroup etc.) is "big" when the group is simple.

In my Ph.D. thesis [Ers09], I worked on a problem of Brian Hartley (1939-1994) on centralizers of finite subgroups in simple locally finite groups. In the following Section 1.3 I give details about this work. In my Phd, I proved that the centralizers of finite subgroups in particular classes of non-linear simple locally finite groups are "big", in the sense they are infinite, also they do not even satisfy minimal condition on subgroups. For the definitions and history of these kind of questions in simple locally finite groups, see Section 1.3 and for the details and the proofs, see [Ers09, EK].

In Section 1.4 I summarize my research after my Ph.D, mostly about the first type of problems I have described above: In all of the five papers presented in this section, we prove structural results on the given finite or locally finite groups having a restriction on the centralizer. Details of these works are presented in Chapters 2, 3,4, 5, 6 and my related papers [Ers12, Ers16, EG, EKS, Ers19] respectively.

In Section 1.5, I summarize my other research on infinite groups. Our paper [ETT] together with my collaborators M. Tota and A. Tortora is presented. This work is not directly related to centralizers, it is on a different problem. On the other hand, classification of finite (minimal) simple groups by Thompson and subgroup structure of finite minimal simple groups were used. Recall that a *minimal simple group* is a finite simple group whose all proper subgroups are solvable. The details can be found in Chapter 7 and our related paper [ETT].

1.1 Basic Information and Notation about Finite Simple Groups

Let us give some basic information and notation about finite simple groups.

By the classification theorem which was accepted to be proved in 1980's, a non-abelian finite simple group is either isomorphic to an alternating group of degree greater than 4, or a "simple group of Lie type", or one of 26 "sporadic groups" which does not belong to any infinite family. Since basically we are trying to extend results about finite simple groups to their infinite analogues, the sporadic groups won't play a role in this research outline.The alternating groups are subgroups of the corresponding symmetric group consisting of even permutations.

Let us define a simple group of Lie type. A **(linear) algebraic group** over k is an affine algebraic variety together with a group structure in which the group operations are morphisms of varieties. In particular we refer to linear algebraic groups over the algebraic closure of finite fields. Throughout this section let p be a prime and k be an algebraic closure of $\mathbb{F}_p$. Let $\overline{G}$ be a simple linear algebraic group of adjoint type over k. A linear algebraic group G is *simple (as an algebraic group)* if it has no closed connected normal subgroups.

Let G, H be two simple algebraic groups. A surjective homomorphism $\phi : G \longrightarrow H$ is called an *isogeny* if $\ker \phi$ is finite. Two simple algebraic groups G, H are called *isogeneous* if there is an isogeny between them. Among the isogeneous groups, the one with trivial center is called *adjoint type* One may see [MT, p.71] for a more technical definition.

Since $\overline{G}$ is a closed subgroup of some general linear group, we can consider

σ_q :

$$\overline{G} \longrightarrow \overline{G}$$
$$(x_{ij}) \longrightarrow (x_{ij})^q$$

The map σ_q is called a *standard Frobenius map* on $\overline{G}$. If $\psi : \overline{G} \longrightarrow \overline{G}$ is an endomorphism with $\psi^k = \sigma_q$ for some k, then ψ is called a Frobenius map. For details, see [MT, Section 21.1]. By [Ste68-1, Thm 10.13] Frobenius maps are precisely algebraic endomorphisms with finite set of fixed points of linear algebraic groups.

Definition 1.4. *Let $\overline{G}$ be a simple linear algebraic group of adjoint type, let σ be a Frobenius map on $\overline{G}$. Let $\overline{G}_\sigma$ denote the group of fixed points of σ in $\overline{G}$. By [Ste68-1, Thm 10.13], $\overline{G}_\sigma$ is finite. The groups obtained in this way are called finite groups of Lie type. On the other hand, generally these groups are not simple. On the other hand, the group*

$$G = O^{p'}(\overline{G}_\sigma)$$

is simple where $\overline{G}$ is of adjoint type and G has order greater than or equal to 60*. The groups obtained in this way are called the finite "simple groups of Lie type".*

Here recall that $O^{p'}(G)$ is the normal subgroup generated by all p-elements of G.

One can find information about finite simple groups in general in [Wil] or the collected volumes of the Classification of Finite Simple Groups, by Gorenstein-Solomon-Lyons, in particular volume 3 [GLS97].

1.2 Simple Locally Finite Groups

After the Classification of Finite Simple Groups was done in 1980's, the next goal was to investigate infinite simple groups, which share some common properties of finite simple groups. In this section we will present some general properties of infinite simple locally finite groups.

Let G be a group and Σ be a set of subgroups of G such that $G = \bigcup_{S \in \Sigma} S$. If for every pair $S, T \in \Sigma$ there is a subgroup $H \in \Sigma$ with $S, T \leq H$ (see [KW73, p.8]) then Σ is called to be a *local system* of G. By [KW73, Theorem 4.4] any infinite simple group has a local system Σ consisting of countably infinite simple groups. This will allow us to work firstly on countable simple locally finite groups, to investigate the structure of all simple locally finite groups.

First, let us give some examples of infinite simple locally finite groups:

Example 1.5. The group of even permutations on an infinite set $\mathbb{N}$, namely $G = Alt(\mathbb{N})$ is an infinite simple locally finite group. Obviously, one can observe that G can be written as a union of a chain of finite alternating groups, embedded into each other naturally.

A field is called a *locally finite field* if every finite subset generate generates a finite subfield. Observe that (infinite) locally finite fields are simply (infinite) algebraic extensions of finite fields.

Example 1.6. Let us how we constructed finite simple groups of Lie type in Definition 1.4. Now let $\overline{G}$ be a simple linear algebraic group of adjioit type, σ a Frobenius map on $\overline{G}$ and a sequence $n_i \in \mathbb{N}$ with $n_i | n_{i+1}$ then, by [**?**],

$$G = \bigcup_{i \in \mathbb{N}} O^{p'}(\overline{G}_{\sigma^{n_i}})$$

is an infinite locally finite simple group of Lie type over a locally finite field, and all infinite locally finite simple groups of Lie type are obtained in this way.

Belyaev, Borovik, Hartley-Shute and Thomas proved independently that a linear simple locally finite group is a simple groups of Lie type over locally finite fields (see [Bel84, Bor83, HS84, Tho]).

Clearly, Example 1.5 is non-linear. Mal'cev proved that if every finitely generated subgroup of a group G is linear of degree at most n, then G is linear of degree n (see [Har95, Theorem 2.7]). The following result will be useful to construct Example 1.8:

Lemma 1.7. *Let p be a prime, $q = p^m$. The center of the special linear group $SL_n(q)$ is isomorphic to the cyclic group of order $(n, q-1)$.*

Proof. Recall that $Z(SL_n(q))$ consists of diagonal matrices $\lambda.I_{n\times n}$ where $\alpha^n = 1$. Now, $\lambda \in \mathbb{F}_q^*$, so $\lambda^{q-1} = 1$. Therefore, $\lambda^{(n,q-1)} = 1$. On the other hand, all matrices $\lambda.I_{n\times n}$ with $\lambda^{(n,q-1)} = 1$ are contained in $Z(SL_n(q))$. Hence $Z(SL_n(q)) \cong C_{(n,q-1)}$. □

Example 1.8. Let k be a finite field of size $q = p^m$ for some p and $m \in \mathbb{N}$ and

$$\begin{aligned} \psi_n : SL_n(k) &\longrightarrow SL_{n+1}(k) \\ A &\longrightarrow \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Observe that $\{(SL_n(k), \psi_n) : n \in \mathbb{N}\}$ form a direct system and the direct limit G is called **stable special linear group** $SL_0(k)$. Here, infinitely many of n's are relatively prime with $q-1$. One may simply consider $SL_{p^k}(q)$ where k runs through $\mathbb{N}$, which all have trivial center. Therefore, infinitely many of the groups in the direct system have trivial center, so $SL_0(k)$ is a union of a chain of finite simple groups, so it is simple.

The following example is known as P. Hall's universal locally finite group.

Example 1.9. Let $G_0 = S_3$, the alternating group on 3 letters. Obviously, $|S_3| = 6$. Embed G_0 into $G_1 = S_6$, by the right regular representation of G_0. Embed G_1 into $G_2 = S_{360}$, also by right regular representation. So, consider the embeddings $A_n \longrightarrow A_{n!}$, with regular embeddings. The direct limit gives us a countable locally finite groups, in which any countable locally finite group embeds, and any isomorphic finite subgroup is conjugate (see [Har95, 1.4]).

By [Har95, 1.4], U can be written as a union of finite simple groups too.

Regarding these examples are all unions of finite simple groups, it is natural to ask the following:

Question 1.10. *Let G be an infinite simple locally finite group. Does G necessarily have a local system consisting of finite simple groups?*

Zalesski and Serezhkin in 1981 (see [ZS]) answered this question negatively. They proved that the stable symplectic group over a field of odd order, which is constructed similarly with Example 1.8 can not be written as a union of finite simple groups.

On the other hand, there is still a close connection between finite simple groups and infinite simple locally finite groups, called *Kegel Sequences* and *Kegel covers*. Here, we will define just the Kegel sequences for countable groups, but for the general definiton of a Kegel cover, see [Har95, 2.1].

Definition 1.11. *[Har95, Definition 2.1, Definition 2.2] Let G be a countable locally finite group, $\{(G_i, N_i) : \ i \in \mathbb{N}\}$ be a set of pair of subgroups such that*

1. *G_i's are finite,*
2. *N_i's are maximal normal subgroups of G_i's,*
3. *$G_i \cap N_{i+1} = 1$ for all $i \in \mathbb{N}$. Then the collection $\{(G_i, N_i) : \ i \in \mathbb{N}\}$ is called a Kegel sequence for G.*

By [Har95, Lemma 2.4], every (countable) simple locally finite group has a Kegel sequence. Moreover, by [Har95, Corollary 2.5] any countably infinite simple locally finite group has a Kegel sequence consisting of perfect G_i's and hence G_i/N_i's form a set of finite simple groups of unbounded orders. The groups G_i/N_i's are called the *Kegel factors*, and N_i's are called *Kegel kernels*.

Now, since the orders of sporadic groups are bounded by the order of the Monster, for any infinite simple locally finite group there exists a Kegel sequence such that the factors are either all alternating groups or a simple group of Lie type. Therefore, if G is an infinite locally finite group then G has a Kegel cover consisting of one of the following factors:

1. G_i/N_i's are isomorphic to alternating groups A_{n_i}, with $n_i < n_{i+1}$,
2. G_i/N_i's are isomorphic to a simple group of the same classical Lie type, bounded rank,
3. G_i/N_i's are isomorphic to a simple group of the same classical Lie type, unbounded rank,
4. G_i/N_i's are isomorphic to a simple group of the same exceptional Lie type.

By [Har95, Theorem 2.6] in Cases 2 and 4, the group G is linear, and in the other cases, it is non-linear.

On the other hand, we are far beyond to obtain a full classification for infinite simple locally finite groups. By [Har95, Corollary 1.16], there exists uncountably many non-isomorphic countable simple linear algebraic groups having Kegel factors isomorphic to alternating groups and Kegel kernels are all 1. The approach is rather using the information about finite simple groups to prove further structural results about simple locally finite groups. The key point in this approach is related to the centralizers.

1.3 Centralizers in simple locally finite groups

After the classification of finite simple groups was achieved by using knowledge on the centralizers of involutions, a lot of information about centralizers in general were obtained. Since simple locally finite groups are related to finite simple groups, it was a good idea to work on centralizers in simple locally finite groups.

Kegel and Wehrfritz asked the following problem:

Question 1.12. *([KW73, Question II.4]) Let G be an infinite simple locally finite group of cardinality κ. Does the centralizer of every element have cardinality κ?*

This problem is still open for higher cardinals, but the answer is positive for countable groups by the following result by Hartley and Kuzucuoğlu:

Theorem 1.13. *[HK91, Theorem A2] In a simple locally finite group, the centralizer of every element is infinite.*

Later Hartley also proved the following:

Theorem 1.14. *[Har92] Let G be a locally finite group with a finite centralizer of an element. Then G is almost locally solvable.*

Hartley asked the following problem:

Question 1.15. *Is the centralizer of every finite subgroup, in a simple non-linear locally finite group necessarily infinite?*

This question is just about "non-linear" simple locally finite groups since for every linear simple locally finite group, there exists a finite subgroup with finite centralizer. This can be observed as an application of Hilbert's Basis Theorem: Now, linear simple locally finite groups are subgroups of simple linear algebraic groups over algebraic closure of finite fields and centralizers are closed sets. In particular, centralizers of finite subgroups form a descending chain of closed sets. Let g_0 be an arbitrary element in G where G is a simple linear algebraic group. Denote $C_0 = C_G(g_0)$. Pick $g_1 \in G \backslash C_0$ and denote $C_1 = C_{C_0}(g_1)$. Inductively, in every step, choose $g_n \in G \backslash C_{n-1}$ and let $C_n = C_{C_{n-1}}(g_n)$. Since the subgroups C_i's form a descending series of closed sets and the subgroup generated by finitely many elements of G is always finite, one ends up with a finite subgroup F with finite centralizers. Therefore, Question 1.15 is naturally just asked for non-linear groups.

The counterpart of Question 1.15, whether the centralizer of every finite subgroup in a simple non-linear locally finite group, involves an infinite non-linear simple group is resolved negatively by Meierfrankenfeld in [Mei07]. He showed in [Mei07, Corollary 7] that, for a given non-empty set Π of primes, there exists a non-linear, locally finite simple group G such that

(a) The centralizer of every non-trivial Π-element has a locally soluble Π-subgroup of finite index.

(b) There exists an element whose centralizer is a locally soluble Π-group.

In particular in the above groups there are elements whose centralizers can not involve even finite non-abelian simple groups.

In my Phd thesis [Ers09] I worked on a version of the Hartley's question, that is the following:

Question 1.16. *Determine all non-linear simple locally finite groups in which the centralizer of every finite subgroup has an abelian subgroup isomorphic to a direct product of cyclic groups of order p_i for infinitely many distinct prime p_i.*

It is a well known theorem of Hall and Kulatilaka that every infinite locally finite group contains an infinite abelian subgroup [HK64]. Observe that a stronger version of the Hall-Kulatilaka Theorem is true here: The centralizer $C_G(F)$ has an infinite abelian subgroup which has elements of order p_i for infinitely many distinct primes p_i.

Consider the set $C_{p^n} = \{x \in \mathbb{C} : \; x^{p^n} = 1\}$. Here, $(C_{p^n}, .)$ defines a group isomorphic to a cyclic group of order p^n. Observe that if m divides n then $C_{p^m} \leq C_{p^n}$, and with the inclusion maps, these sets form a direct system, where the direct limit

$$\lim_{n \in \mathbb{N}} C_{p^n}$$

is denoted by C_{p^∞}, consists of all complex p^n-th roots of unity, and forms a group under complex multiplication. This group is called the quasi-cylic p-group.

Definition 1.17. *A group is called a Chernikov group if it is a finite extension of a direct product of finitely many copies of some quasi-cyclic p_i-groups, for possibly distinct primes p_i.*

Recall that a group G satisfies *minimal condition* on subgroups if every non-empty set of subgroups of G partially ordered by inclusion has a minimal element ([KW73, Chapter 1, Section E]). Shunkov and Kegel-Wehrfritz independently proved that a locally finite group satisfying minimal condition on subgroups is Chernikov (see [Sun71] and [KW70]). Clearly, a Chernikov group has elements of order divisible by only finitely many distinct primes.

Recall that, by [KW73, Theorem 4.5] for every countably infinite locally finite simple group there exists a Kegel sequence $\mathcal{K} = (G_i, N_i)$ where G_i's form a tower of finite subgroups of G satisfying $G = \bigcup_{i=1}^{\infty} G_i, \;\; N_i \lhd G_i,$ such that G_i/N_i is a finite simple group and $G_i \cap N_{i+1} = 1$ for each i. By [KW73, Theorem 4.6], if G is an infinite linear locally finite simple group one can always choose an infinite subsequence (G_j, N_j) such that $N_j = 1$ for all j. By using the classification of finite simple groups one can find that every locally finite simple group is either linear or G_i/N_i are all alternating or a fixed type of classical linear group over various fields with unbounded rank parameter. See [HK91], [Har95] or Section 1.3 for more details about Kegel sequences.

In [Ers09] and [EK] we answered the stronger version of Hartley's question for finite $\mathcal{K}$-semisimple subgroups of simple non-linear locally finite groups.

First let us define a $\mathcal{K}$-semisimple subgroup for a given Kegel sequence $\mathcal{K}$.

Definition 1.18. *Let G be a countably infinite simple locally finite group and F be a finite subgroup of G. The group F is called a $\mathcal{K}$-semisimple subgroup of G, if G has a Kegel sequence $\mathcal{K} = \{(G_i, M_i) \ : i \in \mathbb{N}\ \}$ such that $(|M_i|, |F|) = 1,\ \ M_i$ are soluble for all i and if G_i/M_i is a linear group over a field of characteristic p_i, then $(p_i, |F|) = 1$.*

Theorem 1.19. *[EK, Ers09] Let G be a non-linear simple locally finite group which has a Kegel sequence $\mathcal{K} = \{(G_i, 1) :\ \ i \in \mathbb{N}\}$ consisting of finite simple subgroups. Then for any finite $\mathcal{K}$-semisimple subgroup F, the centralizer $C_G(F)$ has an infinite abelian subgroup A isomorphic to the restricted direct product of $\mathbb{Z}_{p_i}$ for infinitely many distinct primes p_i.*

For the second question in view of the above counterexample of Meierfrankenfeld we prove that if the field is a splitting fields for the simple groups of classical type, then the centralizer of every finite $\mathcal{K}$-semisimple subgroup involves an infinite simple group.

Theorem 1.20. *[EK, Ers09] Let G be a non-linear simple locally finite group which has a Kegel sequence $\mathcal{K} = \{(G_i, 1) :\ i \in \mathbb{N}\}$ consisting of finite simple subgroups G_i. Let F be a finite $\mathcal{K}$-semisimple subgroup of G. Then $C_G(F)$ involves an infinite simple non-linear locally finite group provided that k_i's are splitting field for G_i for all $i \in \mathbb{N}$.*

Then, we have extended Theorem 1.20 for $\mathcal{K}$-semisimple subgroups in non-linear simple locally finite groups with N_i are not necessarily trivial.

Corollary 1.21. *[EK, Ers09] Let G be a non-linear simple locally finite group and $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbb{N}\}$ be a Kegel sequence of G. Then for any finite $\mathcal{K}$-semisimple subgroup F, the centralizer $C_G(F)$ has a subgroup A containing elements of order p_i for infinitely many prime p_i. In particular $C_G(F)$ is infinite.*

In the proofs of Theorem 1.19, Theorem 1.20 and Corollary 1.21 the general method was investigating the structure of the centralizer of a finite subgroup in the Kegel factors, by using the related information about centralizers in finite simple groups. Then, we considered the embeddings between the centralizers. In particular, to prove the existence of an infinite abelian subgroup containing infinitely many elements of distinct prime orders in $C_G(F)$, we have chosen Kegel factors with large enough order and we proved first that the centralizer in that factor G_i/N_i had an element of prime p_i for distinct primes at each step. The information we used in alternating groups was counting the orbits, as well as we used the number of representations of F in large enough rank matrix group and by Maschke's theorem, we concluded that for large enough rank, there is an element of order p_i in the corresponding factor.

1.4 Finite and locally finite groups with certain restrictions on centralizers

In this section I summarize my research on finite and locally finite groups with given conditions on centralizers.

1.4.1 Anticentral elements

First of the papers that we worked on these type of problems is [Ers12], "Infinite groups with an anticentral element". Let G be a non-perfect group. An element $a \in G \backslash G'$ is called an anticentral element if the conjugacy class of a is equal to the coset of the commutator subgroup containing a, that is,

$$aG' = a^G.$$

If G has an anticentral element of order n, then for every $x \in G'$ the identity

$$x.x^a.x^{a^2 \ldots x^{a^{n-1}}} = 1$$

holds, which means a induces a *splitting automorphism* on G'. In this case, $C_{G'}(a)$ has exponent dividing n, so for a group, having an anticentral element and also having a splitting automorphism are conditions on centralizers. In particular these elements (automorpisms resp.) have centralizers (fixed points resp.) of bounded exponent, dividing the order of the element (resp. automorphism).

In [Lad08, Theorem 4.3], Ladisch proved that every finite group with an anticentral element is solvable. On the other hand, there are examples of infinite groups with an anticentral element which are not solvable. In [Ers12], we worked on infinite groups containing an anticentral element.

Chillag and Herzog asked the following question:

Question 1.22. *Is every locally finite group with an anticentral element locally solvable?*

In [Ers12] I answered this question for specific types of locally finite groups, but later, after I have proved that a finite group with a splitting automorphism of odd order is necessarily solvable in [Ers16], as a corollary of this, I also obtained in [Ers16] that a locally finite group with an anticentral element of odd order is locally solvable.

In [Ers12], we also worked on *Camina group*, that is, non-perfect groups with every element outside the commutator subgroup is anticentral. Finite Camina groups were studied by many authors as A. Camina in [Cam78], MacDonald in [Mac81, Mac86], Chillag and MacDonald in [CM], Chillag, Mann and Scoppola in [CMS], Dark and Scoppola in [DS], Isaacs in [Isa89], Ren in [Ren], Lewis in [Lew10]. A complete classification of finite non-abelian Camina groups was given in [DS]. A finite non-abelian Camina group is either a Camina p-group of nilpotency class at most 3 or a Frobenius group whose complements are either cyclic or isomorphic to the quaternion group (see [Lew10, Theorem 1]).

In [HLM11], Herzog, Longobardi and Maj studied infinite Camina groups. They proved in [HLM11, Theorem 7] that an infinite non-abelian Camina group with finite commutator subgroup is a nilpotent p-group of class 2, of exponent dividing p^2 with $Z(G) = G'$. The classification of residually finite Camina groups by Herzog, Longobardi and Maj is as follows:

Theorem 1.23. *[HLM11, Proposition 11]* ***(Herzog, Longobardi, Maj)*** *If G is a non-abelian residually finite Camina group then one of the following holds:*

1. *G is a finite p-group of nilpotency class at most 3,*

2. *G is an infinite p-group of nilpotency class 2 and exponent dividing p^2, with $G' = Z(G)$,*

3. *G is a Frobenius group with Frobenius kernel equal to G' and it is nilpotent of class depending on the order $|G/G'|$ and finite cyclic complements,*

4. *G is a Frobenius group with abelian kernel and complements isomorphic to Q_8.*

They proved in [HLM11, Theorem 16] that a finitely generated solvable Camina group is either abelian or finite. They also proved in [HLM11, Theorem 8] that if G is a locally finite Camina group, then either G/G' is a $p-$group for a suitable prime p, or G' is nilpotent and either G' is a p-group for some prime p or G/G' is locally cyclic.

There are non-solvable finitely generated infinite Camina groups constructed by Olshanskii (see [HLM11]). In [Ers12], we gave a method to construct infinite Camina groups which are not locally solvable. We proved the following theorem:

Theorem 1.24. *[Ers12] For each connected algebraic group H over an algebraically closed field of characteristic p with Frobenius map σ on H, there exist countably many non-isomorphic infinite Camina groups G with $G' \cong H$. In particular, if H is semisimple then G is not locally solvable.*

By using Theorem 1.24, we constructed various non-solvable infinite Camina groups. The key point of this construction was Lang's Theorem (see [SS, 2.2 Theorem]). In particular I constructed examples whose commutator subgroups are algebraic groups over algebraic closures of $\mathbb{F}_p$ as well as other example whose commutator subgroups are non-linear simple locally finite groups (see Chapter 2). These were also interesting in that area since before them only known non-solvable examples were the finitely generated examples constructed by Olshanskii.

