CENTRALIZERS OF FINITE SUBGROUPS IN SIMPLE LOCALLY FINITE GROUPS

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CENTRALIZERS OF FINITE SUBGROUPS IN SIMPLE LOCALLY FINITE GROUPS

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ABSTRACT

CENTRALIZERS OF FINITE SUBGROUPS IN SIMPLE LOCALLY FINITE GROUPS

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A group G is called locally finite if every finitely generated subgroup of G is finite. In this thesis we study the centralizers of subgroups in simple locally finite groups. Hartley proved that in a linear simple locally finite group, the fixed point of every semisimple automorphism contains infinitely many elements of distinct prime orders. In the first part of this thesis, centralizers of finite abelian subgroups of linear simple locally finite groups are studied and the following result is proved: If G is a linear simple locally finite group and A is a finite d-abelian subgroup consisting of semisimple elements of G, then $C_G(A)$ has an infinite abelian subgroup isomorphic to the direct product of cyclic groups of order p_i for infinitely many distinct primes p_i.

Hartley asked the following question: Let G be a non-linear simple locally finite group and F be any subgroup of G. Is $C_G(F)$ necessarily infinite? In the second part of this thesis, the following problem is studied: Determine the non-linear simple locally finite groups G and their finite subgroups F such that $C_G(F)$ contains an infinite abelian subgroup which is isomorphic to the direct product of cyclic groups of order p_i for infinitely many distinct primes p_i. We prove the following: Let G be a non-linear simple locally finite group with a split Kegel cover $\mathcal{K}$ and F be any finite subgroup consisting of $\mathcal{K}$-semisimple elements of G. Then the centralizer $C_G(F)$ contains an infinite abelian subgroup isomorphic to the direct product of cyclic groups of order p_i for infinitely many distinct primes p_i.

Keywords: Locally finite group, simple group, centralizer.

ÖZ

BASİT YEREL SONLU GRUPLARDA SONLU ALTGRUPLARIN MERKEZLEYENLERİ

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Sonlu sayıda eleman tarafından üretilen her altgrubu sonlu olan bir G grubuna yerel sonlu grup denir. Bu tez basit yerel sonlu gruplarda altgrupların merkezleyenleriyle ilgilidir. Hartley, lineer basit yerel sonlu bir grupta her yarı-basit otomorfizmanın sabit noktalarının birbirinden farklı asal mertebeleri olan sonsuz sayıda eleman içerdiğini kanıtladı. Bu tezin ilk bölümünde, lineer basit yerel sonlu gruplarda sonlu değişmeli altgrupların merkezleyenleri incelenmiş ve aşağıdaki sonuç elde edilmiştir: Eğer G lineer basit yerel sonlu bir grup ve A yarı basit elemanlardan oluşan, sonlu, d-değişmeli bir altgrupsa $C_G(A)$'nın sonsuz sayıda birbirinden farkli p_i asalı için, p_i mertebeli devirli grupların direk çarpımına eşyapılı sonsuz, değişmeli bir altgrubu vardır.

Hartley aşağıdaki soruyu sordu: G non-lineer, basit, yerel sonlu bir grup olsun ve F altgrubu G'nin herhangi bir altgrubu olsun. $C_G(F)$ her zaman sonsuz bir grup mudur? Bu tezin ikinci kısmında çalışılan problem: Hangi non-lineer basit yerel sonlu G gruplarında ve hangi sonlu altgruplary F için $C_G(F)$'in sonsuz sayıda birbirinden farklı p_i asalı icin, mertebesi p_i olan devirli grupların direk çarpımıyla eşyapılı değişmeli bir altgrubu vardır sorusuna cevap bulmaktır. Bu soru ile ilgili olarak aşağıdaki sonuç kanıtlandı: G grubu $\mathcal{K}$ split Kegel örtüsüne

sahip, non-lineer, basit, yerel sonlu bir grup ve F altgrubu $\mathcal{K}$-yarıbasit (semisimple) elemanlardan oluşan sonlu bir altgrup olsun. O zaman, $C_G(F)$'nin sonsuz mertebeli öyle bir değişmeli altgrubu vardır ki, bu altgrup sonsuz sayıda birbirinden farklı p_i asalı icin, p_i mertebeli devirli grupların direk ̧carpımına eşyapılıdır.

Anahtar Sözcükler: Yerel sonlu grup, basit grup, merkezleyen.

To my wife Esra

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CHAPTER 1

INTRODUCTION

A group G is called locally finite if every finitely generated subgroup of G is finite. In this thesis, we prove some results on the centralizers of finite subgroups in simple locally finite groups.