1.4.2 Splitting Automorphisms

The second paper of us that we would like to present in this section is about finite groups with a splitting automorphism. Let G be a group and α be an automorphism of G. An element $x \in G$ is called a fixed-point of α if $x^\alpha = x$. We denote the set of fixed points of α in G by $C_G(\alpha)$. An automorphism α of G is called *fixed-point-free* if the identity element is the only fixed-point of α, that is $C_G(\alpha) = 1$.

Yu. M. Gorchakov defined in [Grc65] a *splitting automorphism*:

Definition 1.25. *An automorphism α of order n is called a splitting automorphism if for every $x \in G$, we have*

$$xx^{\alpha}x^{\alpha^2}\dots x^{\alpha^{n-1}} = 1.$$

By [Rob95, 10.5.1], if G is a finite group, a fixed-point-free automorphism is a splitting automorphism.

As we recalled in this section, one can observe that if $a \in G$ is an anticentral element, then a induces a splitting automorphism of G'. Moreover, if α is a fixed-point-free automorphism of a finite group G, then α is an anticentral element of $H = G\langle\alpha\rangle$.

Thompson proved that in [Th59, Theorem 1] a finite group with a fixed-point-free automorphism of prime order is nilpotent. Moreover, Kegel proved in [Keg61, Satz 1] that a finite group with a splitting automorphism of prime order is nilpotent. Rowley proved in [Row, Theorem] that a finite group with a fixed-point-free automorphism is solvable. Later, Ladisch proved in [Lad08] that a finite group with an anticentral element is solvable. We worked on the following question:

Question 1.26. *Is a finite group admitting a splitting automorphism necessarily solvable?*

The answer of this question in the full generality is negative. In Chapter 2 we provide examples of non-solvable groups having a splitting automorphism of order n, particularly when n is a natural number which is divisible by the exponent of a finite simple group.

These kind of examples motivated the following question:

Question 1.27. *Let n be a natural number which is not divisible by the exponent of any finite non-abelian simple group. Is a finite group admitting a splitting automorphism of order n necessarily solvable?*

By Kegel's result, the answer of this question is positive for prime n. E. Jabara proved in [Jab] that a finite group with a splitting automorphism of order 4 is solvable. In [Ers12], I also gave a partial answer to the question, indeed we prove the following result:

Theorem 1.28. *[Ers12] A finite group with a splitting automorphism of odd order is solvable.*

This result has an immediate consequence about locally finite groups:

Corollary 1.29. *A locally finite group with a splitting automorphism of odd order is locally solvable.*

An immediate consequence of this result answers [Ers12, Question 1.1] partially, but in a more general setting:

Corollary 1.30. *A locally finite group with an anticentral element of odd order is locally solvable.*

I studied splitting automorphisms since I wanted to go on my research on anticentral elements. At the time when I wrote [Ers16], the only finite non-solvable groups I was able to construct were the ones with a splitting automorphism whose order is divisible by the exponent of a finite simple group. Later, for every natural number n divisible by 12, we were able to construct other examples of finite non-solvable groups with a splitting automorphisms of order n. So, we showed also that in general the answer to Question 1.27 is negative. This result is now unpublished, it will be part of a joint work with C.K. Gupta and E. Jabara, about finite groups with a splitting automorphism of order 2^n. We will present this result also here.

Proposition 1.31. *Let n be any natural number divisible by 12. Then there exists a finite non-solvable group G with a splitting automorphism of order n.*

Proof. Observe first that any element in $S_5 \backslash A_5$ has order either $2, 4$ or 6. Then, for any $y \in S_5 \backslash A_5$, one has $y^{12} = 1$. Let H be a finite abelian group with a fixed point free automorphism of order n. In particular, let $H = \mathbb{Z}_3^{\frac{n}{2}}$ and $\beta : H \longrightarrow H$ is a map sending $(x_1, x_2, \ldots, x_{\frac{n}{2}})$ to $(x_2, x_3, \ldots, x_{\frac{n}{2}}, -x_1)$. Now, β is a fixed point free automorphism of H of order n. Let $G = A_5 \times H$. Pick an element $y \in S_5 \backslash A_5$ and define $\alpha : G \longrightarrow G$ as $\alpha(g,h) = (g^y, h^\beta)$. Observe that α is a splitting automorphism of order n of the non-solvable group G. Clearly, n need not be divisible by the exponent of a finite simple group. $\square$

In the ongoing joint work with C.K. Gupta (started before she passed away) and E. Jabara, we want to describe the structure of finite non-solvable groups with a splitting automorphisms of order 2^n. Our conjecture is that the only finite non-abelian simple group having a splitting automorphism of order 2^n is A_6.

1.4.3 Centralizers of Finite Rank

We noted that certain conditions on centralizers and fixed points of automorphism give structural restrictions on the group. In the section about centralizers in locally finite groups, we have already noted that Question 1.12, which is still open in full generality, was answered positively for countable simple locally finite groups, by Hartley and Kuzucuoğlu. In [HK91, Theorem A2], they proved that in an infinite simple locally finite group, every element has infinite centralizer. Later, Hartley proved in [Har92, Corollary A1] that if G is a locally finite group with a finite centralizer of an element, then G has a locally solvable normal subgroup of finite index. It is natural to ask questions of following type:

Question 1.32. *Describe the structure of an infinite locally finite groups G with some conditions on the centralizer of an element of G.*

Let us start with some definitions:

Definition 1.33. *A group G is called a Chernikov group if it is a finite extension of direct product of finitely many Prfer p-groups for possibly distinct primes p.*

Definition 1.34. *If any chain of subgroups of a group G has a minimal element, then G satisfies minimal condition (min). If any chain of p-subgroups of a group G has a minimal element then G satisfies min-p.*

Shunkov and Kegel-Wehrfritz proved independently that a locally finite group satisfying minimal condition (*min*) is a Chernikov group (see [Sun70, KW70]). In

[Har82, Theorem B] Hartley proved that if G is a periodic locally solvable group admitting an involutory automorphism ϕ with Chernikov centralizer, then $[G, \phi]'$ and $G/[G, \phi]$ are also Chernikov. Moreover, Asar proved in [Asa82, Theorem] that a locally finite group with a Chernikov centralizer of an involution is almost locally solvable. Later, Belyaev and Hartley proved that if G is a simple locally finite group with Chernikov centralizer of an element, then G is finite (see [Har95, Theorem 3.2] and [Shu07, Theorem 5.5]). Hartley proved in [Har82, Theorem 1] that a locally finite group with a Chernikov centralizer of an element of prime power order is almost locally solvable.

Definition 1.35. *Let G be a group. If every finitely generated subgroup of G is generated by at most r elements, then G is called a group of finite rank r. If every finitely generated p-subgroup of G is generated by at most r_p-elements, G is called a group of finite p-rank r_p.*

In particular, Chernikov groups have finite rank. However, a locally finite group of finite rank need not be Chernikov. But, by a result of Blackburn (see [Bla62, Theorem 4.1]), if a locally finite p-group G has a finite maximal abelian subgroup, then G is Chernikov. Blackburn proved in [Bla62] that locally finite p-group of finite rank is Chernikov.

In [EG], we, together with C.K. Gupta, investigated the following problem:

Question 1.36. *Describe infinite locally finite groups G with an element α of prime order such that $C_G(\alpha)$ has finite rank.*

Here, one can not expect to obtain G locally solvable, since there exists simple locally finite groups G with an element of prime order whose centralizer has finite rank. In particular, let $PSL_2(k)$ where k is an infinite locally finite field of odd characteristic $q \neq p$. Let $x \in G$ be a diagonal (hence semisimple) element of prime order p in G. Then $C_G(x)$ is isomorphic to a split torus if $p \neq 2$ and $C_G(x)^0$ is a torus if $p = 2$. In both cases, $C_G(x)$ has finite rank. However, any unipotent element u in an infinite locally finite simple group of Lie type has a centralizer of infinite rank.

In [KS04, Theorem 1.1], Kuzucuoğlu and Shumyatsky obtained a detailed answer for Question 4.6 for the case $p = 2$:

Theorem 1.37. *[KS04, Theorem 1.1] Let G be an infinite locally finite group with an involution ι such that $C_G(\iota)$ has finite rank. Then $G/[G,\iota]$ has finite rank. Moreover, $[G,\iota]'$ has a characteristic subgroup B such that*

1. *B is a product of finitely subgroups isomorphic to either $PSL(2,K)$ or $SL(2,K)$, which are normal in $[G,\iota]$, for some infinite locally finite fields K of odd characteristic,*

2. *$[G,\iota]'/B$ has finite rank.*

We first classified infinite simple locally finite groups with an automorphism α of prime order p such that $C_G(\alpha)$ has finite rank. We proved the following result:

Theorem 1.38. *[EG] Let G be an infinite simple locally finite group with an automorphism α such that $C_G(\alpha)$ has finite rank. Then, G is isomorphic to one of the following groups:*

1. *$G \cong PSL(l+1,k)$ or $PSU(l+1,k)$ for some infinite locally finite field k of characteristic $q \neq p$ and $p > l$*

2. *G has type $B_l(k), C_l(k)$ or ${}^2B_2$ (that is $l = 2$) over an infinite locally finite field k of characteristic $q \neq p$ (and $q = 2$ in the case of ${}^2B_2(k)$) and $p > 2l-1$.*

3. *$G \cong D_l(k)$ or ${}^2D_l(k)$ or ${}^3D_4(k)$ for some infinite locally finite field k of characteristic $q \neq p$ and $p > 2l-3$*

4. *$G \cong E_6(k)$ or ${}^2E_6(k)$ over an infinite locally finite field of characteristic $q \neq p$, and $p > 11$.*

5. *$G \cong E_7(k), F_4(k)$ or ${}^2F_4(k)$ over an infinite locally finite field of characteristic $q \neq p$, and $p > 17$.*

6. *$G \cong E_8(k)$ over an infinite locally finite field of characteristic $q \neq p$, and $p > 29$.*

7. *$G \cong G_2(k)$ or ${}^2G_2(k)$ over an infinite locally finite field of characteristic $q \neq p$, and $p > 5$.*

The case $p = 2$ is important since there is a wide literature on centralizers of involutions, which were key tools of classification of finite simple groups. By Feit-Thompson Theorem, for any locally finite group G, the group $O_{2'} \leq R(G)$ where $R(G)$ denotes the locally soluble radical of G. For other primes, the first difficulty is the existence of non-soluble p'-groups. However, $p = 3$ deserves special attention since one can list all the simple $3'$-groups. It is a well-known corollary of the classification of finite simple groups that the only finite simple groups whose orders are relatively prime with 3 are Suzuki groups ${}^2B_2(2^{2m+1})$ (see the orders of finite simple groups in [**?**]).

Hence we restricted our attention to the case $p = 3$ and proved the following result:

Corollary 1.39. *[EG] Let G be an infinite simple locally finite group with an automorphism α of order 3 such that $C_G(\alpha)$ has finite rank. Then $G \cong PSL(2,k), PSL(3,k)$ or $PSU(3,k)$ over an infinite locally finite field k of characteristic $q \neq 3$ and $\alpha \in InnDiagG$.*

Therefore, the main result of this paper was a description of an infinite locally finite group with an automorphism α of order 3, whose set of fixed points has finite rank.

Theorem 1.40. *[EG] Let G be an infinite locally finite group with an automorphism α of order 3 with $C_G(\alpha)$ of finite rank. Then*

1. *If G is not almost locally soluble then $[G, \alpha]$ is infinite.*
2. *$O = O_{3'}(G)$ is an almost locally soluble group, with $O/R(O)$ has all minimal normal subgroups isomorphic to direct products of ${}^2B_2(q)$, for possibly distinct fields F_q*
3. *$O_{3'}(G)$ has normal subgroups $N \geq M$ such that N/M is nilpotent and M and G/N has finite rank*
4. *$K = R(G)O_{3'}(G)$ is a finite extension of $R(G)$ and $C_{G/K}(\alpha)$ has finite rank.*
5. *G/K has minimal normal subgroups, any of which is a product of finitely many groups of the form $PSL(2,k), PSL(3,k)$ or $PSU(3,k)$ over some possibly infinite locally finite fields k of characteristic $q \neq 3$.*

1.4.4 Centralizers of Elementary Abelian Subgroups

Another work we have done about locally finite groups with certain conditions on centralizers is the paper "Locally finite groups and their subgroups with small centralizers", together with M. Kuzucuoğlu and P. Shumyatsky. In that paper we worked on following type of problem:

Question 1.41. *Let G be a locally finite group containing a finite subgroup A such that $C_G(A)$ is small in some sense. What can be said about the structure of G?*

These type of results were obtained before, indeed imposing certain conditions on centralizers, some significant information about G can be deduced. Hartley and Meixner proved that if $|A| = 2$ and $C_G(A)$ is finite, then G has a nilpotent subgroup of class at most two with finite index bounded by a function of $|C_G(A)|$ [HM80]. If G contains an element of prime order p whose centralizer is finite of order m, then G contains a nilpotent subgroup of finite (m, p)-bounded index and p-bounded nilpotency class. This result for locally nilpotent periodic groups is due to Khukhro [Khu90] while the reduction to the nilpotent case was obtained combining a result of Hartley and Meixner [HM81] with that of Fong [Fon76]. uses the classification of finite simple groups. One needs to mention Hartley's theorem that if G has an element of order n with finite centralizer of order m, then G contains a locally soluble subgroup with finite (m, n)-bounded index [Har92].

Recall that a group G is Chernikov if it has a subgroup of finite index that is a direct product of finitely many groups of type C_{p^∞} for various primes p (quasi-cyclic p-groups, or Prüfer p-groups). By a deep result obtained independently by Shunkov [Sun70] and Kegel and Wehrfritz [KW70] Chernikov groups are precisely the locally finite groups satisfying the minimal condition on subgroups, that is, any non-empty set of subgroups possesses a minimal subgroup. In the literature there are many results on Chernikov centralizers in locally finite groups. Hartley proved in [Har88] that if a locally finite group contains an element of prime-power order with Chernikov centralizer, then it is almost locally soluble. A group is said to almost have certain property if it contains a subgroup of finite index with that property.

Infinite locally finite groups containing a non-cyclic subgroup with finite centralizer can be simple. One example is provided by the group $PSL(2,k)$, where k is an infinite locally finite field of odd characteristic. This group contains a non-cyclic subgroup of order four with finite centralizer. In [Shu01] Shumyatsky proved that if a locally finite group G contains a non-cyclic subgroup A of order p^2 for a prime p such that $C_G(A)$ is finite and $C_G(a)$ has finite exponent for all $a \in A\backslash\{1\}$, then G is almost locally soluble and has finite exponent.

If G and T are groups, we say that G involves T if there are subgroups $K \leq H \leq G$, with K normal in H, such that $H/K \cong T$.

In Chapter 5 we proved the following theorem:

Theorem 1.42. *Let p be a prime and G a locally finite group containing an elementary abelian p-subgroup A of rank at least 3 such that $C_G(A)$ is Chernikov and $C_G(a)$ involves no infinite simple groups for any $1 \neq a \in A$. Then G is almost locally soluble.*

By Hartley"s result in [Har88], the theorem remains valid also in the case where A is of prime order. On the other hand, the theorem is no longer valid if we allow A to be of rank 2. In particular, this is illustrated by the example of the group $PSL_2(k)$. In particular, we established the following characterization of the groups $PSL_p(k)$.

Theorem 1.43. *An infinite simple locally finite group G admits an elementary abelian p-group of automorphisms A such that $C_G(A)$ is Chernikov and $C_G(a)$ involves no infinite simple groups for any $1 \neq a \in A$ if and only if G is isomorphic to $PSL_p(k)$ for some locally finite field k of characteristic different from p and A has order p^2.*

Here, if G is a simple locally finite group acted on by an elementary abelian group A in such a way that A has rank greater than 2, the centralizer $C_G(A)$ is Chernikov and for every non-identity $\alpha \in A$ the set of fixed points of α involves no infinite simple groups, then G is finite. This result is proven in Chapter 5 by using advanced results by Springer and Steinberg about centralizers of commuting semisimple elements in linear algebraic groups, as well as the classification of finite simple groups and the classification of periodic linear simple groups. Recall that, by the theorem proved independently by Belyaev, Borovik, Hartkey-Shute and Thomas, (see [Bel84, Bor83, HS84, Tho]), an infinite periodic linear simple group is a simple group of Lie type over some locally finite field (i.e., an infinite algebraic extension of a finite field).

1.4.5 Centralizers of subgroups of exponent p

Later I extended the result proven in Chapter 5 for elementary abelian p-groups to any subgroup of the automorphism group having exponent p. This result which will appear in Glasgow Mathematical Journal, is presented in Chapter 6. In particular we made the following progress:

Recall that in Chapter 5, we proved the following result:

Theorem 1.44. *[EKS, Theorem 1.1] Let p be a prime and G a locally finite group containing an elementary abelian p-subgroup A of rank at least 3 such that $C_G(A)$ is Chernikov and $C_G(a)$ involves no infinite simple groups for any $a \in A^{\#}$. Then G is almost locally soluble.*

To prove Theorem 6.1, we gave the following characterization of $PSL_p(k)$ where $char k \neq p$.

Theorem 1.45. *[EKS, Theorem 1.2] An infinite simple locally finite group G admits an elementary abelian p-group of automorphisms A such that $C_G(A)$ is Chernikov and $C_G(a)$ involves no infinite simple groups for any $a \in A^{\#}$ if and only if G is isomorphic to $PSL_p(k)$ for some locally finite field k of characteristic different from p and A has order p^2.*

In Chapter 6, I extended the result in Chapter 5 and I have proved a similar result without assuming A is elementary abelian, but instead for any subgroup of exponent p, the following result was obtained.

Theorem 1.46. *[Ers19] Let G be an infinite simple locally finite group, P a subgroup of automorphisms of exponent p such that*

1. *$C_G(P)$ is Chernikov,*

2. *For every $\alpha \in P \backslash \{1\}$, the set of fixed points $C_G(\alpha)$ does not involve an infinite simple group.*

Then $G \cong PSL_p(k)$ where k is an infinite locally finite field of characteristic p and P has a subgroup Q of order p^2 such that $C_G(P) = C_G(Q) = Q$.

The proof of this result uses our former result with Kuzucuoğlu and Shumyatsky in [EKS] as well as an analysis on the structure of centralizers of p-subgroups consisting of semisimple elements in linear algebraic groups. I have also needed our joint result with C.K. Gupta in [EG, Theorem 4.8], since the elements in Theorem 1.46 were first shown to be inner automorphisms of order p with centralizers of finite rank.

1.5 More on Infinite Groups

In Chapter 7 we summarize our other work on infinite group. See [ETT] we worked on locally graded groups whose each subgroup is either subnormal or soluble of bounded derived length. The first thing to do was investigating finite groups with this condition. Let G be a finite non-soluble group whose all non-soluble subgroups are subnormal of bounded defect. Let R be the soluble radical of this group. We first observe that a minimal normal subgroup of G/R must be a minimal simple group, since if it involves product of at least two simple groups, then there will be a non-soluble non-subnormal subgroup. We know that any finite simple group contains a minimal simple group, which reduces our analysis to minimal simple groups.

W. Möhres in [Möh90] proved that a group with all subgroups subnormal is soluble. Also, C. Casolo ([Cas01]) and H. Smith ([Smi01-3]) shows that such a group is nilpotent if it is also torsion-free. Smith later proved in [Smi01-2], that a locally (soluble-by-finite) group with all subgroups subnormal or nilpotent is soluble, and the same holds for a locally graded group whose non-nilpotent subgroups are subnormal of bounded defect. Also, in both cases, the nilpotence follows if the group is torsion-free (see [S01-1]).

In our paper [ETT], we, together with A. Tortora and M. Tota, we studied groups with all subgroups subnormal or soluble. Since the Tarski monsters constructed by A. Yu. Olshanskii in [Ols91] are infinite finitely generated periodic simple groups whose every proper subgroup is cyclic, we restricted our attention to locally graded groups with all subgroups subnormal or soluble Recall that a group is called *locally graded* if every finitely generated subgroup has a normal subgroup of finite index. The first problem that arises in the locally graded case is the presence of non-soluble locally graded groups in which every proper subgroup is soluble. In fact, the finite minimal simple groups are non-abelian simple groups with this property. Using the classification of minimal simple groups by Thompson ([Th68]), we got all the finite non-abelian simple groups having each proper subgroup metabelian.

Another difficulty was due to infinite locally graded groups with all proper subgroups soluble. Such groups are both hyperabelian (see [FdGN]) and locally soluble (see [DES]), but it is still an open question whether they are soluble. However, there is a positive answer if we bound the derived length of subgroups (see [DE]). Motivated by this result, we worked with groups whose subgroups are either subnormal or soluble of bounded derived length. In our analysis, almost minimal simple groups show up. These are groups which fit between a minimal simple group and its automorphism group.

We proved the following main results in [ETT].

Theorem 1.47. *[ETT] Let G be a locally (soluble-by-finite) group and suppose that, for some positive integer d, every subgroup of G is either subnormal or soluble of derived length at most d. Then either*

(i) G is soluble, or

(ii) $G^{(r)}$ is finite for some integer r and G is an extension of a soluble group of derived length at most d by a finite almost minimal simple group.

Theorem 1.48. *[ETT] Let G be a locally graded group and suppose that, for some positive integers n and d, every subgroup of G is either subnormal of defect at most n or soluble of derived length at most d. Then either*

(i) G is soluble of derived length not exceeding a function depending on n and d, or

(ii) $G^{(r)}$ *is finite for some integer* $r = r(n)$ *and* G *is an extension of a soluble group of derived length at most* d *by a finite almost minimal simple group.*

CHAPTER 2

Infinite Groups with an Anticentral Element

[1]

2.1 Introduction

Let G be a non-perfect group. An element a is called an *anticentral element* if $aG' = a^G$. In [Lad08, Theorem 4.3], Ladisch proved that every finite group with an anticentral element is solvable. However, there are examples of infinite groups with an anticentral element which are not solvable. In this work our objects of interest are infinite groups containing an anticentral element. In particular, it is natural to ask the following question:

Question 2.1. *Is every locally finite group with an anticentral element locally solvable?*

First, we will present some results on periodic linear and finitary groups containing an anticentral element. Then we will show that if a locally finite group containing an anticentral element is residually finite, then it is locally solvable.

A non-perfect group G is called a *Camina group* if every element outside its commutator subgroup is anticentral. Finite Camina groups were studied by many

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authors as A. Camina in [Cam78], MacDonald in [Mac81, Mac86], Chillag and MacDonald in [CM], Chillag, Mann and Scoppola in [CMS], Dark and Scoppola in [DS], Isaacs in [Isa89], Ren in [Ren], Lewis in [Lew10]. A complete classification of finite non-abelian Camina groups was given in [DS]. Indeed, a finite non-abelian Camina group is either a Camina p-group of nilpotency class at most 3 or a Frobenius group whose complements are either cyclic or isomorphic to the quaternion group (see [Lew10, Theorem 1]).

In [HLM11], Herzog, Longobardi and Maj studied infinite Camina groups. They proved in [HLM11, Theorem 7] that an infinite non-abelian Camina group with finite commutator subgroup is a nilpotent p-group of class 2, of exponent dividing p^2 with $Z(G) = G'$. They classified residually finite Camina groups in [HLM11, Proposition 11], indeed they proved that if G is a non-abelian residually finite Camina group then one of the following holds:

1. G is a finite p-group of nilpotency class at most 3,
2. G is an infinite p-group of nilpotency class 2 and exponent dividing p^2, with $G' = Z(G)$,
3. G is a Frobenius group with Frobenius kernel equal to G' and it is nilpotent of class depending on the order $|G/G'|$ and finite cyclic complements,
4. G is a Frobenius group with abelian kernel and complements isomorphic to Q_8.

They proved in [HLM11, Theorem 16] that a finitely generated solvable Camina group is either abelian or finite. They also proved in [HLM11, Theorem 8] that if G is a locally finite Camina group, then either G/G' is a $p-$group for a suitable prime p, or G' is nilpotent and either G' is a p-group for some prime p or G/G' is locally cyclic.

There are non-solvable finitely generated infinite Camina groups constructed by Olshanskii (see [HLM11]). In this work, we will also give a method to construct infinite Camina groups which are not locally solvable. Indeed, we will prove the following theorem:

Theorem 2.2. *For each connected algebraic group H over an algebraically closed field of characteristic p with Frobenius map σ on H, there exist countably many non-isomorphic infinite Camina groups G with $G' \cong H$. In particular, if H is semisimple then G is not locally solvable.*

2.2 Some properties of groups with anticentral elements

First, observe that, if G is a group with an anticentral element a, then $G' = \{[a, g] : \quad g \in G\}$. Indeed, for every $x \in G'$ there exists $g \in G$ such that $ax = a^g$ by the definition of an anticentral element. Hence, $x = [a, g]$. Moreover, $|G'| = |aG'| = |a^G| = |G : C_G(a)|$. It is easy to see that if G has an anticentral element a and N is a normal subgroup of G, then aN is anticentral in G/N.

Recall that the lower central series of G is defined as $\gamma_{n+1}(G) = [\gamma_n(G), G]$ where $\gamma_1(G) = G$ and the upper central series of G is defined as $Z_{n+1}(G)/Z_n(G) = Z(G/Z_n(G))$ where $Z_1(G)$ is the center of G.

The following result and Corollary 2.4 will be useful later.

Lemma 2.3. *Let G be a group with an anticentral element such that $\gamma_{n+1}(G)$ is finite for some n. Then G is solvable.*

Proof. Philip Hall showed that if G is a group such that $\gamma_{n+1}(G)$ is finite, then $G/Z_{2n}(G)$ is finite (see [Rob95, 14.5.3]). Since G contains an anticentral element, G/Z_{2n} is a finite group with an anticentral element, hence, it is solvable by [Lad08, Theorem 4.3]. But $Z_{2n}(G)$ is also solvable since it is nilpotent, so, G is solvable. $\square$

The following result follows from Lemma 2.3 since $G' = \gamma_2(G)$.

Corollary 2.4. *A group with an anticentral element with finite commutator subgroup is solvable.*

The following result is a consequence of the basic properties of a group with an anticentral element:

Lemma 2.5. *Let G be a group with an anticentral element $a \in G$. If $a \in Z_\alpha(G)$ for some ordinal α (where $Z_\alpha(G)$ denotes the α-th term of the transfinitely extended upper central series of G), then G is hypercentral.*

Proof. Since $Z_\alpha(G) \trianglelefteq G$, the element aZ_α is anticentral in G/Z_α. Assume $a \in Z_\alpha(G)$. Then $aZ_\alpha(G) = Z_\alpha(G)$ is anticentral in $G/Z_\alpha(G)$. Then $G/Z_\alpha(G)$ is abelian, hence, $G = Z_{\alpha+1}(G)$. □

Now, the following result is a consequence of a theorem of Kegel on groups with splitting automorphisms of prime order:

Proposition 2.6. *Let G be a group with an anticentral element a of prime order p.*

1. *If $p = 2$ then G' is abelian.*
2. *If $p = 3$ then G' is nilpotent of class at most 3.*
3. *If G is locally finite, then G' is locally nilpotent.*

Proof. Let G be a group with an anticentral element a of order p. Then for every $c \in G'$, the element ca^{-1} is conjugate to a^{-1}, so $|ca^{-1}| = p$. Then

$$\begin{aligned} 1 = (ca^{-1})^p &= ca^{-1}ca^{-1}\dots ca^{-1} \\ &= c(a^{-1}ca)(a^{-2}ca^2)\dots(a^{1-p}ca^{p-1})a^{-p} \\ &= cc^a c^{a^2}\dots c^{a^{p-1}}. \end{aligned}$$

Hence, conjugation by a is a splitting automorphism of G'.

If $p = 2$, then for each $c \in G'$

$$cc^a = 1 \text{ so } c^a = c^{-1}.$$

Then G' is abelian.

If $p = 3$, then G' is a group with a splitting automorphism of order 3. By [Zhu, Lemma 6], G' is nilpotent of class at most 3.

Now, let p be not necessarily 2 or 3 and assume G is locally finite. Let H be a finitely generated subgroup of G'. Then $K = \langle H, a\rangle$ is a finitely generated subgroup of G, so K is finite. Now, $M = H^{\langle a \rangle} \leq K$. Clearly, a is a splitting automorphism of M and since $M \leq K$, the group M is finite. By [**?**, Satz 6], a finite group with a splitting automorphism of prime order is nilpotent, hence M is nilpotent, so G' is locally nilpotent. □

2.3 Locally finite groups with an anticentral element

Let G be a group with an anticentral element. The cardinality of G is the product of cardinalities of G' and $C_G(a)$. We observed that if G' is finite, then G is solvable. Now, we will show that if G is a locally finite group and $C_G(a)$ is finite, then G is locally solvable. Indeed, we have the following consequence of a theorem of Hartley on centralizers in locally finite groups:

Proposition 2.7. *Let G be a locally finite group with an anticentral element a. If there exists $x \in G$ such that $C_G(x)$ is finite, then G is locally solvable.*

Proof. Assume that G is a locally finite group containing an element x with finite centralizer. By [Har92, Corollary A1], G has a normal locally solvable subgroup N of finite index. Then G/N is a finite group with an anticentral element, hence, G/N is solvable by [Lad08, Theorem 4.3]. Let M be a finitely generated subgroup of G. Here, MN/N is solvable since it is contained in G/N. Moreover, $MN/N \cong M/(M \cap N)$ where N is locally solvable and M is finite. Hence, M is an extension of a solvable group by a solvable group, that is, M is solvable. Therefore, G is locally solvable. □

Proposition 2.8. *Let G be a group with locally finite commutator subgroup. Assume that G has an anticentral element a of finite order such that $C_{G'}(a)$ is finite. Then $\langle a \rangle G'$ is locally solvable.*

Proof. Let $H = \langle a \rangle G'$. Since G' is locally finite, H is locally finite. Let $h \in C_H(a)$. Then $h = g'a^k$ for some $g' \in G'$ and some integer k. Now, $ha = ah$. Then $ag'a^k = g'a^{k+1}$, that is, $g' \in C_{G'}(a)$. Hence, $C_H(a) = C_{G'}(a)\langle a \rangle$ is finite.

Now, H is a locally finite group with an element with finite centralizer, by [Har92, Corollary A1], H has a locally solvable normal subgroup N of finite index.

Consider $K = \langle N^g : \ g \in G \rangle$. Clearly K is a normal subgroup of G and since N is locally solvable, the subgroup K is locally solvable. Now, G/K is a group with an anticentral element. Here, $(G/K)' = G'K/K \leq HK/K \cong H/(H \cap K)$. Since $H \cap K \geq N$ and H/N is finite, the quotient $H/(H \cap K)$ is finite. Hence, G/K is a group with finite commutator subgroup and G/K contains an anticentral element. By Corollary 2.4, it is solvable.

Let M be a finitely generated subgroup of H. Since H is locally finite, M is finite. Clearly, $MK/K \leq G/K$ is solvable. But $MK/K \cong M/(M \cap K)$ and $M \cap K$ is a finite solvable group since M is finite and K is locally solvable. Hence, M is solvable, that is, H is locally solvable. □

Now, we will consider linear groups with anticentral elements. Recall that a group is called **linear** if it has a faithful representation in some $GL_n(\mathbb{F})$ where $\mathbb{F}$ is any field. Schur proved that periodic linear groups are locally finite (see [KW73, 1.L.1]).

Locally finite fields of characteristic p are exactly the subfields of the algebraic closure of $\mathbb{F}_p$. Now, if a group G is contained in $GL_n(\mathbb{F})$ for some locally finite field $\mathbb{F}$, then G is contained in $GL_n(\overline{\mathbb{F}})$. So, without loss of generality, we will consider periodic groups with anticentral elements which have a faithful linear representation in some $GL_n(\mathbb{F})$ where $\mathbb{F}$ is the algebraic closure of $\mathbb{F}_p$.

Theorem 2.9. *Let G be a group with an anticentral element a of order m such that G' is a periodic $\mathbb{F}-$linear group where $\mathbb{F}$ has characteristic p. Then one of the following cases occurs:*

1. *$C_{G'}(a)$ is finite and G is solvable.*

2. *$C_{G'}(a)$ has an infinite abelian subgroup of exponent p^k where p^k divides m.*

Proof. Since G' is a periodic linear group, it is locally finite. Without loss of generality, we consider the algebraic closure $\mathbb{K}$ of $\mathbb{F}$, so we may assume G has a faithful linear representation in $GL_n(\mathbb{K})$.

Consider $C_{G'}(a)$. If $C_{G'}(a)$ is finite, by Proposition 2.8, G is locally solvable. By a result of Zassenhaus (see [Rob95, 15.1.3]), a linear locally solvable group is solvable, that is, G is solvable.

Assume that $C_{G'}(a)$ is infinite.

Now, let $x \in C_{G'}(a)$. Since ax is conjugate to a, the order of ax is m. Hence, $1 = (ax)^m = a^m x^m = x^m$. So, $C_{G'}(a)$ has exponent dividing m. By Hall-Kulatilaka Theorem (see [Rob95, 14.3.7]), $C_{G'}(a)$ has an infinite abelian subgroup. So, $C_{G'}(a)$ has an infinite abelian subgroup of exponent dividing m. Since m is divisible by at most finitely many primes, there exists a prime q such that $A = Dr_{i=1}^{\infty}\mathbb{Z}_{q^k}$ is contained in $C_{G'}(a)$ where q^k divides m. Assume $q \neq p$. Then, since A is a set of commuting semisimple elements of $GL_n(\mathbb{F})$, by [SS, 5.8 Theorem (c)], A is contained in a maximal torus T of $GL_n(\mathbb{F})$. But T is a direct product of n copies of $\mathbb{F}^*$ and $\mathbb{F}^*$ is locally cyclic. Hence, A can not be contained in T. Therefore, $q = p$, that is, $C_{G'}(a)$ has an infinite abelian unipotent subgroup of exponent p^k for some k such that p^k divides m. $\square$

The following result is a consequence of Theorem 2.9.

Corollary 2.10. *If G is a group with an anticentral element a of finite order such that G' is periodic $\mathbb{F}-$linear group and $(|a|, p) = 1$ where $char\mathbb{F} = p$. Then G is solvable.*

In particular, periodic linear groups over fields of characteristic p with semisimple anticentral elements are solvable.

Proof. Assume that $C_{G'}(a)$ is infinite. Then by Theorem 2.9, $C_{G'}(a)$ has an infinite abelian subgroup of exponent p^k where p^k divides $|a|$. But $(|a|, p) = 1$. Therefore, $C_{G'}(a)$ is finite and G is solvable. $\square$

The following result is on periodic groups with an anticentral element which have a faithful representation in $GL_n(\mathbb{C})$ for some n.

Proposition 2.11. *If G is a periodic subgroup of $GL_n(\mathbb{C})$ and G has an anticentral element of order n, then G is solvable.*

Proof. By Schur's Theorem, G is locally finite since it is a periodic linear group. Since $|a| = n$, the group $C_{G'}(a)$ has exponent dividing n. Burnside showed that if $chark = 0$, a k-linear group of finite exponent is finite (see [Rob95, 8.1.11.ii]). So, $C_{G'}(a)$ is finite and by Proposition 2.8, G is locally solvable. But since G is linear, by Zassenhaus' Theorem [Rob95, 15.1.3], it is solvable. $\square$

Theorem 2.12. *Let $G \leq Sym(\Omega)$ be a group with an anticentral element a such that $supp(a)$ is finite. Then G' is locally finite and locally solvable.*

Proof. Let $G \leq Sym(\Omega)$ and $a \in G$ be anticentral in G. Since $supp(a)$ is finite, a has finite order and $a \in FSym(\Omega)$. Then, every conjugate of a in G is contained in $FSym(\Omega)$, since $FSym(\Omega)$ is a normal subgroup of $Sym\Omega$. Hence,

$$a^G = aG' \subseteq FSym(\Omega).$$

Since $a \in FSym\Omega$, we have $G' \leq FSym\Omega$. Hence, G' is locally finite.

Now,

$$C_{G'}(a) = G' \cap C_{FSym\Omega}(a) \quad \text{and} \quad C_{FSym\Omega}(a) = HK$$

where $H = C_{Sym(supp(a))}(a)$ and $K = FSym(\Omega \backslash supp(a))$. Hence,

$$C_{G'}(a) = G' \cap HK.$$

Let $x \in G' \cap K$. Then x and a have disjoint supports and $ax = a^g$ for some $g \in G$. Since ax and a are conjugate finitary permutations, they have the same cycle type, that is, both of them are products of n_l cycles of lenght l for each natural number l. But, since a and x have disjoint supports, we have $x = 1$. Hence, $G' \cap K = 1$.

Let $hk_1, hk_2 \in G' \cap HK$. Then $(hk_1)^{-1}(hk_2) = k_1^{-1}k_2 \in G' \cap K = 1$, so $k_1 = k_2$. Therefore, for each $h \in H$ there exists unique a $k \in K$ such that $hk \in G' \cap HK = C_{G'}(a)$. But since H is finite, $C_{G'}(a)$ is finite. Now, $G' \leq FSym\Omega$, so it is a locally finite group.

Since $C_{G'}(a)$ is finite, by Proposition 2.8, $\langle a \rangle G'$ is locally solvable. □

If G is locally finite, we can say more:

Corollary 2.13. *Let G be a locally finite subgroup of $Sym(\Omega)$. If G contains an anticentral element a with $supp(a)$ finite, then G is locally solvable.*

Proof. By Theorem 2.12, G' is locally solvable. Let H be a finitely generated subgroup of G. Since G is locally finite, H is finite. Then

$$HG'/G' \cong H/(H \cap G').$$

Now, HG'/G' is abelian. Since H is finite and G' is locally solvable, the intersection $H \cap G'$ is solvable. Hence, H is solvable, that is, G is locally solvable. □

Corollary 2.14. *If a group G with an anticentral element has a faithful finitary permutation representation, then it is solvable.*

Proof. Since G has a finitary permutation representation, that is $G \leq FSym\Omega$ for some infinite set Ω, then G acts on the infinite set Ω such that every element has finite support. Since the finitary symmetric group is locally finite, G is locally finite. By Corollary 2.13, since $supp(a)$ is finite, G is locally solvable. Then the normal closure $\langle a^G \rangle$ is locally solvable. Since $FSym\Omega$ can be embedded in $FGL(V)$ for some V, the group G has faithful a finitary linear representation. By [MPP, Proposition 1], if G has a faithful finitary linear representation and x is an element of G such that $\langle x^G \rangle$ is locally solvable, then $\langle x^G \rangle$ is solvable. Hence, $A = \langle a^G \rangle$ is solvable. But $a^G = aG'$. For each $g' \in G'$, we have $ag' = a^g$ for some $g \in G$, hence $ag' \in A$. Since $a \in A$, the commutator subgroup G' is contained in A. Since A is solvable, G' is solvable, that is G is solvable. □

The following observation shows that if a locally finite group with an anticentral element is residually finite, then it is locally solvable.

Proposition 2.15. *Let G be a residually finite and locally finite group with an anticentral element a. Then G is locally solvable.*

Proof. Since G is residually finite, there exists a descending chain of normal subgroups N_i of finite index in G with $\bigcap N_i = 1$. Since G/N_i is a finite group with an anticentral element, by [Lad08, Theorem 4.3], G/N_i is solvable for each i.

Assume that N_i is locally solvable for some i. If T is a finitely generated subgroup of G, then T is finite. TN_i/N_i is solvable since it is contained in G/N_i. But $TN_i/N_i \cong T/(T \cap N_i)$, so, T is solvable. Hence, G is locally solvable.

Then assume each N_i is a non-(locally solvable) group. Then N_i's are locally finite groups which are not locally solvable and

$$N_1 \geq N_2 \geq \ldots.$$

Since N_1 is a locally finite group which is not locally solvable, N_1 has a finite non-solvable subgroup K. Now, since KN_i/N_i is solvable for each i, the group N_i has a finite non-solvable subgroup $K \cap N_i$, say K_i. Then K_i's form a descending chain of finite non-solvable subgroups. This chain must stabilize, so, there exists some n such that $K_n = K_{n+i}$ for each i. Therefore, $K_n \leq \bigcap N_i = 1$, which is a contradiction. Hence, G is locally solvable.

□

2.4 Infinite non-solvable Camina groups

In this section, we will construct some examples of infinite Camina groups which are not locally solvable. In these examples the commutator subgroups are periodic linear groups. Our main tool in constructing these examples is Lang-Steinberg Theorem. To state Lang-Steinberg Theorem, we need some definitions:

Definition 2.16. *Let G be a linear algebraic group over an algebraically closed field $\mathbb{F}$ of characteristic p for some prime p. Let $q = p^m$ with $m \geq 1$ and σ_q be the map given by*

$$\begin{aligned} \sigma_q : GL_n(\mathbb{F}) &\longrightarrow GL_n(\mathbb{F}) \\ (a_{ij}) &\longrightarrow (a_{ij}^q). \end{aligned}$$

*Now, σ_q is a group automorphism of $GL_n(F)$. A homomorphism $\sigma : G \to G$ is called a **standard Frobenius map** if for some n the embedding $i : G \to GL_n(\mathbb{F})$ satisfies $i(\sigma(g)) = \sigma_q(i(g))$ for some $q = p^k$ and for all $g \in G$.*

*A homomorphism α from G to G is called a **Frobenius map** if some positive power of α is a standard Frobenius map.*

Now, by [Car93, page 31], Frobenius maps are algebraic endomorphisms with finite fixed point group. Let $\overline{G}$ be a connected linear algebraic group over an algebraically closed field $\mathbb{F}$ of characteristic p. Let σ be a Frobenius map on $\overline{G}$ and $C_{\overline{G}}(\sigma) = \overline{G}_\sigma$ its fixed point group. The following result by Lang and Steinberg will be useful in our construction.

Theorem 2.17. *[SS, Theorem 2.2] (**Lang-Steinberg**) Let G be a connected algebraic group over an algebraically closed field of characteristic p and let σ be a Frobenius map on G. Then the map L defined as*

$$\begin{aligned} L \ : \ G &\longrightarrow G \\ x &\longrightarrow x^{-1}x^{\sigma} \end{aligned}$$

is surjective.

Now, we will prove the following result:

Theorem 2.18. *Let G be any connected (linear) algebraic group over an algebraically closed field of characteristic p and σ be a Frobenius map on G. Then the group $H = G \rtimes \langle \sigma \rangle$ is an infinite Camina group with a periodic linear commutator subgroup. Moreover, if G is semisimple, then H is not locally solvable.*

Proof. Let G be any connected linear algebraic group over an algebraically closed field of characteristic p. Consider G as a subgroup of $GL_m(\mathbb{F})$ for some m. Now, for each $n \in \mathbb{N}$ let $\sigma_n : (x_{ij}) \longrightarrow (x_{ij}^{p^n})$ be a standard Frobenius map from G to G. Now, σ_n is a group automorphism of the abstract group G, which is also an algebraic endomorphism of the algebraic group G. Observe that σ_n^{-1} is also a group automorphism, but it is not a morphism of varieties, i.e., σ_n is not an algebraic automorphism. For each $k \in \mathbb{N}$, $\sigma_n^k = \sigma_{nk}$ is also a Frobenius map. So, without loss of generality, take $\sigma = \sigma_n$ which sends each matrix entry to its $q-$th power where $q = p^n$. Now, consider the group $H = G \rtimes \langle \sigma \rangle$.

Let $(g, \sigma^{k_1}), (h, \sigma^{k_2})$ be any two elements of H. Their product

$$(g, \sigma^{k_1})(h, \sigma^{k_2}) = ((g)^{\sigma^{k_2}} h, \sigma^{k_1+k_2}).$$

Then, one can observe that $(g, \sigma^k)^{-1} = ((g^{-1})^{\sigma^{-k}}, \sigma^{-k})$.

We first prove the following:

Claim For every $s \neq 0$, the element $(1, \sigma^s)$ is an anticentral element of H.

Proof of the Claim First, consider the conjugacy class of $(1, \sigma^s) \in H$ where

$s<0$. Any conjugate of $(1,\sigma^s)$ is of the form

$$\begin{aligned}((g^{-1})^{\sigma^{-k}},\sigma^{-k})(1,\sigma^s)(g,\sigma^k) &= ((g^{-1})^{\sigma^{-k}},\sigma^{-k})(g,\sigma^{k+s}) \\ &= ((g^{-1})^{\sigma^s}g,\sigma^s).\end{aligned}$$

Now, let $x=g^{\sigma^s}\in G$. Then $(g^{-1})^{\sigma^s}g=x^{-1}x^{\sigma^{-s}}$. Since s is a negative integer, σ^{-s} is also a Frobenius map. Therefore, by Lang-Steinberg Theorem (Theorem **??**), the map $x\longrightarrow x^{-1}x^{\sigma^{-s}}$ is onto G. Hence, $(1,\sigma^s)^H=\{(g,\sigma^s):\ \ g\in G\}$ for each $s<0$.

Now, let

$$\tilde{G}=\{(x,1):\ \ x\in G\}\leq H$$

and, consider H'. Since $H/\tilde{G}$ is cyclic, $\tilde{G}\geq H'$. Now, consider

$$\begin{aligned}[(g,1),(1,\sigma)] &= (g^{-1},1)(1,\sigma^{-1})(g,1)(1.\sigma) \\ &= ((g^{-1})^{\sigma^{-1}},\sigma^{-1})(g^{\sigma},\sigma) \\ &= (g^{-1}g^{\sigma},1)\end{aligned}$$

Clearly, the set $A=\{[(g,1),(1,\sigma)]:\ \ g\in G\}$ is a subset of H'. But, by Theorem **??**, since the map $g\longrightarrow g^{-1}g^{\sigma}$ is surjective from G to G, the sets A and $\tilde{G}$ coincide. Hence, $H'=\tilde{G}$.

Then, we have

$$(1,\sigma^s)H'=\{(1,\sigma^s)(x,1):\ \ x\in G\}=\{(x,\sigma^s):\ \ x\in G\}.$$

Therefore, for each $s<0$, the element $(1,\sigma^s)$ is anticentral in H. But the inverse of an anticentral element is also anticentral, so for each $s\neq 0$, the element $(1,\sigma^s)$ is anticentral. This proves our claim.

Now, we observed also that for each $s<0$ the conjugacy class of $(1,\sigma^s)$ in H consists of the elements of the set $\{(g,\sigma^s):g\in G\}$. Then, each element in these conjugacy classes and their inverses are also anticentral. Therefore (g,σ^k) is an anticentral element of H for all $g\in G$, for all $k\neq 0$. This shows that H is a Camina group. Clearly, $H'\cong G$ is a linear algebraic group over an algebraically closed field of characteristic p, hence it is locally finite. $\square$

Corollary 2.19. *For any connected linear algebraic group G defined over an algebraically closed field $\mathbb{F}$ of characteristic p, there exist countably many non-isomorphic copies of infinite Camina groups whose commutator subgroups are isomorphic to G.*

Proof. Let σ be the Frobenius map on G defined by

$$\begin{aligned} \sigma: \quad & G \longrightarrow G \\ & (x_{ij}) \longrightarrow (x_{ij}^p) \end{aligned}$$

Define $H_n = G \rtimes \langle \sigma^n \rangle$. By Theorem 2.18, H_n is an infinite Camina group whose commutator subgroup is isomorphic to G. Now, we have to show that one can construct countably many non-isomorphic infinite Camina groups in this way.