The study of centralizers of elements in simple groups was motivated by Brauer-Fowler Theorem and it had a key role in the classification of finite simple groups. Since every finite group has a composition series, finite simple groups are the building blocks to understand the structure of a finite group. Feit and Thompson showed in [8] that every group of odd order is solvable. Hence, every finite non-abelian simple group must have even order, so, by Cauchy Theorem, every finite non-abelian simple group has an element of order 2. The elements of order 2 are called involutions. Centralizers of involutions are necessarily proper subgroups of a finite simple group. Brauer and Fowler proved in [3] that the order of a finite simple group is bounded by a function of the order of the centralizer of an involution. This theorem gave direction to group theorists during the 20th century and the classification of finite simple groups was completed in 1980's. According to the classification, a finite simple group belongs to one of the following families:

1. Cyclic groups $\mathbb{Z}_p$ of prime order p,

2. Alternating groups A_n of degree greater than 4,

3. Simple groups of Lie type (Chevalley and twisted Chevalley groups over finite fields),

4. 26 sporadic groups.

Locally finite groups are infinite groups with a finiteness condition, hence it is possible to use some information about finite groups to understand the structure of locally finite groups. Since finite simple groups are classified, it is natural to ask if it is possible to classify all infinite simple locally finite groups too. The experts think that we are far from an answer to this question. By [19, Corollary 6.12], there exist $2^{\aleph_0}$ non-isomorphic countable simple locally finite groups which can be obtained as direct limits of finite alternating groups. However, it may be possible to obtain information about some "nice" families of simple locally finite groups, and try to generalize the scope of this.

A group G is called linear if it has a faithful representation into $GL_n(k)$ for some natural number n and for some field k. Linear simple locally finite groups were classified independently by Belyaev, Borovik, Hartley-Shute and Thomas (see [1, 2, 15, 37]). They proved that a linear simple locally finite group is a Chevalley or a twisted Chevalley group over a locally finite field. By Theorem 2.31, we will see that a linear simple locally finite group over a locally finite field k of characteristic p can be written as a union of finite simple groups of the same Lie type over finite fields of characteristic p.

Hartley and Kuzucuoğu proved in [14, Theorem A2] that in an infinite locally finite simple group, the centralizer of every element is infinite. Hartley proved a generalization of the Brauer-Fowler Theorem in [10, Theorem A′], namely, he proved if G is a finite simple group with an automorphism α of order n with at most k fixed points, then the order of G is bounded by a function of n and k. By using this result, he proved in [10, Corollary A1] that if G is a locally finite group containing an element with finite centralizer, then G contains a locally solvable normal subgroup of finite index. By using the generalization of Brauer-Fowler Theorem ([10, Theorem A′]), Hartley also proved in [10, Theorem C] that if G is a simple locally finite group of Lie type over an infinite locally finite field of characteristic p, and α is an automorphism of coprime order with p, then there are infinitely many elements of distinct prime orders, which are fixed by α.

Hartley asked the following question in [11]:

Question 1.1. *Let G be a non-linear simple locally finite group and F be a finite subgroup of G. Is $C_G(F)$ necessarily infinite?*

In this work, our starting point was this question. First, we considered centralizers of finite subgroups in linear simple locally finite groups. In linear case, by Remark 4.2 we will see that, if G is a linear simple locally finite group, it is always possible to find a finite subgroup F with trivial centralizer. In fact, in Section 4.2, for each n, we will present a method to construct a finite abelian subgroup of $PSL_n(k)$ consisting of semisimple elements whose centralizer is itself.

So, for the linear case the question turns into the following:

Question 1.2. *Let G be a linear simple locally finite group. Determine all the finite abelian subgroups A consisting of semisimple elements such that $C_G(A)$ contains an infinite abelian subgroup isomorphic to the direct product of cyclic groups of order p_i for infinitely many primes p_i.*

For the linear case, we need the following definition:

Definition 1.3. *Let $\overline{G}$ be a simple linear algebraic group. A finite abelian subgroup A consisting of semisimple elements of $\overline{G}$ is called a* **d-*abelian*** *subgroup if it satisfies one of the following:*

1. *The root system associated with $\overline{G}$ has type A_l and Hall-π-subgroup of A is cyclic where π is the set of primes dividing $l+1$*

2. *The root system associated with $\overline{G}$ has type B_l, C_l, D_l or G_2 and the Sylow 2-subgroup of A is cyclic.*