Now, $\mathbb{F}$ is the algebraic closure of $\mathbb{F}_p$, hence it is a locally finite field. Then, there exists a sequence of integers $n_i | n_{i+1}$ such that one can write k as a union of finite fields such as

$$\mathbb{F} = \bigcup_{n_i | n_{i+1}} \mathbb{F}_{p^{n_i}}.$$

Define $G_i = G(\mathbb{F}_{p^{n_i}})$, the $\mathbb{F}_{p^{n_i}}$-rational points of G. Similarly, one can write G as a union of finite groups, that is, G has a local system consisting of $G(\mathbb{F}_{p^{n_i}})$. Indeed, $G = \bigcup G_i$. Here G_i's are fixed points of σ^{n_i} in G. Since $n_i | n_{i+1}$, the subgroups G_i form an increasing chain. Since G_i's are finite and G is infinite, we can choose n_i's so that $G_i < G_{i+1}$ for each i, that is, each element of the chain is a proper subgroup of the other one.

Now, for each n_i, consider the Camina group H_{n_i} constructed as in Theorem **??**. Now, for each i, j with $i \neq j$ we will show that H_{n_i} is not isomorphic to H_{n_j}. Without loss of generality, assume $i < j$. Now, consider an anticentral element of H_{n_j},

$$(g, \sigma_{n_j}^k) \in H_{n_j}.$$

The centralizer $C_{H_{n_j}}((g, \sigma_{n_j}^k))$ is isomorphic to $\cong G_{n_j k}$, so an element of H_{n_j} whose centralizer has minimal order is in the conjugacy class of $(1, \sigma_{n_j})$. But, $C_{H_i}((1, \sigma_{n_i})) \cong G_{n_i}$ has smaller order, so, H_{n_i} is not isomorphic to H_{n_j}. Therefore, we have countably many non-isomorphic Camina groups H_{n_i} such that $H'_{n_i} = G$. $\square$

Now, we can construct some examples of infinite Camina groups:

Example 2.20. Let T be a torus in $GL_n(\mathbb{F})$ where $\mathbb{F}$ is an algebraically closed field of characteristic p and let σ be a standard Frobenius map. Then, $G = T \rtimes \langle \sigma \rangle$ is an infinite solvable Camina group by Theorem 2.18. Here, G' is abelian and $C_{G'}(\sigma)$ is finite. By [HLM11, Theorem 16], a finitely generated solvable Camina group is either abelian or finite. Now, G is infinite and non-abelian, so, G is not finitely generated.

Example 2.21. Let $H = SL_n(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field of characteristic p. Then H is a simple linear algebraic group. Let

$$\begin{aligned} \sigma_k : H &\longrightarrow H \\ (x_{ij}) &\longrightarrow (x_{ij}^{p^k}) \end{aligned}$$

Then for each $k \in \mathbb{N}$, the group $G_k = H \rtimes \langle \sigma_k \rangle$ is an infinite Camina group which is not locally solvable. Here, if $k_1 \neq k_2$ the group G_{k_1} is not isomorphic to G_{k_2}. To see this, assume without loss of generality that $k_1 < k_2$. We have $C_{G_{k_2}}((g, \sigma_{k_2})) \cong SL_n(p^{k_2})$ and $C_{G_{k_2}}((g, \sigma_{k_2}^s)) \cong SL_n(p^{sk_2})$ for each $s \geq 0$. Hence, the minimal size of a centralizer in G_{k_2} is the order of $SL_n(p^{k_2})$. But, in G_1, the order of the centralizer of $(1, \sigma_{k_1})$ is smaller. Therefore, G_{k_1} is not isomorphic to G_{k_2}.

Remark 2.22. In Theorem 2.9, we showed that if G is a group with an anticentral element a of order n such that G' is a periodic $\mathbb{F}-$linear group where $\mathbb{F}$ has characteristic p then either G is solvable or $C_{G'}(a)$ has an infinite elementary abelian $p-$subgroup. Now, observe that by Theorem 2.18, there are infinite Camina groups G, whose anticentral elements have infinite order such that G is neither solvable nor locally solvable and the centralizer of any anticentral element is finite.

The following example is a Camina group whose commutator subgroup is a non-linear simple locally finite group, hence, the commutator subgroup contains every finite group.

Example 2.23. Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. Let

$$\begin{aligned} \phi_n : SL_n(\mathbb{F}) &\longrightarrow SL_{n+1}(\mathbb{F}) \\ A \quad &\longrightarrow \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

be the embedding of $SL_n(\mathbb{F})$ into $SL_{n+1}(\mathbb{F})$. Consider the direct limit G of $SL_n(\mathbb{F})$'s via these embeddings ϕ_n's. Here, G is a non-linear simple locally finite group. Let α be the automorphism of G defined as $\alpha|_{SL_n(\mathbb{F})} : (x_{ij}) \longrightarrow (x_{ij}^q)$ where $q = p^k$. Hence, $\alpha|_{SL_n(\mathbb{F})}$ is a standard Frobenius map $\sigma_{n,q}$ for $SL_n(\mathbb{F})$. Consider the map

$$\begin{aligned} L : G &\longrightarrow G \\ x \quad &\longrightarrow x^{-1}x^{\alpha}. \end{aligned}$$

Since for each $x \in G$ the map α sends $x \in SL_n(\mathbb{F})$ to an element in $SL_n(\mathbb{F})$, the map $L|_{SL_n(\mathbb{F})}$ is equal to the Lang map $x \longrightarrow x^{-1}x^{\sigma_{n,q}}$, so it is surjective. Therefore, by the same argument as in Theorem 2.18, $G \rtimes \langle \alpha \rangle$ is a Camina group, whose commutator subgroup is isomorphic to G, a non-linear simple locally finite group. Moreover, for each k where $q = p^k$ we obtain a non-isomorphic infinite non-linear Camina group whose commutator subgroup is isomorphic to G. Also, since every finitely generated subgroup of $G \rtimes \langle \alpha \rangle$ is contained in an extension of $SL_n(\mathbb{F})$ by $\langle \sigma \rangle$, the group $G \rtimes \langle \alpha \rangle$ is not finitely generated.

CHAPTER 3

Finite Groups with a Splitting Automorphism of Odd Order

3.1 Introduction

Let G be a group and α be an automorphism of G. An element $x \in G$ is called a fixed-point of α if $x^\alpha = x$. We denote the set of fixed points of α in G by $C_G(\alpha)$. An automorphism α of G is called **fixed-point-free** if the identity element is the only fixed-point of α, namely $C_G(\alpha) = 1$.

Yu. M. Gorchakov defined in [Grc65] the following:

Definition 3.1. *An automorphism α of order n is called a splitting automorphism if for every $x \in G$, we have*

$$xx^\alpha x^{\alpha^2} \dots x^{\alpha^{n-1}} = 1.$$

By [**?**, 10.5.1], if G is a finite group, a fixed-point-free automorphism is a splitting automorphism.

An element $a \in G$ is called an anticentral element of G if $aG' = a^G$ (see [Lad08] or [Ers12]). One can observe that if $a \in G$ is an anticentral element, then a induces a splitting automorphism of G'. Moreover, if α is a fixed-point-free automorphism of a finite group G, then α is an anticentral element of $H = G\langle\alpha\rangle$.

Indeed, one can observe the following implications:

Remark 3.2. Let G be a group.

1. If G is finite and α is a fixed point free automorphism of G then α is anticentral in $G\langle\alpha\rangle$

2. If a is an anticentral element of G then a induce a splitting automorphism of G'

Thompson proved that in [Th59, Theorem 1] a finite group with a fixed-point-free automorphism of prime order is nilpotent. Moreover, Kegel proved in [Keg61, Satz 1] that a finite group with a splitting automorphism of prime order is nilpotent.

Rowley proved in [Row, Theorem] that a finite group with a fixed-point-free automorphism is solvable. Later, Ladisch proved in [Lad08] that a finite group with an anticentral element is solvable. It is natural to ask the following question:

Question 3.3. *Is a finite group admitting a splitting automorphism necessarily solvable?*

The answer of this question in the full generality is negative. In particular, let H be a cyclic group of order 31. Consider the map

$$\begin{aligned} \beta : H &\longrightarrow H \\ x &\longrightarrow 11x. \end{aligned}$$

One can observe easily that β is a fixed-point-free automorphism of H, and $|\beta| = 30$. Now, consider the direct product of H with the alternating group of degree 5. Now, $\alpha : H \times A_5 \longrightarrow H \times A_5$ with

$$\alpha(x, y) = (x^{\beta}, y)$$

is a splitting automorphism of $H \times A_5$, and $|\alpha| = 30$.

These kind of examples motivate the following question:

Question 3.4. *Let n be a natural number which is not divisible by the exponent of any finite non-abelian simple group. Is a finite group admitting a splitting automorphism of order n necessarily solvable?*

By Kegel's result, the answer of Question 3.4 is positive for prime n. E. Jabara proved in [Jab] that a finite group with a splitting automorphism of order 4 is solvable. In this paper, we also give a partial answer to Question 3.4, indeed we prove the following result:

Theorem 3.5. *A finite group with a splitting automorphism of odd order is solvable.*

This result has an immediate consequence about locally finite groups:

Corollary 3.6. *A locally finite group with a splitting automorphism of odd order is locally solvable.*

Proof. Let G be a locally finite group with a splitting automorphism α of odd order n and let H be a finitely generated subgroup of G. Observe that $K = \langle H, H^{\alpha}, H^{\alpha^2}, \dots H^{\alpha^{n-1}} \rangle$ is a finite group with a splitting automorhism of odd order. Hence, by Theorem 3.5, K and H are solvable. □

An immediate consequence of Corollary 3.6 answers [Ers12, Question 1.1] partially, but in a more general setting:

Corollary 3.7. *A locally finite group with an anticentral element of odd order is locally solvable.*

3.2 Preliminaries

Let us give some properties of splitting automorphisms. The following easy observation will be useful in the proof of Theorem **??**:

Proposition 3.8. *Let n be a natural number, let G be group and α be an automorphism of G such that*

1. *$|\alpha|$ divides n, and,*
2. *$xx^{\alpha}x^{\alpha^2} \dots x^{\alpha^{n-1}} = 1$ for every x in G.*

Then $C_G(\alpha)$ has exponent dividing n and for every $x \in G$, the element $x\alpha^{-1} \in G\langle\alpha\rangle$ has order dividing n.

Proof. Let $x \in C_G(\alpha)$. Then $1 = x.x^{\alpha}x^{\alpha^2} \dots x^{\alpha^{n-1}} = x^n$. Therefore, $C_G(\alpha)$ has exponent dividing n.

Moreover,

$$(x\alpha^{-1})^n = x(\alpha^{-1}x\alpha)(\alpha^{-2}x\alpha^2)\dots(\alpha^{-(n-1)}x\alpha^{n-1})\alpha^n$$

$$= xx^{\alpha}x^{\alpha^2}\dots x^{\alpha^{n-1}} = 1.$$

□

Lemma 3.9. *Let n be an odd natural number and let S be a finite non-abelian simple group with an automorphism α such that*

1. *$|\alpha|$ divides n, and,*
2. *$xx^{\alpha}x^{\alpha^2}\dots x^{\alpha^{n-1}} = 1$ for every x in S.*

Then $\alpha \in AutS \backslash InnS$ and S is a simple group of Lie type.

Proof. Assume that $\alpha \in InnS \cong S$. In particular, let $g \in S$ be the element of S inducing α, that is, $x^{\alpha} = g^{-1}xg$. Then, by Proposition **??**, the element xg^{-1} has order dividing n. Then, the image of the map

$$\begin{aligned} \pi_g : S &\longrightarrow S \\ x &\longrightarrow xg^{-1} \end{aligned}$$

has exponent dividing n. But π_g is a bijection of S, since $g \in S$. But by the Feit-Thompson theorem, the finite non-abelian simple group S can not have odd exponent, hence α can not be an inner automorphism.

Then, α induces an outer automorphism of S. It is well known that if $S \cong Alt(\Omega)$ where Ω is a finite set with $|\Omega| \geq 5$, then $|Out(S)| = 2$ if $|\Omega| \neq 6$ and $|OutS| = 4$ if $|\Omega| = 6$. Therefore, if S is an alternating group, S can not have an outer automorphism of odd order. One can observe the same result for sporadic groups in [ATLAS, Table 1], in particular, the outer automorphism group of any sporadic group has order dividing 2. Hence, S must be a simple group of Lie type. □

Let us give some notation about simple groups of Lie type. Let S be a finite simple group of Lie type. Then, by [Car93], there exists a simple linear algebraic group $\overline{S}$ of adjoint type over the algebraic closure of a finite field of characteristic p, and a Frobenius map σ on $\overline{S}$ such that $S = O^{p'}(\overline{S}_\sigma)$ where $\overline{S}_\sigma$ denotes the set of fixed points of σ on $\overline{S}$.

By [Ste68-2], automorphisms of a finite simple group of Lie type are well known. In particular, [Ste68-2, Theorem 30], if α is an automorphism of S where $S = O^{p'}(\overline{S}_\sigma)$ is a finite simple group of Lie type then

$$\alpha = g\phi\delta$$

where g is an element of $InnDiagS = \overline{S}_\sigma$ (an inner-diagonal automorphism), ϕ is induced by an automorphism of the underlying field k, and δ is a graph automorphism. Indeed ϕ is induced by a Frobenius map φ on $\overline{S}$ such that $\varphi^k = \sigma$, hence $\varphi|_S = \phi$. We will denote $\overline{S}_\sigma/S$ by $OutDiagS$.

3.3 Main result

First we will prove the following result about finite simple groups to prove Theorem 3.5:

Theorem 3.10. *Let S be a non-abelian finite simple group with an automorphism α of order dividing n with*

$$x.x^{\alpha}.x^{\alpha^2}\dots.x^{\alpha^{n-1}} = 1$$

for every $x \in G$. Then n is even.

Proof. Assume that n is odd. Then by Lemma 3.9, S is of Lie type and α is not an inner automorphism. Then, $\alpha = g\theta$ where $1 \neq \theta = \phi\delta$ where ϕ is a field automorphism of S and δ is a graph automorphism.

By [GLS97, Theorem 2.5.12, (a) and (e)], subgroup of inner diagonal automorphisms is normal in $Aut(S)$ and field automorphisms and graph automorphisms commute with each other. Therefore, $|\alpha|$ must be divisible by $|\theta|$ and $|\theta|$ must be divisible by $|\delta|$. Since n is odd by assumption, $|\delta| \neq 2$, hence, either $\delta = 1$ and $\theta = \phi$ or $S \cong D_4(q)$ for the prime power q and δ is a graph automorphism of order 3.

1. First assume that $S = D_4(q)$ and $\alpha = g\phi\delta$ where δ is a graph automorphism of order dividing 3 (in particular δ maybe be equal to 1). Observe that $H = C_S(\phi) \cong D_4(q_1)$ where $q_1|q$ and H is δ-invariant. By [Har92, Lemma 3.1], $C_H(\delta)$ involves a finite simple group, so fix an involution $\iota \in C_H(\delta) = C_S(\phi, \delta)$. Now, ι is fixed by θ. Clearly, $\alpha = g\phi\delta = g\theta = \theta g^\theta$. Denote $g^\theta = g_1$. Definitely, g_1 has odd order, and one has $\iota^\alpha = \iota^{g_1}$. But by [**?**, Theorem 2.5.12 (h)] $OutDiagD_4$ is either 1 or an elementary abelian group of order 4, so g_1 must be an element of S.

   Then $x = \iota g_1 \in S$, so, by Proposition 3.8

   $$x\alpha^{-1} = \iota g_1 g_1^{-1}\theta^{-1} = \iota\delta^{-1}\phi^{-1}$$

   has order dividing n.

   Since ι commutes with ϕ and δ, one has $\iota^n = \iota = \phi^n\delta^n = 1$, which is a contradiction.

   Hence, S is not isomorphic to $D_4(q)$.

2. Then, $\alpha = g\phi$ where $g \in InnDiagS = \overline{S}_\sigma$ and ϕ is a field automorphism. Assume $g \in S$. Observe that $\overline{S}_\phi \leq \overline{S}_\sigma$ is a finite group of Lie type (and simple except $PSL_2(2)$, $PSL_2(3)$ and ${}^2B_2(2)$), so it has even order. Choose an involution in $\iota \in \overline{S}_\phi$. Necessarily, $\iota^\phi = \iota$ and $g\iota \in S$. So, $(g\iota\alpha^{-1})^n = 1$. But $g\iota\alpha^{-1} = g\iota\phi^{-1}g^{-1}$ is conjugate to $\iota\phi^{-1}$, which has even order since ι and ϕ commute.

3. Then suppose $g \notin S$. Since g has odd order, this is possible only if $\overline{S}_\sigma/S$ is not a 2-group. By [GLS97, Theorem 2.5.12 (c)], in this case $\overline{S}$ has type A_l or $E_6(q)$ with $3|q-1$. Assume that $\overline{S}$ has type $E_6(q)$ where $3|(q-1)$. Let $\alpha = g\phi = xy\phi$ where $x \in \overline{S}_\sigma \backslash S$, the map ϕ is a field automorphism of order k and $y \in S$. Clearly, $k|n$. Since $g \notin S$, we know $x \neq 1$.

   By [GLS97, Theorem 2.5.12 (b)], we may assume $|x| = 3$. Now since $(3, q) = 1$, we conclude that x is a semisimple element of $\overline{S}$ fixed by σ. By [GLS97, Theorem 2.5.12 (b)] $\overline{S}_\sigma/S$ is normalized by the group of field automorphisms, hence $x^\phi = x$ or $x^\phi = x^2 = x^{-1}$. If $x^\phi = x^{-1}$, one has

$x = x^{\phi^k} = (x^{-1})^{\phi^{k-1}}$ where $|\phi| = k$. Since k is odd, x must have order 2, which is a contradiction. Therefore, x is fixed by ϕ. Hence, x is an inner-diagonal automorphism of $S_0 = O^{p'}(\overline{S}_\phi)$ where $S_0 \cong E_6(q_0)$ with $q_0|q$. By [**?**, Lemma 3.4], since x is a semisimple element of $\overline{S}$ fixed by ϕ, the group $C_{S_0}(x)$ contains a simple group of rank greater than $\frac{6-3}{3+[\frac{4}{3}]}$, which is necessarily positive. Hence, there exists an involution $\iota \in S$ fixed by x and ϕ. Take $y^{x^{-1}}\iota \in S$. By Proposition 3.8, $(y^{x^{-1}}\iota\alpha^{-1})^n = 1$. Now, $\alpha = xy\phi = y^{x^{-1}}x\phi$, hence

$$y^{x^{-1}}\iota\alpha^{-1} = y^{x^{-1}}\iota\phi^{-1}x^{-1}(y^{-1})^{x^{-1}}$$

has order t dividing n. But this element is conjugate to $\iota\phi^{-1}x^{-1}$ which has even order.

4. Therefore, we end up with the case $\overline{S}$ is a simple linear algebraic group of adjoint type A_l. Let $S = O^{p'}(\overline{S}_\sigma)$ with $\alpha = g\phi$. Still, $|\alpha| = n$ is odd, $g \in \overline{S}_\sigma$ and ϕ is a field automorphism of order k. Again, $k|n$. Indeed, ϕ is induced by a Frobenius map ψ on $\overline{S}$ with $\psi^k = \sigma$. Observe that $S_0 = O^{p'}(\overline{S}_\psi) \leq O^{p'}(\overline{S}_\sigma)$ is a finite group of Lie type over, hence it has even order. Fix an involution $\iota \in S_0$. Write $\alpha = g\phi = xy\phi$ where $x \in \overline{S}_\sigma \backslash S$ and $y \in S$. By Case 2 we may assume $x \neq 1$.

   (a) Let g_s and g_u be the semisimple and the unipotent parts of the Jordan decomposition of g. Since $x \neq 1$, we get $g_s \neq 1$. Clearly $gS = g_sg_uS = g_sS$ since $g_u \in S$. On the other hand, $gS = xS$, so without loss of generality, one may assume that x is semisimple. By the choice of the coset representative, one may also assume that $x^\phi = x$ (indeed x is $\mathbb{F}_p$-rational).

   If x is not a regular semisimple element, $C_{\overline{S}}(x)$ involves a ψ invariant simple linear algebraic group, hence, there exist an involution $\iota \in \overline{S}$ with $\iota^\phi = \iota$ and $[\iota, x] = 1$. Choose $w = (\iota y)^\phi \in S$ such that $w\alpha^{-1} = \phi^{-1}\iota y\phi\phi^{-1}y^{-1}x = \phi^{-1}\iota x$. This element has even order.

   (b) Next, consider the case $\alpha = g\phi = xy\phi$ where $y \in S$ and $x \in OutDiagS$ a regular semisimple element where S is of type A_l. Hence, by [GLS97, Theorem 2.5.1 (b), Definition 2.5.13 (a)] x is a diagonal automorphism, without loss of generality, $x = diag(\lambda_1, \dots, \lambda_{l+1})$ where λ_i's are distinct eigenvalues fixed by σ. Now, there exists $x_1 = diag(\prod_{i=1}^{l+1} \lambda_i, 1, 1, \dots 1)$ and $y_1 \in S$ such that $xy = x_1 y_1$. If $l \geq 2$, x_1 is not regular and we are done as in Case 4a.

(c) Finally, $\overline{S}$ is of type A_1, the map $\alpha = xy\phi$ where $x \in OutDiagS$ and $y \in S$ and ϕ is a field automorphism. In this case x has order 2 and $xy = g_s g_u$. Since $(xy)^2 \in S$ one has $g_s^2 \in S$ which contradicts with $|g_s|$ dividing $|g|$ and $|g|$ odd.

Hence, if a finite non-abelian simple group S admits an automorphism α of order k dividing n with

$$x.x^{\alpha}.x^{\alpha^2 \dots x^{\alpha^{n-1}}} = 1$$

for every $x \in S$ then n must be even.

□

Now, we can prove the main result of this paper, namely Theorem 3.5:

Proof of Theorem 3.5. Let G be a finite non-solvable group with a splitting automorphism α of odd order n. Let R be the solvable radical of G. Then R^{α} is a solvable normal subgroup, so $R = R^{\alpha}$. Consider G/R. Define $\overline{\alpha} : G/R \longrightarrow G/R$ as $(xR)^{\overline{\alpha}} = x^{\alpha}R$ for all x in G. Now, for every $xR \in G/R$, one has

$$(xR).(xR)^{\overline{\alpha}}.(xR)^{\overline{\alpha}^2} \dots (xR)^{\overline{\alpha}^{n-1}} = x.x^{\alpha}.x^{\alpha^2} \dots x^{\alpha^{n-1}}R = R. \qquad (3.3.1)$$

Therefore, $\overline{\alpha}$ is an automorphism of G/R, satisfying (3.3.1) for every $xR \in G/R$ and $|\overline{\alpha}|$ divides n. We may assume that $R = 1$ and hence $\overline{\alpha} = \alpha$. Now, let G be a finite group having trivial solvable radical, with an automorphism α satisfying

$$x.x^{\alpha}.x^{\alpha^2} \dots x^{\alpha^{n-1}} = 1$$

for every $x \in G$ and α is of order dividing n, where n is an odd natural number.

Let M be a minimal normal subgroup of G. Since $R = 1$, the group M is not solvable. By [Rob95, 3.3.15], M is isomorphic to the direct product of finitely many copies of a finite non-abelian simple group S.

Fix $S \leq M$ and consider the action of $\langle\alpha\rangle$ on the group

$$K = S \times S^{\alpha} \times \ldots \times S^{\alpha^{t-1}}$$

where t is the length of an orbit of α. Clearly, t divides $|\alpha| = n$. If $(s_1, s_2^{\alpha}, \ldots, s_t^{\alpha^{t-1}}) \in C_K(\alpha)$, then,

$$(s_1, s_2^{\alpha}, \ldots, s_t^{\alpha^{t-1}})^{\alpha} = (s_t^{\alpha^t}, s_1^{\alpha}, \ldots, s_{t-1}^{\alpha^{t-1}}) = (s_1, s_2^{\alpha}, \ldots, s_t^{\alpha^{t-1}})$$

hence

$$s_1 = s_2 = \ldots = s_t = s_t^{\alpha^t}.$$

Therefore, $C_K(\alpha) \cong C_S(\alpha^t)$. Since $C_K(\alpha)$ must have odd order, $\alpha^t \neq 1$.

Consider $H = \{(s, 1, \ldots, 1) \in K\} \cong S$. Since $H \leq G$, the map α satisfies

$$(s, 1, \ldots, 1).(s, 1, \ldots, 1)^{\alpha} \ldots (s, 1, \ldots, 1)^{\alpha^{n-1}} = (1, 1, \ldots, 1).$$

In particular, $(s.s^{\alpha^t}.s^{\alpha^{2t}} \ldots . s^{\alpha^{t(m-1)}}) = 1$ for all $s \in S$ where $mt = n$. Therefore, $\beta = \alpha^t$ is a splitting automorphism of H and $|\beta|$ is odd.

Since H is isomorphic to S, we conclude that S is a finite simple group with a splitting automorphism β of odd order. By Lemma 3.9, S is a simple group of Lie type and $\beta \in AutS \backslash InnS$, is a splitting automorphism of odd order. This contradicts with Theorem 3.10.

Hence, a finite group with a splitting automorphism of odd order is solvable.

$\square$

Chapter 4

Locally Finite Groups with Centralizers of Finite Rank, *with C.K. Gupta*

4.1 Introduction

Centralizers and fixed points of automorphisms carry a lot structural information in finite and locally finite groups. For finite groups, two famous results of this kind are Brauer-Fowler Theorem and Thompson's theorem on finite groups with a fixed point free automorphism of prime order. The following question of Kegel and Wehrfritz motivated the study of centralizers in (simple) locally finite groups:

Question 4.1. *[KW73, Question II.4] Let G be an infinite simple locally finite group of cardinality κ. Is the cardinality of the centralizer of every element of G equal to κ?*

This question, which is still open in full generality, was answered positively for countable simple locally finite groups, by Hartley and Kuzucuoğlu. Namely, in [HK91, Theorem A2], they proved that in an infinite simple locally finite group, every element has infinite centralizer. Later, Hartley proved in [Har92, Corollary A1] that if G is a locally finite group with a finite centralizer of an element, then G has a locally soluble normal subgroup of finite index. It is natural to ask questions of the following type:

Question 4.2. *Describe the structure of an infinite locally finite groups G with given conditions on the centralizer of an element of G.*

Many authors studied locally finite groups imposing conditions on centralizers. Before mentioning results about structure of locally finite groups with restricted structure of centralizers, we need some definitions:

Definition 4.3. *A group G is called a **Chernikov group** if it is a finite extension of direct product of finitely many Prfer p-groups for possibly distinct primes p.*

Definition 4.4. *If any chain of subgroups of a group G has a minimal element, then G satisfies **minimal condition** (min). If any chain of p-subgroups of a group G has a minimal element then G satisfies min-p.*

Shunkov and Kegel-Wehrfritz proved independently that a locally finite group satisfying minimal condition (*min*) is a Chernikov group (see [**?**, KW70]). In [Har82, Theorem B] Hartley proved that if G is a periodic locally soluble group admitting an involutory automorphism ϕ with Chernikov centralizer, then $[G, \phi]'$ and $G/[G, \phi]$ are also Chernikov. Moreover, Asar proved in [**?**, Theorem] that a locally finite group with a Chernikov centralizer of an involution is almost locally soluble. Later, Belyaev and Hartley proved that if G is a simple locally finite group with Chernikov centralizer of an element, then G is finite (see [Har95, Theorem 3.2] and [Shu07, Theorem 5.5]). Hartley proved in [Har88, Theorem 1] that a locally finite group with a Chernikov centralizer of an element of prime power order is almost locally soluble.

Definition 4.5. *Let G be a group. If every finitely generated subgroup of G is generated by at most r elements, then G is called a **group of finite rank** r. If every finitely generated p-subgroup of G is generated by at most r_p-elements, G is called a **group of finite p-rank** r_p.*

In particular, Chernikov groups have finite rank. However, a locally finite group of finite rank need not be Chernikov. But, by a result of Blackburn (see [Bla62, Theorem 4.1]), if a locally finite p-group G has a finite maximal abelian subgroup, then G is Chernikov. Indeed, Blackburn proved in [Bla62] that a locally finite p-group of finite rank is Chernikov.

In this paper, we will investigate the following problem:

Question 4.6. *Describe infinite locally finite groups G with an element α of prime order p such that $C_G(\alpha)$ has finite rank.*

Here, one can not prove that G is locally soluble, since there exists simple locally finite groups G with an element of prime order p whose centralizer has finite rank. In particular let $G = PSL_2(k)$ where k is an infinite locally finite field of odd characteristic $q \neq p$. Let $x \in G$ be a diagonal (hence semisimple) element of prime order p in G. Then $C_G(x)$ is isomorphic to a split torus if $p \neq 2$, and $C_G(x)^0$ is a torus if $p = 2$. In both cases, $C_G(x)$ has finite rank. However, any unipotent element u in an infinite locally finite simple group of Lie type has a centralizer of infinite rank.

In [KS04, Theorem 1.1], Kuzucuooğlu and Shumyatsky obtained a detailed answer for Question 4.6 for the case $p = 2$:

Theorem 4.7. *[KS04, Theorem 1.1] Let G be an infinite locally finite group with an involution ι such that $C_G(\iota)$ has finite rank. Then $G/[G,\iota]$ has finite rank. Moreover, $[G,\iota]'$ has a characteristic subgroup B such that*

1. *B is a product of finitely subgroups isomorphic to either $PSL(2,K)$ or $SL(2,K)$, which are normal in $[G,\iota]$, for some infinite locally finite fields K of odd characteristic,*

2. *$[G,\iota]'/B$ has finite rank.*

We will first classify infinite simple locally finite groups with an automorphism α of prime order p such that $C_G(\alpha)$ has finite rank. Indeed, we will prove the following result:

Theorem 4.8. *Let G be an infinite simple locally finite group with an automorphism α of order p such that $C_G(\alpha)$ has finite rank. Then, G is isomorphic to one of the following groups:*

1. *$G \cong PSL(l+1,k)$ or $PSU(l+1,k)$ for some infinite locally finite field k of characteristic $q \neq p$ and $p > l$*

2. *G has type $B_l(k), C_l(k)$ or ${}^2B_2$ (that is $l = 2$) over an infinite locally finite field k of characteristic $q \neq p$ (and $q = 2$ in the case of ${}^2B_2(k)$) and $p > 2l-1$.*

3. $G \cong D_l(k)$ *or* ${}^2D_l(k)$ *or* ${}^3D_4(k)$ *for some infinite locally finite field* k *of characteristic* $q \neq p$ *and* $p > 2l - 3$

4. $G \cong E_6(k)$ *or* ${}^2E_6(k)$ *over an infinite locally finite field of characteristic* $q \neq p$*, and* $p > 11$.

5. $G \cong E_7(k), F_4(k)$ *or* ${}^2F_4(k)$ *over an infinite locally finite field of characteristic* $q \neq p$*, and* $p > 17$.

6. $G \cong E_8(k)$ *over an infinite locally finite field of characteristic* $q \neq p$*, and* $p > 29$.

7. $G \cong G_2(k)$ *or* ${}^2G_2(k)$ *over an infinite locally finite field of characteristic* $q \neq p$*, and* $p > 5$.

The case $p = 2$ is important since there is a wide literature on centralizers of involutions, which were key tools of classification of finite simple groups. By Feit-Thompson Theorem, for any locally finite group G, the group $O_{2'}(G) \leq R(G)$ where $R(G)$ denotes the locally soluble radical of G. For other primes, the first difficulty is the existence of non-soluble p'-groups. However, $p = 3$ deserves special attention since one can list all the simple $3'$-groups. Indeed, Lemma **??** indicates that only simple $3'$-groups are exactly Suzuki groups ${}^2B_2(q)$. Hence, in this paper, we will investigate the structure of locally finite groups G with an automorphism α of order 3 such that $C_G(\alpha)$ has finite rank. First, let us observe the following consequence of Theorem 4.8 for $p = 3$:

Corollary 4.9. *Let* G *be an infinite simple locally finite group with an automorphism* α *of order* 3 *such that* $C_G(\alpha)$ *has finite rank. Then* $G \cong PSL(2,k), PSL(3,k)$ *or* $PSU(3,k)$ *over an infinite locally finite field* k *of characteristic* $q \neq 3$ *and* $\alpha \in InnDiagG$.

By using Corollary 4.9 we will prove the following the following result on locally finite groups with an automorphism of order 3 whose centralizer has finite rank:

Theorem 4.10. *Let* G *be an infinite locally finite group with an automorphism* α *of order* 3 *with* $C_G(\alpha)$ *of finite rank. Then*

1. *If G is not almost locally soluble then $[G,\alpha]$ is infinite.*

2. *$O = O_{3'}(G)$ is an almost locally soluble group, with $O/R(O)$ has all minimal normal subgroups isomorphic to direct products of ${}^2B_2(q)$, for possibly distinct fields F_q*

3. *$O_{3'}(G)$ has normal subgroups $N \geq M$ such that N/M is nilpotent and M and G/N has finite rank*

4. *$K = R(G)O_{3'}(G)$ is a finite extension of $R(G)$ and $C_{G/K}(\alpha)$ has finite rank.*

5. *G/K has minimal normal subgroups, any of which is a product of finitely many groups of the form $PSL(2,k), PSL(3,k)$ or $PSU(3,k)$ over some possibly infinite locally finite fields k of characteristic $q \neq 3$.*

4.2 Preliminaries

The following result of Shunkov helps us understand the structure of locally finite groups of finite rank:

Theorem 4.11. *[Sun71, Corollary 2] A locally finite group of finite rank is almost locally soluble.*

Therefore, if G is a locally finite group with an automorphism α such that $C_G(\alpha)$ has finite rank, by Theorem 4.11 $C_G(\alpha)$ is almost locally soluble.

We also observe that a locally finite group of finite rank satisfies *min-p* for every prime p.

Lemma 4.12. *A locally finite group of finite rank satisfies min-p for each prime p.*

Proof. Let G be a locally finite group of finite rank. Then G has finite p-rank for each prime p. By [**?**, Theorem 4.1], since G has maximal elementary abelian p-subgroups, any Sylow p-subgroup of G is Chernikov. Hence, G satisfies *min-p* for each prime p. □

The following well-known result of Kegel and Wehrfritz will be useful in our proofs:

Theorem 4.13. *[KW73, 3.2 Corollary] Let G be a locally finite group containing an elements of order p. Then the following are equivalent:*

1. *There exists $x \in G$ such that $|x| = p$ and $C_G(x)$ satisfies min-p.*

2. *G satisfies min-p.*

Remark 4.14. Therefore, if G is a locally finite group with an element of order p such that $C_G(x)$ has finite rank, then by Lemma 4.12, $C_G(x)$ satisfies *min-p*, hence, by Theorem 4.13, G satisfies *min-p*. Infinite simple locally finite groups satisfying *min-p* are classified independently by Belyaev, Borovik, Hartley-Shute and Thomas. In particular, simple locally finite groups satisfying *min-p* are precisely simple groups of Lie type over infinite locally finite fields of characteristic $q \neq p$ (see [Bel84, Bor83, HS84, Tho]).

The following result is used to deal with coprime automorphisms:

Lemma 4.15. *[KS04, Lemma 2.1] Let A be a finite group of automophisms of a locally finite group G. If N is an A-invariant normal subgroup of G with $(|A|, |N|) = 1$ then $C_{G/N}(AN/N) = C_G(A)N/N$.*

Remark 4.16. By Lemma 4.15, we deduce that if G is a locally finite group with an automorphism α of order p with $C_G(\alpha)$ has finite rank, then $C_{G/O_{p'}(G)}(\alpha)$ has also finite rank.

Lemma 4.17. *Let G be a periodic almost locally soluble group with an element x of order p such that $C_G(x)$ satisfies min-p. Then $G/O_{p'}(G)$ is Chernikov.*

Proof. By [KW73, 3.17 Theorem], if a periodic almost locally soluble group G satisfies *min-p*, then $[G : O_{p'p}(G)]$ is finite where $O_{p'p}(G)$ denotes the preimage of $O_p(G/O_{p'}(G))$. Now, since $C_G(x)$ satisfies *min-p*, by Theorem 4.13, G satisfies *min-p*, hence $O_p(G/O_{p'}(G))$ is a locally finite p-group satisfying *min-p*. Then, it is Chernikov by [Sun71, KW70], hence $G/O_{p'}(G)$ is Chernikov. $\square$

Here, we need some background information on simple groups of Lie type. Let G be a simple linear algebraic group of adjoint type over an algebraically closure of $\mathbb{F}_q$. An algebraic group is an algebraic variety which has a group structure, such that group operations are variety morphisms. In particular, if this variety is affine, or equivalently if $G \leq GL(n, K)$ for some $n \in \mathbb{N}$ and an algebraically closed field K, then G is called a linear algebraic group. There are classical and exceptional types of linear algebraic groups, arising from Lie theory, namely, types A_l, B_l, C_l and D_l are classical types, where E_6, E_7, E_8, F_4, G_2 are the exceptional types. For each Lie type, there are several groups, all of which is a perfect central extension of the one called adjoint type, and the universal central extension is called the simply connected group. For example, for type A_l, the simply connected group is $SL(l+1, K)$ and the adjoint group is $PGL(l+1, K)$. For more information, see [Hum75] or [**?**].

Now, let G be a simple locally finite group of Lie type over a locally finite field K of characteristic q.

By [Tur], there exist a simple linear algebraic group $\overline{G}$ of adjoint type over the algebraic closure of K, a Frobenius map σ on $\overline{G}$ and a sequence of integers $n_i | n_{i+1}$ such that

$$G = \bigcup_{i \in \mathbb{N}} O^{p'}(\overline{G}_{\sigma^{n_i}})$$

where $\overline{G}_{\sigma^k}$ denotes the set of fixed points of the Frobenius map σ^k on $\overline{G}$ and $O^{p'}(H)$ denotes the subgroup generated by p-elements of H (see [HK91, Lemma 4.3]). Hence, a locally finite, simple group of Lie type has a local system consisting of finite simple groups of same type.

Automorphisms of simple groups of Lie type over perfect fields are classified by Steinberg in [Ste68-2, Theorem 30]. Indeed, if α is an automorphism of a simple group G of Lie type over a locally finite field K, then there exists an element g of $\bigcup \overline{G}_{\sigma^{n_i}} = InnDiagG$, an automorphism ϕ induced by an automorphism of K and an automorphism δ induced by the symmetries of the Dynkin diagram such that

$$\alpha = g\phi\delta.$$

These are called inner-diagonal, field and graph automorphisms.

Let G be a simple group of Lie type defined over a locally finite field of characteristic q. Recall that an automorphism α of G, is called a **semisimple automorphism** if $(|\alpha|, q) = 1$. An element $x \in G$ is called a **semisimple**

element if $(x,q)=1$.

We need the following result by B. Hartley:

Lemma 4.18. *[Har92, Lemma 3.1] Let $\overline{G}$ be an adjoint type simple linear algebraic group over the algebraic closure of $\mathbb{F}_q$. Let $G = O^{q'}(\overline{G}_\sigma)$ where σ is a Frobenius map on $\overline{G}$. If α is an automorphism of G of prime order $p \neq q$ with $\alpha \in AutG \backslash InnDiagG$, then $C_G(\alpha)$ involves a finite non-abelian simple group H such that $|G|$ is bounded by a function of $|H|$ and p.*

We will use this result of Hartley to show that if G is an infinite locally finite group with an automorphism α of prime order whose centralizer has finite rank, then G is of Lie type and α is an inner-diagonal automorphism.

Lemma 4.19. *Let G be an infinite simple locally finite group with an automorphism α of prime order p such that $C_G(\alpha)$ is of finite rank. Then G is a simple group of Lie type over an infinite locally finite field of characteristic $q \neq p$ and $\alpha \in InnDiagG$.*

Proof. Let G be an infinite simple locally finite group with an automorphism α of order p such that $C_G(\alpha)$ has finite rank. By Lemma 4.12, $C_G(\alpha)$ satisfies *min-p*. Then, by Theorem 4.13, G itself satisfies *min-p*. By Remark 4.14, G is an infinite simple group of Lie type over a locally finite field of characteristic $q \neq p$. Then, by [HK91, Theorem 4.3], G is a union of finite simple groups $G_i = O^{q'}(\overline{G}_{\sigma^{n_i}})$, each of which is invariant under α, that is, $G = \bigcup_{i\in\mathbb{N}} G_i$. Now, α is an automorphism of G_i, of order p. If $\alpha \in AutG_i \backslash InnDiagG_i$, then by Lemma **??**, there exists a finite simple group H_i which is involved in $C_{G_i}(\alpha)$ such that $|G_i|$ is bounded in terms of $|H_i|$ and p. Since p is constant, if $|H_i|$'s are bounded, then $|G_i|$'s must be bounded, hence G should be finite, which is not the case. So, if α is not an inner-diagonal automorphism, then $C_G(\alpha)$ involves finite simple groups of arbitrarily large orders. Hence, $C_G(\alpha)$ can not be almost locally soluble, and can not be of finite rank. □

Hence, if G is a simple locally finite group with an automorphism α of prime order such that $C_G(\alpha)$ has finite rank, then G is linear and α is a semisimple inner-diagonal automorphism. The following result is a direct consequence of this fact and [Har92, Theorem C].

Proposition 4.20. *Let G be an infinite simple locally finite group with an automorphism α of prime order such that $C_G(\alpha)$ has finite rank. Then $C_G(\alpha)$ has infinitely many elements of distinct prime orders.*

One can observe that $[G, \alpha] = \langle g^{-1}g^{\alpha} \ : \ g \in G\rangle$ is the intersection of all normal subgroups N of G such that α acts trivially on G/N.

Lemma 4.21. *Let G be a locally finite group with an automorphism α such that $C_G(\alpha)$ has finite rank. If G is not almost locally soluble, then $[G, \alpha]$ is infinite.*

Proof. By Theorem 4.11, $C_G(\alpha)$ is almost locally soluble. Since G is not almost locally soluble by assumption, $[G : C_G(\alpha)]$ is infinite. Assume $[G, \alpha]$ is finite, that is, let $|[G, \alpha]| = n$. Let $x_1, x_2, \dots x_{n+1} \in G$. Clearly, there exists $i, j \in \{1, 2, \dots n\}$ such that $x_i^{-1}x_i^{\alpha} = [x_i, \alpha] = [x_j, \alpha] = x_j^{-1}x_j^{\alpha}$.

Hence, $x_i x_j^{-1} \in C_G(\alpha)$. Then

$$x_i C_G(\alpha) = x_j C_G(\alpha).$$

So, $[G : C_G(\alpha)]$ must be finite, which contradicts our assumption G not being almost locally soluble. □

The following two observations might be easy but interesting. The first one is a consequence of the solution of Restricted Burnside Problem:

Proposition 4.22. *Let G be a locally finite group of finite rank and bounded exponent. Then G is finite.*

Proof. Let G be a locally finite group of finite rank r and bounded exponent m. By the solution of Restricted Burnside Problem (see [ZV]), for each r and m, there are only finitely many finite groups generated by r elements and have exponent m. Then, since any finite subgroup of G are generated by at most r-elements and have exponent m, it has order less than some constant C. Assume that $H \leq G$ is a finite subgroup of maximal order. If G is infinite, then there exists $g \in G \backslash H$, but then $\langle H, g\rangle$ is a finite subgroup of G, whose order exceeds $|H|$, which is a contradiction. Hence, G must be finite. □

So, the following easy result follows from Proposition 4.22:

Proposition 4.23. *Let G be a locally finite group with an automorphism α of finite order such that $C_G(\alpha)$ has finite rank. If $C_G(\alpha)$ has bounded exponent, then G is almost locally soluble.*

Proof. Assume that $C_G(\alpha)$ has finite rank r and bounded exponent m. By Proposition 4.22, $C_G(\alpha)$ must be finite. Then $H = G\langle\alpha\rangle$ is a locally finite group with an element α such that $C_H(\alpha)$ is finite. By [Har92, Corollary A1], H is almost locally soluble, and hence, G is almost locally soluble. □

The following observation is a consequence of the classification of finite simple groups:

Lemma 4.24. *A non-abelian finite simple group of order coprime with $3'$ order is ${}^2B_2(2^{2k+1})$.*

Proof. Clearly the order of any alternating group is divisible by 3. By [**?**, Table 1, p.vii], the same holds for sporadic groups. By [**?**, Table 5, p. xvi] the order of any finite simple group G of Lie type over a finite field of size q is divisible by $q(q^2-1)$ unless $G \cong^2 B_2(2^{2k+1})$. If q is not a power of 3, one gets $3|q^2-1$, so the result follows. □

The next result is related to the case $p = 3$, which we will mainly deal with:

Lemma 4.25. *Let G be an infinite locally finite $3'$-group with an automorphism α of order 3 and $C_G(\alpha)$ has rank r. Then G is almost locally soluble. If R denotes the locally soluble radical of G, then $soc(G/R)$ is generated by k-copies of ${}^2B_2(q_i)$, where q_i's and k are bounded by r.*

Proof. Khukhro and Mazurov proved in [KM] that a locally finite p'-group H having an automorphism β of order p such that $C_H(\beta)$ has finite rank r has a locally soluble normal subgroup of $\{p, r\}$-bounded index. Hence, in our case, a locally finite $3'$-group G with an automorphism α of order 3 with $C_G(\alpha)$ of rank r is almost locally soluble and G/R is bounded by r. Hence $|soc(G/R)|$ is also bounded by r. Since G/R has no soluble normal subgroups, all of its minimal normal subgroups are isomorphic to products of ${}^2B_2(q_i)$'s since the only simple $3'$-group is ${}^2B_2(q_i)$ for some $q_i = 2^{2m_i+1}$, by Lemma **??**.

□

Lemma 4.26. *Let G be an infinite locally finite group with an automorphism α of order 3 and $C_G(\alpha)$ has rank r. Let R be the locally soluble radical of G and $K = O_{3'}(G)R$. Then*

1. *$K/R = O_{3'}(G)R/R$ is a finite group, any of whose minimal normal subgroups is generated by k-copies of ${}^2B_2(q)$, where q and k are bounded by r.*

2. *$O_{3'}(G)$ has normal subgroups $N \geq M$ such that N/M is nilpotent and M and G/N has finite rank.*

3. *G/K has a minimal normal subgroup.*

Proof. We will prove the first and third step, since the second follows from a result of Khukhro:

1. By Lemma 4.25, $O_{3'}(G)$ is almost locally soluble, hence $O_{3'}(G)/R(O_{3'}(G))$ is a finite non-soluble $3'$-group. By Lemma 4.24, $O_{3'}(G)/R(O_{3'}(G))$ has m minimal normal subgroups, any of which is generated by k_i-copies of ${}^2B_2(q_i)$, where q_i and k_i are bounded by r, $i = 1, \dots m$. Since, K is a product of an almost locally soluble group and R, which is the locally soluble radical of G, one can conclude easily that K is almost locally soluble. But $R(K) = R(O_{3'}(G))R$. Since $R(O_{3'}(G)) char O_{3'}(G)$ and $O_{3'}(G) \trianglelefteq G$, we get $R(O_{3'}(G)) \trianglelefteq G$, hence $R(K) = R$. Hence, $K/R = K/R(K)$ is finite and any of its minimal normal subgroups is generated by k-copies of ${}^2B_2(q)$, where q and k are bounded by r.

2. Follows from [Khu07, Corollary 3].

3. Observe that G/K has no normal locally soluble subgroups and G/K has no normal $3'$-subgroups. Denote $G_0 = G/K$. Since $C_{G/K}(\alpha)$ is a quotient of $C_{G/O_{3'}(G)}(\alpha)$ by a finite group and the later has finite rank by Lemma 4.15, $C_{G/K}(\alpha)$ has finite rank, in particular it satisfies *min*-3. By Theorem 4.13, G_0 satisfies *min*-3. Let S be a Sylow 3-subgroup of G_0. Since G_0 satisfies *min*-3, one gets S is Chernikov. Let $\Gamma = \{T \leq S : \text{ there exists } N \trianglelefteq G_0 \text{ with } N \cap S = T\}$. Since S is Chernikov, Γ has a minimal element

Q. If $Q = 1$, any normal subgroup of G_0 must be $3'$-group, but since G_0 has no normal $3'$-subgroups by assumption, G_0 must be simple, hence it is the minimal normal subgroup of itself. Otherwise, if $Q \neq 1$, let M be the intersection of all normal subgroups in Γ whose intersections with S contains T. Then M is a minimal normal subgroup of G_0.

□

4.3 Main results

The following result presents the main idea of our proofs.

Lemma 4.27. *Let $G = PSL(n,k)$ where k is an infinite locally finite field of characteristic q. Let $g \in PGL(n,k)$ be an inner-diagonal automorphism of G such that $C_G(g)$ has finite rank. Then g is a regular semisimple element of $\overline{G} = PGL(n,\overline{k})$ and $n \leq |g|$.*

Proof. Let $g \in PGL(n,k) \leq \overline{G}$ be an inner-diagonal automorphism of G. Now, $C_G(g)$ has finite rank and $PGL(n,k)/PSL(n,k)$ has finite rank. Then

$$C_{PGL(n,k)}(g)/C_G(g) = C_{PGL(n,k)}(g)/(C_{PGL(n,k)}(g) \cap G)$$

$$\cong C_{PGL(n,k)}(g)G/G \leq PGL(n,k)/PSL(n,k),$$

so, $C_{PGL(n,k)}(g)/C_G(g)$ and $C_{PGL(n,k)}(g)$ has finite rank. Since $C_{PGL(n,k)}(g) = \bigcup_{i\in\mathbb{N}} C_{G_i}(g)$ where $G_i = \overline{G}_{\sigma^{n_i}}$ for some $n_i | n_{i+1}$, in our case $C_{\overline{G}}(g)$ has finite rank, so it must be almost locally soluble.

Consider the Jordan decomposition of $g \in \overline{G}$. Namely, there exists $s, u \in \overline{G}$, with $su = us = g$ and $|s|$ is coprime with $char k = q$ and $|u|$ is a power of q. Moreover, $C_G(g) = C_G(s) \cap C_G(u)$.

If $s = 1$, then $g = u$ and $C_G(u)$ can not be of finite rank for any unipotent element since the center of a Sylow q-subgroup of G contains an infinite elementary abelian q-subgroup. Then $s \neq 1$. Now, assume $g = su = us$ with $u \neq 1$ and $s \neq 1$. In this case, neither of s or u can be regular. Since g is conjugate to its Jordan form, there exist a matrix of the form $x = \begin{pmatrix} J_s & 0 \\ 0 & J_u \end{pmatrix} Z \in \overline{G}$ which is conjugate to g, where J_s is a semisimple $k \times k$ block and J_u is a unipotent $m \times m$ block with $k + m = n$. Hence $C_{\overline{G}}(g) \cong C_{\overline{G}}(x)$. Since $s \neq 1$ and $u \neq 1$, the unipotent block $J_u \neq 1$ and $m \geq 2$. Therefore, there exists a unipotent subgroup consisting of matrices of the form

$$\{\begin{pmatrix} I_k & 0 \\ 0 & U_\lambda \end{pmatrix}\}$$

which is contained in $C_G(x)$ and can not be of finite rank. Hence, $u = 1$. Now, $C_G(g)$ has finite rank, then $C_G(g)$ must be almost locally soluble by Shunkov's result. Here $g = s$ is semisimple, so it is a diagonalizable element in $PGL(n, k)$ with all distinct eigenvalues, that is s is a regular semisimple element (see [SS, E50, 1.7 Corollary]) . Indeed, if two eigenvalues of g are equal, then $C_G(g)$ involves $PSL(2, k)$.Then, since all eigenvaules of g are distinct roots of $x^{|g|} = 1$, the size of the matrix g must be less than or equal to $|g|$. Hence, $n \leq |g|$.

□

Proof of Theorem 4.8. Let G be a simple locally finite group with an automorphism α of prime order such that $C_G(\alpha)$ has finite rank. Then $C_G(\alpha)$ satisfies *min-p*. By Remark 4.14 ([Bel84, Bor83, HS84, Tho]), if G is an infinite simple locally finite group satisfying *min-p*, then G is a simple group of Lie type over an infinite locally finite field of characteristic $q \neq p$.

Moreover, by Lemma 4.19, α is an inner-diagonal automorphism of G. Since G is a simple group of Lie type over an infinite locally finite field of characteristic q, by [Tur], there exist a simple linear algebraic group $\overline{G}$ of adjoint type, a Frobenius map σ on $\overline{G}$ and a sequence of integers $n_i | n_{i+1}$ such that

$$G = \bigcup_{i \in \mathbb{N}} O^{p'}(\overline{G}_{\sigma^{n_i}})$$

where $\overline{G}_{\sigma^k}$ denotes the set of fixed points of the Frobenius map σ^k on $\overline{G}$ and $O^{p'}(H)$ denotes the subgroup of a group H generated by p-elements of H (see [HK91, Lemma 4.3]). Since $\alpha \in InnDiagG$, in fact $\alpha = g \in \bigcup_{i \in \mathbb{N}} \overline{G}_{\sigma^{n_i}}$.

Denote $H_i = \overline{G}_{\sigma^{n_i}}$. Since $n_i | n_{i+1}$, the groups $H_i = \overline{G}_{\sigma^{n_i}}$'s and $G_i = O^{p'}(\overline{G}_{\sigma^{n_i}})$'s form increasing chains of subgroups. Hence, G is a union of finite simple groups G_i's of Lie type. Since $\alpha = g \in \bigcup H_i$, by passing a subsequence of n_i's, one may assume that each of G_i's is invariant under g. Since H_i/G_i is finite, and $C_G(\alpha)$ has finite rank, so does $C_H(\alpha)$ where $H = \bigcup_{\imath\in\mathbb{N}} \overline{G}_{\sigma^{ni}}$.

Now, $C_H(g) = \bigcup C_{\overline{G}}(g)_{\sigma^{n_i}}$. Since g is a semisimple element of $\overline{G}$, by [SS], the identity component $C_{\overline{G}}(g)^0$ is a connected reductive linear algebraic group, that is, $C_{\overline{G}}(g)^0 = TS_1S_2 \ldots S_k$ where T is a central torus and S_i's are simple linear algebraic groups by [SS, E3 1.4].

By passing a subsequence of n_i's, one can assume that $\alpha \in \overline{G}_{\sigma^{n_i}}$ for every i. Now, $g \in InndiagG = \bigcup \overline{G}_{\sigma^{n_i}}$. The endomorphisms σ^{n_i}'s are Frobenius maps on $\overline{G}$, by [SS, E10, 3.2], $C_{\overline{G}}(\alpha)$ is σ^{n_i}-invariant for each $i \in \mathbb{N}$.

Now, we need to analyse the cases where $C_{\overline{G}}(g)$ has finite rank, hence, almost locally soluble. Indeed $C_G(g)$ is almost locally soluble if and only if $C_{\overline{G}}(g)$ is almost locally soluble. But, since $C_{\overline{G}}(g)$ has its identity component which is a reductive normal subgroup of finite index, the only possible case is $C_{\overline{G}}(g)^0$ is a torus. In particular, if $\overline{G}$ is a simple linear algebraic group and $C_{\overline{G}}(g)$ is almost locally soluble, then $C_{\overline{G}}(g)$ must be abelian by finite. Indeed, by [SS, E50, 1.7 Corollary], this is equivalent to say that g is a regular semisimple element of the simple algebraic group $\overline{G}$.

Hence, we end up with $q \in \overline{G}$, a regular semisimple element of order p. If $\overline{G}$ has type A_l, namely, if $\overline{G} \cong PGL(n,k)$ for some infinite locally finite field of characteristic q, then by Lemma **??**, $n \leq p$.

Assume that $\overline{G}$ has Lie type other than A_l. Let $\overline{H}$ be the universal central extension of $\overline{G}$. Then $\overline{H}$ is simply connected simple algebraic group of the same type with $\overline{G}$. Let $\bar{g}$ be a preimage of $g \in \overline{H}$ under the canonical map. Then $C_H(\bar{g}) \cong C_{\overline{G}}(g)^0$ and $\bar{g}^p \in Z(H)$.

Now, H is a simply connected simple linear algebraic group and $\bar{g}$ is a semisimple element whose order modulo $Z(H)$ is p. Let $r = \sum m_i r_i$ be the highest root of the corresponding root system. By [HK91, Theorem D (i)], if $p \leq \sum m_i$, then $C_H(\overline{G}) \cong C_{\overline{G}}(g)^0$ involves an infinite simple group, hence it can not have finite rank.

Now, we need the list sums of coefficients of highest roots of root systems. For the details about root systems of simple linear algebraic groups, see [Car93]:

For type A_l, the sum of the coefficients of the highest root is $\sum_i^l m_i = l$.

For classical types B_l, C_l, and D_l the sums of the coefficients of the highest root are $2l-1, 2l-1$ and $2l-3$ respectively.

For exceptional types E_6, E_7, E_8, F_4 and G_2, the sums $\sum m_i$ are $11, 17, 29, 11$ and 5 respectively.

Hence, if G is an infinite locally finite simple group with an automorphism α of order p such that $C_G(\alpha)$ has finite rank, then one of the following cases about the underlying simple algebraic group $\overline{G}$ holds:

- $\overline{G}$ has type A_l, namely $G \cong PSL(l+1,k)$ or $PSU(l+1,k)$ for some infinite locally finite field k of characteristic $q \neq p$ and $p > l$ (by Lemma **??**).

- $\overline{G}$ has type B_l or C_l, namely G has type $B_l(k), C_l(k)$ or ${}^2B_2$ (that is $l = 2$) over an infinite locally finite field k of characteristic $q \neq p$ (and $q = 2$ in the case of ${}^2B_2(k)$) and $p > 2l-1$.

- $\overline{G}$ has type D_l, namely $G \cong D_l(k)$ or ${}^2D_l(k)$ or ${}^3D_4(k)$ for some infinite locally finite field k of characteristic $q \neq p$ and $p > 2l-3$

- $\overline{G}$ is an exceptional simple linear algebraic group of type E_6, namely $G \cong E_6(k)$ or ${}^2E_6(k)$ over an infinite locally finite field of characteristic $q \neq p$, and $p > 11$.

- $\overline{G}$ is an exceptional simple linear algebraic group of type E_7 of F_4, namely $G \cong E_7(k), F_4(k)$ or ${}^2F_4(k)$ over an infinite locally finite field of characteristic $q \neq p$, and $p > 17$.

- $\overline{G}$ is an exceptional simple linear algebraic group of type E_8, namely $G \cong E_8(k)$ over an infinite locally finite field of characteristic $q \neq p$, and $p > 29$.

- $\overline{G}$ is an exceptional simple linear algebraic group of type G_2, namely $G \cong G_2(k)$ or ${}^2G_2(k)$ over an infinite locally finite field of characteristic $q \neq p$, and $p > 5$.

□

The next result will give restrictions of minimal normal subgroups of a locally finite group with an element of order p whose centralizer has finite rank.

Theorem 4.28. *Let M be a commuting product of infinite simple locally finite groups $G_1, G_2, \dots G_m$ and α be an automorphism of order p such that $C_M(\alpha)$ has finite rank r. Then each G_i is α-invariant, G_i's belong to the list given in Theorem 4.8. Moreover, $Out(M)$ is soluble by finite.*

Proof. Consider $M = \prod G_i$. The automorphism α acts on the product of simple groups, so, consider an orbit of α on M, and consider the product of the groups in this orbit, namely $M \geq K = G_i \times G_i^{\alpha} \times \dots G_i^{\alpha^{t-1}}$, where t is the length of the orbit. Clearly t divides p. But, if $t = p$ then, $C_K(\alpha) \cong C_{G_i}(\alpha^p) \cong G_i$. But, since $C_M(\alpha)$ has finite rank, it should be almost locally soluble by Theorem 4.11. Therefore, $t = 1$, that is, each simple factor in M are α-invariant. Hence, each G_i belongs to the list given in Theorem 4.8 for each p. Hence, for each p, there are finitely many possible simple factors G_i.

Let $A = AutM$ and $I = InnM$. Clearly, A permutes the isomorphic factors in the product $G = G_1 \dots G_m$. So, there exists a homomorphism ψ from A to the product $S_{n_1} \times \dots \times S_{n_k}$ where n_i denotes the numbers of isomophic simple groups in G, and k is the number of isomorphism types. Then, $K = ker\psi$ has finite index in G. Since each G_i is normal in G, the kernel K contains I. Let K_i be the subgroup of K which consists of all automorphisms β of G that restricts as an inner automorphism of G_i. Now, $I = InnG = \cap K_i$ and the map

$$f: K/\cap K_i \longrightarrow \prod_{i=1}^{m} K/K_i$$
$$x(\cap K_i) \longrightarrow (xK_1, \dots, xK_m)$$

is an isomorphism. Now, we need to show that K/K_i is soluble. But K/K_i embeds in $OutG_i$, which is soluble by the Classification of finite simple groups. Hence, K/I is soluble and A/I is soluble by finite. □

For $p = 3$, we will have a special case of this result:

Corollary 4.29. *Let $M = G_1 \times \dots \times G_m$ be a direct product of simple locally finite groups G_i. Assume that M has an automorphism α of order 3, such that $C_M(\alpha)$*

has finite rank. Then G_i's are isomorphic to $PSL(2,k), PSL(3,k)$ or $PSU(3,k)$ for some infinite locally finite field of characteristic different from 3. Moreover, $OutM$ is abelian by finite.

Proof. The first part follows from Corollary 4.9 and Theorem 4.28. Now, $OutM$ embeds in the direct product of finitely many copies of outer automorphism groups of $PSL(2,k), PSL(3,k)$ or $PSU(3,k)$ with $char k \neq 3$ by the proof of Theorem 4.28. Now, $Out(PSL(2,k))$ is an extension of $Aut(k)$ by degree 2. For $PSL(3,k)$ and $PSU(3,k)$, the group of outer automorphisms is isomorphic to an extension of $Aut(k)$ by degree 6 and 3 respectively. □

Proof of Theorem 4.10. The first, second and fourth steps are consequences of some results that was proved in this paper:

1. Follows from Lemma 4.21.

2. Follows from Lemma 4.25.

3. Follows from [Khu07, Corollary 3].

4. Follows from Lemma 4.26.

5. G/K has minimal normal subgroups by Lemma 4.26. Any minimal normal subgroup MK/K of G/K has an automorphism α of order 3 with $C_{MK/K}(\alpha)$ has finite rank.

   Denote G/K by G_0 and MK/K by M_0. Since M_0 is a minimal normal subgroup of a locally finite group, by [**?**, 3.3.15 (ii)] it is isomorphic to a product of the same simple locally finite groups H, that is, M_0 is isomorphic to $\prod_{i \in I} H$. If I is infinite, $C_{M_0}(\alpha)$ contains an infinite product of the group H, which has infinite rank. Hence, $M_0 \cong \prod_{i=1}^{m} H$ for some $m \in \mathbb{N}$. By Corollary **??**, H is isomorphic to one of $PSL_2(k), PSL_3(k)$ or $PSU_3(k)$ where k is a locally finite field of characteristic $q \neq 3$.

□

CHAPTER 5

Locally Finite Groups with Abelian Subgroups Whose Centralizers Are Small, *with M. Kuzucuoğlu and P. Shumyatsky*

5.1 Introduction

A group is locally finite if every finite subset of the group generates a finite subgroup. In the theory of locally finite groups centralizers play an important role. In particular the following family of problems has attracted great deal of attention in the past. Let G be a locally finite group containing a finite subgroup A such that $C_G(A)$ is small in some sense. What can be said about the structure of G? In some situations quite significant information about G can be deduced. For example if $|A| = 2$ and $C_G(A)$ is finite, then G has a nilpotent subgroup of class at most two with finite index bounded by a function of $|C_G(A)|$ [HM80]. If G contains an element of prime order p whose centralizer is finite of order m, then G contains a nilpotent subgroup of finite (m, p)-bounded index and p-bounded nilpotency class. This result for locally nilpotent periodic groups is due to Khukhro [Khu90] while the reduction to the nilpotent case was obtained combining a result of Hartley and Meixner [HM81] with that of Fong [Fon76]. The latter uses the classification of finite simple groups. Another important result in this direction is Hartley's theorem that if G has an element of order n with finite centralizer of order m, then G contains a locally soluble subgroup with finite (m, n)-bounded index [Har92]. The interested reader should consult two excellent survey articles due to Hartley [Har95, **?**] and the paper [BH] due to Belyaev and Hartley for the comprehensive description of the developments in this area in the twentieth century.

Recall that a group G is Chernikov if it has a subgroup of finite index that is a direct product of finitely many groups of type C_{p^∞} for various primes p (quasi-cyclic p-groups, or Prüfer p-groups). By a deep result obtained independently by Shunkov [Sun70] and Kegel and Wehrfritz [KW70] Chernikov groups are precisely the locally finite groups satisfying the minimal condition on subgroups, that is, any non-empty set of subgroups possesses a minimal subgroup. In the literature there are many results on Chernikov centralizers in locally finite groups. By and large, they resemble the corresponding results on finite centralizers. In particular, Hartley proved in [Har88] that if a locally finite group contains an element of prime-power order with Chernikov centralizer, then it is almost locally soluble. A group is said to almost have certain property if it contains a subgroup of finite index with that property.

Infinite locally finite groups containing a non-cyclic subgroup with finite centralizer can be simple. One example is provided by the group $PSL(2, k)$, where k is an infinite locally finite field of odd characteristic. This group contains a non-cyclic subgroup of order four with finite centralizer. In [Shu01] the third author proved that if a locally finite group G contains a non-cyclic subgroup A of order p^2 for a prime p such that $C_G(A)$ is finite and $C_G(a)$ has finite exponent for all $a \in A^\#$, then G is almost locally soluble and has finite exponent. Here the symbol $A^\#$ stands for the set of the nontrivial elements of A.

If G and T are groups, we say that G involves T if there are subgroups $K \leq H \leq G$, with K normal in H, such that $H/K \cong T$. The main purpose of the present article is to prove the following theorem.

Theorem 5.1. *Let p be a prime and G a locally finite group containing an elementary abelian p-subgroup A of rank at least 3 such that $C_G(A)$ is Chernikov and $C_G(a)$ involves no infinite simple groups for any $a \in A^\#$. Then G is almost locally soluble.*

In view of the aforementioned result of Hartley the theorem remains valid also in the case where A is of prime order. On the other hand, the theorem is no longer valid if we allow A to be of rank 2. In particular, this is illustrated by the example of the group $PSL_2(k)$. More precisely, we will establish the following characterization of the groups $PSL_p(k)$.

Theorem 5.2. *An infinite simple locally finite group G admits an elementary abelian p-group of automorphisms A such that $C_G(A)$ is Chernikov and $C_G(a)$ involves no infinite simple groups for any $a \in A^{\#}$ if and only if G is isomorphic to $PSL_p(k)$ for some locally finite field k of characteristic different from p and A has order p^2.*

Of course, this implies that if G is a simple locally finite group acted on by an elementary abelian group A in such a way that GA satisfies the hypothesis of Theorem 5.1, then G is finite. This provides the main tool for the proof of Theorem 5.1. In turn, the proof of Theorem 5.2 uses a number of sophisticated tools. In particular, it depends on the classification of finite simple groups and the classification of periodic linear simple groups. The latter was obtained independently in [Bel84, Bor83, HS84, Tho]. The result says that an infinite periodic linear simple group is of Lie type over some locally finite field. It seems unlikely that one could prove Theorem 5.1 without using this.

5.2 Proof of Theorem 5.2

If π is a set of primes, we denote by $O_\pi(G)$ the maximal normal π-subgroup of a group G and by $O^{\pi'}(G)$ the subgroup generated by all π-elements. Recall that a group satisfies min-p if every descending chain of p-subgroups has only finitely many members.

The next lemma is well-known, see for example [Shu07, Proposition 3.6].

Lemma 5.3. *Let G be a locally finite group and let A be a finite p-group of automorphisms of G such that $C_G(A)$ satisfies min-p. Then G satisfies min-p, too.*

The following short lemma reduces the study of simple locally finite groups satisfying our assumptions to the simple groups of Lie type over locally finite fields.

Lemma 5.4. *Let G be an infinite simple locally finite group admitting an elementary abelian p-group A of automorphisms such that $C_G(A)$ is a Chernikov group. Then G is simple of Lie type over a locally finite field of characteristic $q \neq p$.*

Proof. Since $C_G(A)$ is Chernikov, it satisfies min-p. Thus by Lemma **??**, G satisfies min-p too. Theorem B of [HS84] tells us that a locally finite simple group satisfying min-p is a group of Lie type over an infinite locally finite field of characteristic $q \neq p$. □

Let G be a simple locally finite group of Lie type. By [HK91, Lemma 4.3], there exists a simple linear algebraic group $\overline{G}$ of adjoint type, a Frobenius map σ on $\overline{G}$ and a sequence of natural numbers $n_i | n_{i+1}$ such that

$$G = \bigcup_{i=1}^{\infty} O^{q'}(\overline{G}_{\sigma^{n_i}}) \tag{$*$}$$

Here $\overline{G}_\sigma$ denotes the subgroup of fixed points of σ in $\overline{G}$. By [**?**, Theorem 30], if α is an automorphism of a simple group G of Lie type over a locally finite field k, then

$$\alpha = g\phi\delta,$$

where g is an element of $\bigcup_{i=1}^{\infty} \overline{G}_{\sigma^{n_i}} = InndiagG$ (an inner-diagonal automorphism), ϕ is induced by an automorphism of the field k, and δ is a graph automorphism.

Lemma 5.5. *Let G be a simple group of Lie type over an infinite locally finite field of characteristic q and let a be an automorphism of G of prime order $p \neq q$. Assume that $C_G(a)$ involves no infinite simple groups. Then a is an inner-diagonal automorphism.*

Proof. By (*), we have $G = \bigcup_{i \in \mathbb{N}} G_i$ where $G_i = O^{q'}(\overline{G}_{\sigma^{n_i}})$ and $\overline{G}$ is a simple linear algebraic group of adjoint type, σ is a Frobenius map on $\overline{G}$ and $n_i | n_{i+1}$ are natural numbers. Observe that each G_i is a finite simple group of Lie type.

Hartley's argument in the proof of Theorem C of [Har92] shows that if necessary by passing to a subsequence of G_i, one may assume each G_i is a-invariant. Assume that a is not an inner-diagonal automorphism. By [Har92, Lemma 3.1], each $C_{G_i}(a)$ involves a non-abelian simple group. Therefore, their union $C_G(a)$

is nonsoluble. By [Har92, Theorem 4.4], there exists an infinite reductive algebraic group H and a Frobenius map ψ of $\overline{G}$ which leaves H invariant, such that, $C_{\overline{G}_\sigma}(a) = H_\psi$, and hence $C_G(a) = \bigcup_{i\in\mathbb{N}} H_{\psi^{n_i}}$.

Since H is a reductive group, by [Hum75, p. 168], H^0 is isomorphic to a commuting product of a torus and finitely many simple linear algebraic groups. We write $H^0 = TS_1S_2\dots S_m$ for some $m \geq 1$. Here, the group T is a central torus and each S_i is a simple linear algebraic group. Now, ψ acts on this product. Let $S = S_1$.

Take an orbit $S, S^\psi, S^{\psi^2} \dots S^{\psi^{k-1}}$. Let $M = S \times S^\psi \times S^{\psi^2} \dots \times S^{\psi^{k-1}}$ in H^0. Observe that $H_\psi \geq M_\psi \cong S_{\psi^k}$.

Therefore, S_{ψ^k} embeds in $H_\psi = C_{\overline{G}_\sigma}(a)$. In particular, for every n_i one may assume $S_{\sigma^{kn_i}} \leq C_{\overline{G}_{\sigma^{n_i}}}(a)$ and since $n_i | n_{i+1}$, we have

$$\bigcup_{i\in\mathbb{N}} S_{\psi^{kn_i}} \leq \bigcup_{i\in\mathbb{N}} C_{\overline{G}_{\sigma^{n_i}}}(a) \leq C_Q(a)$$

where $Q = \bigcup_{i\in\mathbb{N}} \overline{G}_{\sigma^{n_i}}$. Recall that $G = O^{q'}(Q)$. Consider the union $K = \bigcup_{i\in\mathbb{N}} S_{\psi^{kn_i}}$ and set $K_0 = O^{q'}(K)$. Since $K \leq Q$, it follows that $K_0 \leq O^{q'}(Q) = G$. Hence $K_0 \leq G \cap C_Q(a)$, and so $K_0/Z(K_0)$ is an infinite simple group involved in $C_G(a)$. This contradiction shows that a is inner diagonal. □

Hence, we restrict our attention to inner-diagonal automorphisms. Recall that an element $x \in G$ is called a semisimple element if $(|x|, q) = 1$. An automorphism $\alpha \in Aut(G)$ is called a semisimple automorphism if $(|\alpha|, q) = 1$. Observe that under hypothesis of Lemma 5.4 the elements of A induce semisimple, inner-diagonal automorphisms on G.

Lemma 5.6. *Let $\overline{G}$ be a simple linear algebraic group over an algebraically closed field of positive characteristic and let x be a semisimple element of $\overline{G}$. Then $C_{\overline{G}}(x)$ involves no infinite simple linear algebraic groups if and only if $C_{\overline{G}}(x)$ is metabelian.*

Proof. Assume that $C_{\overline{G}}(x)$ involves no simple linear algebraic groups, By [SS, II, Theorem 4.1] the connected component $H = (C_{\overline{G}}(x))^0$ is reductive. A connected reductive group is isomorphic to a commuting product of a torus with finitely many simple linear algebraic groups (see for example [Hum75, p. 168]). Next, we deduce from [MT, Corollary 8.22] that H' is perfect. Therefore, either H' is a product of simple linear algebraic groups or $H' = 1$. If H' is a product of simple linear algebraic groups, then $C_{\overline{G}}(x)$ involves a simple linear algebraic group by the argument in Lemma 5.5. Hence $H' = 1$ and H is a torus. On the other hand, by [SS, 4.4 Corollary], $C_{\overline{G}}(x)/H$ is abelian. So $C_{\overline{G}}(x)$ is metabelian.

Clearly, if $C_{\overline{G}}(x)$ is metabelian, it does not involve a non-abelian simple group. $\square$

Recall that torsion primes of linear algebraic groups are defined as follows. For type A_l, these are the primes that divide $l+1$. For types B_l, C_l, D_l, G_2 the prime is 2. For types E_6, E_7, F_4 the primes are 2 and 3, and for type E_8 the primes are $2, 3, 5$ (see [Ste75]).

Proposition 5.7. *Let G be a simple group of Lie type over an infinite locally finite field k of characteristic q and let A be an elementary abelian p-group of inner-diagonal automorphisms where $p \neq q$. Then the following are equivalent:*

1. *$C_G(A)$ is finite.*
2. *$C_G(A)$ is Chernikov*
3. *$C_G(A)$ does not contain a torus of G.*

Proof. Obviously (1) implies (2). Let us show that (2) implies (3). Assume that $C_G(A)$ contains a torus T. By Hartley's argument in [Har92, Theorem C], T contains elements of infinitely many distinct prime orders, hence T cannot be Chernikov. Finally, assume that $C_G(A)$ is infinite. Let $\overline{G}$ be the corresponding simple linear algebraic group, where G is written as in (*). Clearly $C_{\overline{G}}(A)$ is infinite. By [Ste75, 2.18 Corollary] the identity component $C_{\overline{G}}(A)^0$ is an infinite connected reductive group. By definition, it contains a torus $\overline{T}$ of $\overline{G}$, hence $C_G(A)$ contains $T = \overline{T} \cap G$, which is a torus of G. $\square$

We are now ready to embark on the proof of the main result of this section.

Proof of Theorem 5.2 Suppose that G admits an elementary abelian p-group A of automorphisms such that the hypotheses are satisfied. By Lemma 5.4 G is of Lie type.

Write

$$G = \bigcup_{i=1}^{\infty} O^{q'}(\overline{G}_{\sigma^{n_i}})$$

where $\overline{G}$ is the corresponding simple linear algebraic group of adjoint type and σ is a Frobenius map on $\overline{G}$. By Lemma 5.5 A is an elementary abelian p-subgroup of semisimple, inner-diagonal automorphisms of G.

Assume that for some $a \in A^{\#}$ the centralizer $C_{\overline{G}}(a)$ involves an infinite simple linear algebraic group $\overline{H}$. By definition of a Frobenius map, some power of σ is a standard Frobenius map. Therefore, there exists $k \in \mathbb{N}$ such that $\overline{H}$ is σ^k-invariant. Let $\sigma^k = \psi$.

Now, one can show as in Lemma 5.5 that $C_G(a)$ involves an infinite simple locally finite group. Hence we get a contradiction. In view of Lemma **??**, we conclude that $C_{\overline{G}}(a)$ is metabelian.

Obviously, $A \leq C_G(A)$. Since $C_G(A)$ is a Chernikov group, it does not contain an infinite elemantary abelian subgroup, so A is finite, say $|A| = p^r$. By [Har92, Theorem C], if A is cyclic, then $C_G(A)$ is not Chernikov. Hence, $r \geq 2$.

If p is not a torsion prime, then by [Ste75, Theorem 2.28] A is contained in a maximal torus and by [SS, Lemma 5.9], indeed A is contained in a σ-invariant maximal torus, say T of $\overline{G}$. In that case, $T_0 = \bigcup T_{\sigma^{n_i}}$ is a maximal torus of G which is contained in $C_G(A)$. By Proposition 5.7, $C_G(A)$ is not Chernikov.

Hence, p must be a torsion prime.

Case 1. Suppose that $\overline{G}$ is of type A_l. Then the torsion primes are the primes dividing $l+1$. Consider $a \in A^{\#}$. If a is not a regular semisimple element, then by [SS, Theorem 4.1] $C_{\overline{G}}(a)$ involves an infinite simple group. By [Har92, Theorem 4.4], there exists an infinite reductive algebraic group H and a Frobenius map ψ on H such that $C_{\overline{G}_\sigma}(a) = H_\psi$. Hence, $C_G(a) = \bigcup_{i=1}^{\infty} O^{q'}(H_{\psi^{k_i}})$ for some $k_i | k_{i+1}$. Since $C_G(a)$ involves no infinite simple groups, H, being reductive, must be a torus. Since $A \leq C_G(a)$, one has $A \leq H$, so $C_G(A)$ contains $\bigcup\limits_{i=1}^{\infty} O^{q'}(H_{\psi^{k_i}})$ which is a torus. In view of Proposition 5.7, this contradicts the hypothesis that $C_G(A)$ is Chernikov.

Consequently a must be a regular semisimple element. Hence, the characteristic polynomial and the minimal polynomials of a are equal, and all eigenvalues of a are distinct.

Since $|a| = p$, we conclude that $p = l + 1$. Hence, $\overline{G} \cong PGL_p(\overline{k})$ and $G \cong PSL_p(k)$ or $PSU_p(k)$ depending on the Frobenius map σ, where k is a locally finite field of characteristic $q \neq p$. Here, as usual, $\overline{k}$ is the algebraic closure of k.

Since a is a regular semisimple element, by [SS, Chapter III, Corollary 1.7] a is contained in a unique maximal torus T of $\overline{G}$ and $C_{\overline{G}}(a)^0$ is a torus. Therefore, $C_{\overline{G}}(a)^0 = T$. We have $A \leq C_G(a) \leq N_G(T)$. Since $\overline{G}$ is simple, it is connected and so, by [**?**, p.28], we have $C_{\overline{G}}(T) = T$. If $A \leq T$, then $C_{\overline{G}}(A)$ contains the torus T. In this case, $\bigcup_{i=1}^{\infty} T_{\sigma^{n_i}} \leq C_G(A)$. Again, in view of Proposition 5.7, this is a contradiction with the hypothesis that $C_G(A)$ is Chernikov. Therefore, $A \backslash T$ must be non-empty. Recall that $N_{\bar{G}}(T)/T$ is isomorphic to the Weyl group of $PGL_p(\overline{k})$, which is isomorphic to the symmetric group S_p. Since a Sylow p-subgroup of S_p is cyclic, $A/(A \cap T)$ is cyclic. Let $y \in A \backslash T$. If we prove that $C_T(y)$ is also of order p, then we conclude that $|A| = p^2$.

Recall that, $\overline{G} = PGL_p(\overline{k}) \cong PSL_p(\overline{k})$. It is sufficient to show that if S is a Sylow p-subgroup of the Weyl group W, then $|C_T(S)| = p$.

Since all maximal tori are conjugate, we may assume without loss of generality that T consists of the matrices

$$\begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \lambda_p \end{pmatrix} Z$$

where $Z = Z(SL_p(\overline{k}))$ (scalar matrices) and λ_i are elements of $\overline{k}$ such that $\prod_{k=1}^{p} \lambda_k = 1$. Choose a generator w of S which corresponds to the cycle $(1\,2\,3 \ldots p) \in S_p$.

Let $s \in C_T(w)$. Then s can be chosen of the form $s = \begin{pmatrix} s_1 & & & & \\ & s_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & s_p \end{pmatrix} Z$

with $s_p = 1$.

We have

$$s = s^w = (\begin{pmatrix} s_1 & & & & \\ & s_2 & & & \\ & & . & & \\ & & & . & \\ & & & & s_p \end{pmatrix} Z)^w = \begin{pmatrix} s_2 & & & & \\ & s_3 & & & \\ & & . & & \\ & & & s_p & \\ & & & & s_1 \end{pmatrix} Z.$$

Hence, $1 = s_p = s_1 z = s_2 z^2 = \ldots = s_{p-1} z^{p-1}$ for some $z \in Z$. So, to each $z \in Z$ corresponds a unique element in $C_T(w)$. Therefore $|C_T(w)| = |Z| = p$, as required. So $|C_{\overline{G}}(A)| \leq p^2$. Hence $|A| = p^2$.

Now, $\overline{G}$ has type A_{p-1} and G is either $PSL_p(k)$ or $PSU_p(k)$, depending on the Frobenius map σ. Let us show that G is necessarily isomorphic to $PSL_p(k)$.

Suppose first that the simple locally finite group G is twisted of type ${}^2A_{p-1}$ with p a prime greater than 3. Write $p - 1 = 2m$ for some $m \geq 2$. By [**?**, 13.3.8] the Dynkin diagram has type B_m and by [**?**, 3.6] the Weyl group has order $2^m m!$. Therefore the Weyl group has no elements of order p. It follows that A cannot be embedded in the group of inner-diagonal automorphisms of $PSU_p(k)$.

Suppose $p = 3$. By [**?**, 13.3.1] the Weyl group of $PSU_3(k)$ is cyclic of order 2, and so it has no elements of order 3.

Finally, assume that $p = 2$. Remark that for any prime power q, the group $PSU_2(q^2)$ is isomorphic to $PSL_2(q)$. Therefore, $PSU_2(k) \cong PSL_2(k_0)$ for some locally finite field k_0 with $[k : k_0] = 2$. Thus, we have shown that $G \cong PSL_p(k)$ whenever $\overline{G}$ is of type A_l.

Case 2. Let $\overline{G}$ be a simple group of Lie type different from A_l. Since p is a torsion prime, p is either 2 or 3 or 5.

Assume that $p = 2$. Recall that G has a local system consisting of finite simple groups G_i of the same type with G, defined over finite fields $\mathbb{F}_{q^{n_i}}$ where $n_i | n_{i+1}$. Any element $a \in A^{\#}$ is an involutory automorphism of G and one can pass to an a-invariant subsequence of G_i, so that a becomes an involutory automorphism of each of them. Table 4.3.1 of [GLS97] shows that the centralizer of any involutory automorphism of a finite simple group of Lie type different from type A_1 involves a simple group of Lie type. More precisely, arguing as in the proof of Lemma **??**, each of $C_{G_i}(a)$ involves a finite non-abelian simple group H_i such that $H_i \leq H_{i+1}$. Again $\bigcup H_i$ is an infinite simple group involved in $C_G(a)$. Thus, $p \neq 2$.

Assume that $p = 3$. Then $\overline{G}$ is of type F_4, E_6, E_7 or E_8. The finite simple groups of these types have Schur multipliers whose orders are relatively prime with 3, so $A \leq G$. In [Aza79] Azad classified the centralizers of semisimple elements of order 3 in finite Chevalley groups. According to his work, the centralizer of the element of order 3 of a group of type F_4, E_6, E_7 or E_8 or possible twisted versions of these groups always contains a finite simple group. Recall that $G = \bigcup_{i=1}^{\infty} G_i$ and we can assume that all subgroups G_i are all of the same type in one of the above families. As above, one can observe that the centralizer $C_{G_i}(a)$ contains a simple subgroup H_i with $H_i \leq H_{i+1}$. The union $\bigcup H_i$ is an infinite simple group involved in $C_G(a)$ and we obtain a contradiction.

Finally, assume that $p = 5$. Then $\overline{G}$ has type E_8. There is just one conjugacy class of elements of order 5 in E_8, and if $a \in E_8$ has order 5 then $C_{\overline{G}}(a)$ is a reductive group of type A_4A_4 (see [Lie94, Theorem 1.17] and the example in [Lie94, Section 1.5]). Hence, $C_G(a)$ involves an infinite simple group. This shows that under our hypothesis $G \cong PSL_p(k)$ and $|A| = p^2$.

On the other hand, the proof shows that the group $PSL_p(k)$ really contains the subgroup A with required properties.

5.3 The main theorem

The first Lemma of this section is immediate from [KW73, Theorem 3.17].

Lemma 5.8. *Let G be a periodic almost locally soluble group satisfying min-p. Then $G/O_{p'}(G)$ is Chernikov.*

The following lemma is taken from Hartley [Har82].

Lemma 5.9. *Let A be a finite π-group of automorphisms of a locally finite group G and let N be a normal A-invariant subgroup of G.*

1. *If N is a π'-subgroup, then $C_{G/N}(A) = C_G(A)N/N$.*

2. If $N/O_{\pi'}(N)$ is Chernikov, then $|C_{G/N}(A) : C_G(A)N/N| < \infty$.

The following theorem can be obtained without using the classification of finite simple groups. The case where A is cyclic of course follows from the famous theorem of J. Thompson [**?**]) and the case where A has order p^2 is due to P. Martineau [Mar72]. Finally, the case where $|A| \geq p^3$ can be easily deduced from results on soluble signalizer functors (see [Gla76], [Gold72], [Ben75]). Deduction of Theorem 5.10 from the classification of finite simple groups is easy in view of the well-known fact that a coprime group of automorphisms of a finite simple group is cyclic (cf. [GS01, Lemma 2.7]).

Theorem 5.10. *Let G be a finite group admitting an elementary abelian p-group of automorphisms A such that $C_G(A) = 1$. Then G is soluble.*

The minimal subgroup of finite index of a Chernikov group T is called the radicable part of T. Suppose the radicable part of T has index i and is a direct product of precisely j groups of type C_{p^∞} (for various primes p). The ordered pair (j, i) is called the size of T. The set of all pairs (j, i) is endowed with the lexicographic order. It is easy to check that if H is a proper subgroup of T, the size of H is necessarily strictly less than that of T. This observation will enable us to use induction on the size of a Chernikov subgroup.

We are now ready to prove the main result of the article.

Proof of Theorem 5.1. Recall that G is a locally finite group containing an elementary abelian p-subgroup A of rank at least 3 such that $C_G(A)$ is Chernikov and $C_G(a)$ involves no infinite simple groups for any $a \in A^{\#}$. We wish to prove that G is almost locally soluble. Let R be the product of all normal locally soluble subgroups in G. Combining Lemmas 5.3 and 5.8 we deduce that $R/O_{p'}(R)$ is Chernikov. Lemma 5.9 now implies that the quotient-group G/R satisfies the hypothesis of the theorem. Thus, without loss of generality we can additionally assume that $R = 1$. By induction on the size of $C_G(A)$ we will show that with this additional assumption G is finite.

It follows from Theorem 5.10 that $C_X(A) \neq 1$ for every normal subgroup $X \leq G$. Since $C_G(A)$ is Chernikov, we deduce that G possesses minimal normal subgroups. Let N be a minimal normal subgroup. Thus, N is a direct product of isomorphic simple groups (see [Rob95, 3.3.15]). Write $N = S_1 \times S_2 \times \ldots$, where each S_i is simple. The subgroup A permutes the simple factors in N.

Suppose that S_1 is infinite. If $a \in A$ does not normalize some S_i, then a centralizes the diagonal of the direct product $S_i \times S_i^a \times \ldots \times S_i^{a^{p-1}}$. Since $C_G(a)$ does not contain infinite simple subgroups, we obtain a contradiction. Therefore A normalizes each of the subgroups S_i. This however leads to a contradiction of Theorem **??**.

Hence, S_1 is finite. It follows that N has finite exponent and, taking into account that $R = 1$, we invoke Theorem 1.2 of [Shu01] and deduce that N is finite. It follows that $C_G(N)$ has finite index. Let $C = C_N(A)$ and observe that because of Theorem **??** $C \neq 1$. We further observe that $C \cap C_G(N) = 1$ because $R = 1$. It follows that $C_G(N) \cap C_G(A)$ is a proper subgroup in $C_G(A)$. Thus, by induction, $C_G(N)$ is finite and so is G. The proof is now complete.

□

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CHAPTER 6

Centralizers of p-Subgroups in Simple Locally Finite Groups

6.1 Introduction

In [EKS] we have proved the following result:

Theorem 6.1. *[EKS, Theorem 1.1] Let p be a prime and G a locally finite group containing an elementary abelian p-subgroup A of rank at least 3 such that $C_G(A)$ is Chernikov and $C_G(a)$ involves no infinite simple groups for any $a \in A^{\#}$. Then G is almost locally soluble.*

To prove Theorem 6.1, we gave the following characterization of $PSL_p(k)$ where $char k \neq p$.

Theorem 6.2. *[EKS, Theorem 1.2] An infinite simple locally finite group G admits an elementary abelian p-group of automorphisms A such that $C_G(A)$ is Chernikov and $C_G(a)$ involves no infinite simple groups for any $a \in A^{\#}$ if and only if G is isomorphic to $PSL_p(k)$ for some locally finite field k of characteristic different from p and A has order p^2.*

In this paper, we will improve Theorem 6.2. Indeed, we will prove a similar result without assuming A is elementary abelian, but instead we prove for any subgroup of exponent p.

Theorem 6.3. *Let G be an infinite simple locally finite group, P a subgroup of automorphisms of exponent p such that*

1. *$C_G(P)$ is Chernikov,*

2. *For every $\alpha \in P \backslash \{1\}$, the set of fixed points $C_G(\alpha)$ does not involve an infinite simple group.*

Then $G \cong PSL_p(k)$ where k is an infinite locally finite field of characteristic p and P has a subgroup Q of order p^2 such that $C_G(P) = C_G(Q) = Q$.

6.2 Preliminaries

Let us recall some definitions of the concepts mentioned in the theorems. First, consider $C_{p^n} = \{x \in \mathbb{C} : \ x^{p^n} = 1\}$. Here $(C_{p^n}, .)$ defines a group isomorphic to a cyclic group of order p^n. Observe that if $m|n$ then $C_{p^m} \leq C_{p^n}$, and with the inclusion maps these sets form a direct system, where the direct limit

$$\lim_{n \in \mathbb{N}} C_{p^n}$$

is denoted by C_{p^∞}, consists of all complex p^n-th roots of unity, and forms a group under complex multiplication. This group is called the quasi-cylic p-group.

Definition 6.4. *A group is called a Chernikov group if it is a finite extension of a direct product of finitely many copies of some quasi-cyclic p_i-groups, for possibly distinct primes p_i.*

Definition 6.5. *Let χ be a group-theoretical property. If a group G has a normal subgroup of finite index satisfying χ, then G is called almost χ.*

Definition 6.6. *Let G and H be two groups. If G has a normal subgroup K such that G/K has a subgroup isomorphic to H then G is said to involve a subgroup isomorphic to H.*

Definition 6.7. *A group satisfies the minimal condition, namely min, if any non-empty set of subgroups has a minimal subgroup. A group satisfies min-p if any non-empty set of p-subgroups, has a minimal subgroup.*

Kegel-Wehrfritz and Sunkov proved independently that a locally finite group satisfying minimal condition is a Chernikov group (see [KW70] and [**?**]). For detailed discussion of groups satisfying *min* and *min-p*, see [KW73].

6.3 Main Results

First, we need the following proposition:

Proposition 6.8. *Let $\overline{G}$ be a simple linear algebraic group of adjoint type over the algebraic closure of $\mathbb{F}_q$, let $g \in \overline{G}$ be an element of prime order $p \neq q$ such that $C_{\overline{G}}(g)$ is a non-abelian group which does not involve any infinite simple groups. Then*

(i) The identity component $C_{\overline{G}}(g)^0$ of the centralizer of g in $\overline{G}$ is a maximal torus of $\overline{G}$,

(ii) $\overline{G} \cong PGL_p(\overline{\mathbb{F}_q})$.

Proof. Since $\overline{G}$ is a simple linear algebraic group of adjoint type over the algebraic closure of $\mathbb{F}_q$ and $g \in G$ a semisimple element, g is contained in a maximal torus T of $\overline{G}$. By [MT, Proposition 14.1, 14.2], $C_{\overline{G}}(g)^0$ is connected reductive, containing a maximal torus T, and involving no infinite simple groups. Hence, $C_{\overline{G}}(g)^0 = T$. By [MT, Proposition 14.20], the exponent of $C_{\overline{G}}(g)/C_{\overline{G}}(g)^0$ divides p, hence either $C_{\overline{G}}(g)$ is connected, and hence a torus, or $C_{\overline{G}}(g)/C_{\overline{G}}(g)^0$ is a finite group of exponent p.

Since $C_{\overline{G}}(g)$ is not abelian, one has $C_{\overline{G}}(g)$ a finite extension of an abelian group T, so it has finite rank. Recall that an infinite group G is said to have finite rank r if every finitely generated subgroup is r-generated. In [EG, Theorem 1.8] we have shown that when a simple linear algebraic group $\overline{G}$ over the algebraic closure of $\mathbb{F}_q$ has an element g of order p with $C_{\overline{G}}(z)$ has finite rank, then one of the following cases occur:

1. $\overline{G}$ is of type A_l and $p > l$

2. $\overline{G}$ is of type B_l, C_l and $p > 2l - 1$

3. $\overline{G}$ has type D_l and $p > 2l - 3$

4. $\overline{G}$ is isomorphic to one of E_6, E_7, E_8, F_4 or G_2 and $p > 11, 17, 29, 17$ or 5 respectively.

On the other hand, since $C_{\overline{G}}(g)/C_{\overline{G}}(g)^0$ has exponent p, by [SS, 4.4. Corollary] and [MT, Proposition 14.20], we get p is a torsion prime. The list of torsion primes of linear algebraic groups are defined as follows: For type A_l, these are the primes that divide $l+1$. For types B_l, C_l, D_l, G_2 the prime is 2.EKS For types E_6, E_7, F_4 the primes are 2 and 3, and for type E_8 the primes are $2, 3, 5$ (see [Ste75]).

Hence, one deduce that the only possible case that may occur is $\overline{G}$ has type A_{p-1}, indeed $\overline{G} \cong PGL_p(\overline{\mathbb{F}_q})$. EG □

Theorem 6.9. *Let G be an infinite simple locally finite group with a finite non-abelian p-group of automorphisms P such that*

1. *$C_G(P)$ is Chernikov,*
2. *For every $\alpha \in P\backslash\{1\}$ the set of fixed points $C_G(\alpha)$ does not involve an infinite simple group*

Then G is isomorphic to $PSL_p(k)$ where k is a locally finite field of characteristic $q \neq p$ and P is metabelian.

Proof. Since P is a finite p-group and $C_G(P)$ satisfies *min-p*, by [EKS, Lemma 2.1], G satisfies *min-p*. Then, by [**?**, Theorem B], G is a simple group of Lie type over a locally finite field k of characteristic q. Now assume that $q = p$. Clearly G contains a root subgroup, which is an infinite elementary abelian p-subgroup. Hence G can not satisfy *min-p*. Hence, $q \neq p$, that is, G is isomorphic to a simple group of Lie type over an infinite locally finite field of characteristic $q \neq p$.

Now, by [HK91, Lemma 4.3], there exists a simple linear algebraic group $\overline{G}$ of adjoint type, a Frobenius map σ on $\overline{G}$ and a sequence of natural numbers $n_i | n_{i+1}$ such that

$$G = \bigcup_{i \in \mathbb{N}} O^{p'}(\overline{G}_{\sigma^{n_i}}).$$

By assumption, the centralizer of any non-identity element does not involve an infinite simple group, so [**?**, Lemma 2.3] implies that P consists of inner-diagonal automorphisms of G. Hence, $P \leq \bigcup_{i \in \mathbb{N}} \overline{G}_{\sigma^{n_i}}$. Therefore, $P \leq \overline{G}_{\sigma^{n_j}}$ for some $j \in \mathbb{N}$.

Choose $1 \neq z \in Z(P)$. Clearly, $P \leq C_{\overline{G}}(z)$. Now, $C_G(z) = \bigcup_{i \in \mathbb{N}} O^{p'}(C_{\overline{G}}(z)_{\sigma^{n_i}})$.

By assumption, $C_G(z)$ does not involve an infinite simple group. Now, suppose that $C_{\overline{G}}(z)$ involves an simple linear group algebraic group H. Consider the union of fixed points of σ^{n_i} on H, denote $H_i = H_{\sigma^{n_i}}$. Clearly $H_i \leq H_{i+1}$ and infinitely many of H involves finite simple groups such that their union form an infinite locally finite simple group. Hence, we get a contradiction and we deduce $C_{\overline{G}}(z)$ does not involve a simple linear algebraic group. By [EKS, Lemma 2.4], $C_{\overline{G}}(z)$ is metabelian. Hence, P is metabelian. On the other hand, since P is not abelian, $C_{\overline{G}}(z)$ is not abelian.

By Proposition 6.8, $\overline{G}$ is isomorphic to $PGL_p(\overline{\mathbb{F}_q})$. Hence G is isomorphic to either $PSL_p(k)$ or $PSU_p(k)$. Following the argument in the proof of Theorem 6.2 in [EKS], since the Weyl group of $PSU_p(k)$ has no elements of order p, and PT/T embeds in the Weyl group, $PSU_p(k)$ has no such non-abelian subgroup P. Therefore, $G \cong PSL_p(k)$ where k is an infinite locally finite field of characteristic $q \neq p$. □

Then, we prove the main result of the paper:EKS-testermanmalle

Proof of Theorem 6.3. Assume first that P is abelian. Then by Theorem 6.2, the result follows with $|P| = p^2$.

Now, assume P is non-abelian. By Theorem **??**, $G \cong PSL_p(k)$ where k is a locally finite field of characteristic $q \neq p$. Let $1 \neq z \in Z(P)$, observe that $P \leq C_G(z) \leq C_{\overline{G}}(z)$ where $\overline{G}$ is the corresponding simple linear algebraic group and σ is the Frobenius map such that $G = \bigcup_{i \in \mathbb{N}} O^{p'}(\overline{G}_{\sigma^{n_i}})$, which exist by [HK91, Lemma 4.3]. Denote the maximal torus of $\overline{G}$ containing z by T. By Proposition 6.8(i), $C_{\overline{G}}(z)^0 = T$. Indeed, by [SS, 1.7.Corollary], T is the unique maximal torus containing z. Since P is not abelian $C_{\overline{G}}(z)/C_{\overline{G}}(z)^0$ can not be 1, hence by [MT, Proposition 14.20], it has exponent p. Let y be any element of $C_{\overline{G}}(z) \backslash C_{\overline{G}}(z)^0$. Then $Q = \langle y, z \rangle$ has order p^2. Indeed, $C_{\overline{G}}(z)^0 = T$, and $y \in N_{\overline{G}}(T)$. Hence, y induces an element w of order p in the Weyl group. Now, $z \in C_T(w)$. The computation in the proof of Theorem 6.2 in [EKS] shows that indeed $C_T(w)$ has order p, hence $C_{\overline{G}}(Q) = Q$. This Q is the required subgroup.

□

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CHAPTER 7

On Groups with All Subgroups Subnormal or Soluble of Bounded Derived Length, *with A. Tortora and M. Tota*

7.1 Introduction

A well known result, due to W. Möhres (see [Möh90]), states that a group with all subgroups subnormal is soluble, while a result proved, separately, by C. Casolo (see [Cas01]) and H. Smith (see [Smi01-3]) shows that such a group is nilpotent if it is also torsion-free. Later, Smith generalized these results to groups in which every subgroup is either subnormal or nilpotent. More precisely, he proved, in [Smi01-2], that a locally (soluble-by-finite) group with all subgroups subnormal or nilpotent is soluble, and the same holds for a locally graded group whose non-nilpotent subgroups are subnormal of bounded defect. Also, in both cases, the nilpotence follows if the group is torsion-free (see [S01-1]).

In this paper, we are interested in studying groups with all subgroups subnormal or soluble. We need to restrict out attention to locally graded groups with all subgroups subnormal or soluble, because of the existence of Tarski monsters, constructed by A. Yu. Olshanskii, namely, finitely generated infinite simple groups, whose every proper subgroup is cyclic of prime order (see [Ols91]). The first problem that arises in the locally graded case is the presence of non-soluble locally graded groups in which every proper subgroup is soluble. In fact, the finite minimal simple groups are non-abelian simple groups with this property.

They have been completely classified by J. G. Thompson ([Th68]). Using this classification, in Section 2, we get all the finite non-abelian simple groups having each proper subgroup metabelian.

Another difficulty is due to infinite locally graded groups with all proper subgroups soluble. Such groups are both hyperabelian (see [FdGN]) and locally soluble (see [DES]), but it is still an open question whether they are soluble. However, there is a positive answer if we bound the derived length of subgroups (see [DE]). Motivated by this result, we deal with groups whose subgroups are either subnormal or soluble of bounded derived length. In our analysis, almost minimal simple groups show up. These are groups which fit between a minimal simple group and its automorphism group.

In line with the Smith's results ([S01-1, Smi01-2]), our main theorems follow. They will be proved in Section 3.

Theorem 7.1. *Let G be a locally (soluble-by-finite) group and suppose that, for some positive integer d, every subgroup of G is either subnormal or soluble of derived length at most d. Then either*

(i) *G is soluble, or*

(ii) *$G^{(r)}$ is finite for some integer r and G is an extension of a soluble group of derived length at most d by a finite almost minimal simple group.*

Theorem 7.2. *Let G be a locally graded group and suppose that, for some positive integers n and d, every subgroup of G is either subnormal of defect at most n or soluble of derived length at most d. Then either*

(i) *G is soluble of derived length not exceeding a function depending on n and d, or*

(ii) *$G^{(r)}$ is finite for some integer $r = r(n)$ and G is an extension of a soluble group of derived length at most d by a finite almost minimal simple group.*

7.2 Minimal simple groups

In this section we focus on locally graded minimal simple groups. By [FdGN, Lemma 2.4] such groups are necessarily finite and they are known:

Theorem 7.3. *[Th68, Corollary 1] Every finite minimal simple group is isomorphic to one of the following groups:*

(i) $PSL(2, 2^p)$*, where* p *is any prime;*

(ii) $PSL(2, 3^p)$*, where* p *is any odd prime;*

(iii) $PSL(2, p)$*, where* $p > 3$ *is any prime such that* $p^2 + 1 \equiv 0$ *(mod 5);*

(iv) $PSL(3, 3)$*;*

(v) $Sz(2^p)$*, where* p *is any odd prime.*

The table below, that will be useful later, shows the outer automorphisms of a finite minimal simple group M. By [ATLAS, p. xv], $|Out\,(M)| = d \cdot f \cdot g$ where, d is the order of the group of diagonal automorphisms, f is the order of the group of field automorphisms and g is the order of the group of graph automorphisms (modulo field automorphisms). For more details, see [ATLAS, Table 5, p. xvi].

M	d	f	g	$\|Out(M)\|$
$PSL(2, 2^p)$	1	p	1	p
$PSL(2, 3^p)$, $p \geq 3$	2	p	1	$2p$
$PSL(2, p)$, $p > 3$ and $5\|(p^2 + 1)$	2	1	1	2
$PSL(3, 3)$	1	1	2	2
$Sz(2^p)$, $p \geq 3$	1	p	1	p

Table 7.1: [ATLAS] Outer automorphisms of a finite minimal simple group

In light of Theorem 7.3, we now classify all the finite non-abelian simple groups whose proper subgroups are metabelian.

Proposition 7.4. *Let* G *be a finite non-abelian simple group with every proper subgroup metabelian. Then* G *is isomorphic to one of the following groups:*

(i) $PSL(2, 2^p)$*, where* p *is any prime;*

(ii) $PSL(2, 3^p)$*, where* p *is any odd prime;*

(iii) $PSL(2, p)$*, where* $p > 3$ *is any prime such that* $p^2 + 1 \equiv 0$ *(mod 5) and* $p^2 - 1 \not\equiv 0$ *(mod 16).*

Proof. It is enough to analyze each case of Theorem 7.3.

Let q be a power of any prime. By [Suz82, Theorem 6.25], $PSL(2,q)$ contains a non-metabelian soluble subgroup if and only if it has a subgroup isomorphic to S_4, the symmetric group of degree 4. Also, by [**?**, Theorem 6.26], this is equivalent to the condition $q^2 \equiv 1$ $(mod\ 16)$. Hence, if $q = 2^p$, then all subgroups of $PSL(2,2^p)$ are metabelian. Suppose $q = 3^p$, with $p = 2k+1$. Since $9^{2k} \equiv 1$ $(mod\ 16)$, S_4 is never contained in $PSL(2,3^p)$ and therefore all subgroups of $PSL(2,3^p)$ are metabelian. Let $q = p > 3$ with $p^2+1 \equiv 0$ $(mod\ 5)$. If $p^2-1 \not\equiv 0$ $(mod\ 16)$, all subgroups of $PSL(2,p)$ are metabelian.

Now, we have to consider $PSL(3,3)$ and $Sz(2^p)$, $p \geq 3$. But $PSL(3,3)$ has a subgroup isomorphic to $SL(2,3)$, which has derived length 3; so we finish with $Sz(q)$ where $q = 2^p$. By [**?**, Theorem 9], $Sz(q)$ contains a Frobenius group F of order $q^2(q-1)$. Moreover, $Sz(q)$ has only one abelian subgroup of order dividing $q^2(q-1)$, that is cyclic of order $q-1$, and its normalizer is a dihedral group of order $2(q-1)$ (see [Suz62], p. 137). Hence, F is not metabelian. □

Remark 7.5. We can observe that every proper subgroup of a minimal simple group has derived length at most 5. By Theorem 7.3 and Proposition 7.4, we need to consider the following cases:

Let $G = PSL(2,p)$, $p > 3$, $p^2+1 \equiv 0$ $(mod\ 5)$ and $p^2-1 \equiv 0$ $(mod\ 16)$. Then, by [Suz82, Theorems 6.25, 6.26], G has a subgroup isomorphic to S_4 and hence soluble of derived length 3. This is also the unique non-metabelian subgroup of G.

Let $G = PSL(3,3)$ and H be a proper subgroup of G. Since H is soluble, it contains a non-trivial normal elementary abelian subgroup. Thus, by [**?**, Theorem 7.1], one of the following holds:

(1) H has a cyclic normal subgroup of index at most 3;

(2) H has an abelian normal subgroup K such that H/K can be embedded into the symmetric group S_3;

(3) H has a normal elementary abelian 3-subgroup K such that H/K can be embedded into $GL(2,3)$. Now, the derived length of $GL(2,3)$ is 4 and so H has derived length at most 5. Indeed, let $Z = Z(SL(3,3))$ and H be the subgroup of G given by

$$\{\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix} Z \mid a,b,c,d,e,f,g \in \mathbb{F}_3,\ \ (ae-bd)g = 1\}.$$

Then

$$K = \{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} Z \ \mid c, f \in \mathbb{F}_3 \}$$

is an elementary abelian 3-subgroup of H such that $H/K \cong GL(2,3)$.

Therefore, every proper subgroup of $PSL(3,3)$ has derived length at most 5 and $PSL(3,3)$ contains a subgroup of derived length 5.

Let $G = Sz(2^p)$ for $p \geq 3$. Then, by [Wil, Theorem 4.1], any maximal subgroup of G has derived length at most 3.

7.3 Main results

We start with some preliminary lemmas.

Lemma 7.6. *Let H be a subgroup of a group G. If every subgroup containing H is subnormal in G, then $G^{(r)} \leq H$ for some $r \geq 0$. In particular, $r = 0$ if and only if $H = G$.*

Proof. We may assume $H < G$. Then there exists a series from H to G, and by [Möh90, Theorem 7], each factor is soluble. Hence, we have an abelian series from H to G, say $H = H_0 \lhd H_1 \lhd \ldots \lhd H_r = G$. As $G^{(i)} \leq H_{r-i}$, for all $i \geq 0$, we get $G^{(r)} \leq H$. □

Remark 7.7. In the previous lemma, if we also assume that the subgroups containing H are subnormal of defect at most n in G, then by [LR, 12.2.8], r depends on n. See also [Cas86].

Lemma 7.8. *Let G be a locally graded group with all subgroups subnormal or soluble, and suppose that N is a minimal non-soluble normal subgroup of G.*

(*i*) *If N is infinite, then G is hyperabelian.*

(*ii*) *If N is finite, then G is an extension of a soluble group by a finite almost minimal simple group.*

Proof. First, notice that every subgroup of G/N is subnormal, so that G/N is soluble, by [Möh90, Theorem 7].

(*i*) By [**?**, Lemma 2.4], N is hyperabelian. Let $K = \langle K_i : K_i < N,\ K_i \lhd G \rangle$. If $K = N$, since each K_i is soluble, N has a G-invariant ascending abelian series. Hence G is hyperabelian, being G/N a soluble group. Now, consider $K < N$ and assume by a contradiction that G is not hyperabelian. Then K is soluble and so by

[LMS, Corollary] G/K is locally graded. Moreover, G/K is not hyperabelian and its normal subgroup N/K is minimal non-soluble. However, N/K is hyperabelian and thus N/K is also infinite. We can therefore restrict to the case $K = 1$.

Let A be a non-trivial normal abelian subgroup of N. Then $N = A^G$, so that N is the product of normal abelian subgroups. Then, by [LR, 12.2.2] N is locally nilpotent. If T is its torsion subgroup, we have either $T = 1$ or $T = N$. Let $T = 1$. As solubility is a countably recognizable property, we have that N is countable. It is also locally nilpotent and torsion-free. Then, by [Möh89, Lemma 2], there exists $M < N$ such that the isolator $I_N(M)$ equals N. Also, $I_N(M)^{(i)} \leq I_N(M^{(i)})$ for all $i \geq 0$ (see, for instance, [LR, 2.3.9]). As M is soluble, so is N, a contradiction. Assume $T = N$. Then N is a locally finite, p-group. Clearly $Z(N) = 1$ and so we may apply [ASS, Lemma 2.1] to N. It follows that, there exists $m > 0$, such that $R = \langle Z(H) : H \lhd N, d(H) > m \rangle$ is a proper subgroup of N, where $d(H)$ denotes the derived length of H. On the other hand, N has a finitely generated soluble subgroup of length greater than m. This means that there is a subgroup L of N generated by finitely many abelian normal subgroups that is necessarily nilpotent but of derived length $> m$. The set J of all such subgroups L is invariant under $Aut\,(N)$. So the subgroup $\bar{R} = \langle Z(L) : L \in J \rangle$ is characteristic in N and normal in G. Furthermore, $\bar{R} \leq R < N$. Thus $\bar{R} = 1$ and this is a contradiction.

(ii) Let S be the soluble radical of N and let S^G be its normal closure in G. Of course S^G is a normal locally soluble subgroup of N and so $S^G = S$. Without loss of generality, assume $S = 1$. Then N is a finite non-abelian simple group, hence $C_G(N) \cap N$ is trivial. This gives that $C_G(N)$ is soluble. Since $G/C_G(N)$ embeds in $Aut(N)$, we get that G is soluble-by-finite. This also implies that the soluble radical of G is soluble. Suppose it is trivial, so that G is finite.

Let M be a minimal normal subgroup of G. Then M is the direct product of copies of a non-abelian simple group A. Obviously, A is subnormal in G and so, by Lemma 7.6, we have $G^{(r)} \leq A$ for some $r > 0$. Thus $A = M$. It follows that M is a minimal simple group and, by [**?**, (1) Lemma], G/M is soluble. Moreover $C_G(M) \cap M = 1$, so that $C_G(M)$ is a normal soluble subgroup of G. By our assumption, $C_G(M) = 1$ and this gives $G \lesssim Aut\,(M)$. □

Lemma 7.9. *Let G be a locally (soluble-by-finite) group with all subgroups subnormal or soluble. Then either*

(i) *G is locally soluble, or*

(ii) *$G^{(r)}$ is finite for some integer r and G is an extension of a soluble group by a finite almost minimal simple group.*

Proof. We have $G^{(\alpha)} = G^{(\alpha+1)}$ for some ordinal α. Let H be a proper subgroup of $G^{(\alpha)}$ and suppose that H is not soluble. Then H is subnormal in G and, by

[Möh90], $G^{(\alpha)}$ has a non-trivial abelian factor. But this is impossible. Hence, either $G^{(\alpha)}$ is soluble or $G^{(\alpha)}$ is a minimal non-soluble group. Furthermore, in this latter case, $G/G^{(\alpha)}$ is soluble by [**?**, (1) Lemma] and therefore α is a finite ordinal. If $G^{(\alpha)}$ is soluble, then G has a descending normal series with abelian factors. So that G is locally soluble, being locally (soluble-by-finite). In the other case, $G^{(\alpha)}$ is a minimal non-soluble group for some integer $\alpha = r$. The claim immediately follows by Lemma 7.8. □

Notice that, proving (*ii*) of Lemma 7.9, we have that the derived series of G ends in finitely many steps. The next lemma, which follows from [Smi, Proposition 1] together with [**?**, 12.2.6], shows that this also happens when G is locally soluble. One may see also [Cas86].

Lemma 7.10. *Let $\mathfrak{X} = \bigcup_{i\in\mathbb{N}} \mathfrak{X}_i$ be a class of groups, where each class $\mathfrak{X}_i$ is closed under taking subgroups and direct limits, and $\mathfrak{X}_i \subseteq \mathfrak{X}_{i+1}$ for all i. Let G be a group with all subgroups subnormal or in $\mathfrak{X}$, and suppose that $G \notin \mathfrak{X}$. If G is locally soluble, then $G^{(r)} = G^{(r+1)}$ for some integer r.*

Now, we can prove Theorem 7.1.

Proof of Theorem 7.1. By Lemma 7.9, G is either locally soluble, or $G^{(r)}$ is finite for some integer r and G is an extension of a soluble group S by a finite almost minimal simple group. Clearly, S must be soluble of length $\leq d$, by [Möh90]. Let G be locally soluble and suppose that it is not soluble. By Lemma 7.10 with $\mathfrak{X}$ the class of soluble groups, we have $G^{(s)} = G^{(s+1)}$ for some $s \geq 0$. Moreover, $G^{(s)}$ is not soluble. It follows, as in the proof of Lemma 7.9, that every proper subgroup of $G^{(s)}$ is soluble of length at most d. Thus $G^{(s)}$ is finite by [DE, Lemma 2.1], a contradiction. □

By a theorem of J. E. Roseblade (see [Ros] and [**?**, 12.2.3]), a group in which every subgroup is subnormal of defect at most $n \geq 1$ is nilpotent of class not exceeding a function depending only on n. Using this, we can generalize Lemma 7.9 to the locally graded case, provided that the subnormal defect is bounded.

Lemma 7.11. *Let G be a locally graded group and suppose that, for some positive integer n, every non-soluble subgroup of G is subnormal of defect at most n. Then G is locally (soluble-by-finite).*

Proof. We may assume that G is finitely generated. Suppose that it is not soluble-by-finite and denote by R its finite residual. As G is locally graded, we have $R < G$. Let N be a normal subgroup of G with finite index. Then every subgroup of G/N is subnormal of defect $\leq n$ and so, by Roseblade's Theorem ([Ros]), G/N is nilpotent of bounded class depending on n. It follows that G/R is nilpotent and R is not soluble. Let S be a proper subgroup of R and suppose that S

is not soluble. Then every subgroup of R/S^R is subnormal of defect $\leq n$, in particular R/S^R is soluble, by [Ros]. This implies that $R' < R$. So G/R' is finitely generated and abelian-by-nilpotent. We get that G/R' is residually finite (see, for instance, [Rob72, Theorem 9.51]) and $R' = R$, a contradiction. Hence every proper subgroup of R is soluble and R cannot be finite: otherwise, G would be finite-by-nilpotent and, consequently, also nilpotent-by-finite. By Lemma 7.8, we obtain that G is hyperabelian. Then G has a finite non-nilpotent homomorphic image G/M (see, for instance, [Rob72, Theorem 10.51]) which is also soluble, being G/M finite and hyperabelian. Therefore M is not soluble and every subgroup of G/M is subnormal of defect $\leq n$. Thus G/M is nilpotent by [Ros], a contradiction. □

Proof of Theorem 7.2. By Lemma 7.11, jointly with Theorem 7.1, we have that G is either soluble, or $G^{(r)}$ is finite for some $r \geq 0$ and G is an extension of a soluble group of derived length at most d by a finite almost minimal simple group. Let G be soluble and denote by e its derived length. We may assume $d < e$. Then $H = G^{(e-(d+1))}$ is soluble of length $d+1$ and every subgroup of G containing H is subnormal of defect $\leq n$. It follows that $G^{(s)} \leq H$ for some s depending on n (see [Cas86] and Remark 7.7). Thus, G is soluble of length at most $s+d+1$. Suppose now that there exists $r \geq 0$ such that $K = G^{(r)}$ is finite and non-soluble. Since every subgroup of G containing K is subnormal of defect $\leq n$, we get, as before, $G^{(t)} \leq K$ for some t depending on n. □

As a final remark we point out that, in (ii) of Theorems 7.1 and 7.2, one cannot expect that G is an extension of a soluble group S by a finite minimal simple group: it suffices to consider the direct product of any abelian group by the symmetric group of degree 5. However, if M/S is a finite minimal simple subgroup of G/S such that $G/S \lesssim Aut\,(M/S)$, then $M \lhd G$ by Lemma 7.6 and we can compute the order of G/M. In fact, $G/M \lesssim Out\,(M/S)$ where $|Out\,(M/S)|$ divides $2p$, with p odd prime, by Table 7.1.

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