3. *The root system associated with $\overline{G}$ has type E_6, E_7 or F_4 and the Hall-$\{2,3\}$-subgroup of A is cyclic.*

4. *The root system associated with $\overline{G}$ has type E_8 and the Hall-$\{2,3,5\}$-subgroup of A is cyclic.*

We proved the following result:

Theorem 1.4. *Let G be a locally finite simple group of Lie type defined over an infinite locally finite field of characteristic p. Let A be a d-abelian subgroup of G. Then $C_G(A)$ contains an infinite abelian subgroup which is isomorphic to a direct product of cyclic groups of order p_i for infinitely many prime p_i*

A group G is called Černikov if it has a normal subgroup H of finite index such that $H \cong Dr_{i=1}^{n} C_{p_i^{\infty}}$ for a finite set of primes $\{p_1, \dots p_n\}$. Šunkov and Kegel-Wehrfritz proved independently in [35] and [19, 20] respectively that a locally finite group satisfying minimal condition on subgroups is necessarily a Černikov group. By its definition, a Černikov group contains only finitely many elements of distinct prime orders. Hence, in Theorem 1.4 we proved that in a locally finite, simple group of Lie type, centralizer of a d-abelian subgroup can not be Černikov, that is, it can not satisfy minimal condition.

The first part (Chapter 4) of this thesis is about this result. In the second part, we study the centralizer of finite subgroups in non-linear simple locally finite groups. Here, we study a different version of Hartley's Question 1.1 :

Question 1.5. *Let G be a non-linear simple locally finite group and F be a finite subgroup. Does $C_G(F)$ contain an infinite abelian subgroup isomorphic to the direct product of cyclic groups of order p_i for infinitely many distinct primes p_i?*

The answer of this question is not positive in the most general case, because Meierfrankenfeld proved in [23] that there exists a non-linear simple locally finite group G with an element x such that the centralizer $C_G(x)$ is a p-group. Hence, we restricted our attention to a smaller class of simple locally finite groups, namely, we studied simple locally finite groups with a **split Kegel cover**. Recall that if G is a locally finite group, a set $\{(G_i, N_i) \mid i \in I\}$ consisting of pairs of subgroups of G satisfying $N_i \trianglelefteq G_i$, is called a **Kegel cover** of G provided that G can be written as the union of G_i's, the factors G_i/N_i are finite simple groups and $G_i \cap N_{i+1} = 1$ (For the details see Section 2.2). Here, observe that G_i/N_i is a finite simple group, so it is either an alternating group, or a simple groups of Lie type or a sporadic group. Since there are finitely many sporadic groups, by passing to a subsequence we may assume that the factors are either alternating groups or simple groups of Lie type. (See Remark 2.29 for details.)

Definition 1.6. *A Kegel cover $\mathcal{K} = \{(G_i, N_i) : \; i \in I\}$ is called a **split Kegel cover** if $C_{G_i/N_i}(KN_i/N_i) = C_{G_i}(K)N_i/N_i$ for every subgroup K of G_i.*

We need a general notion of a semisimple element in a simple locally finite group:

Definition 1.7. *Let G be a non-linear simple locally finite group and*

$$\mathcal{K} = \{(G_i, N_i) \; : \; i \in I\}$$

*be a Kegel cover for G. An element x in G is called $\mathcal{K}$-**semisimple** if $\mathcal{K}$ is a Kegel cover consisting of alternating groups or G_i/N_i is a finite simple group of Lie type and xN_i is a semisimple element of G_i/N_i for every $i \in I$.*

The main result of this thesis is the following:

Theorem 1.8. *Let G be a non-linear simple locally finite group with a split Kegel cover $\mathcal{K}$ and F be any finite subgroup of G consisting of $\mathcal{K}$-semisimple elements. The centralizer $C_G(F)$ contains an infinite abelian subgroup isomorphic to a direct product of cyclic groups of order p_i for infinitely many prime p_i.*

In Chapter 5, the proof of this result is presented.

Hartley studied fixed points of semisimple automorphisms in linear simple locally finite groups. In Chapter 3, we present some results about fixed points of automorphisms in infinite alternating groups.

CHAPTER 2

PRELIMINARIES

In this chapter we will give the basic definitions and primary results which we will use in Chapter 4 and Chapter 5.

2.1 Linear Algebraic Groups

In this section, the main definitions and basic results on linear algebraic groups will be summarized. Let k denote an algebraically closed field of characteristic p.

Definition 2.1. *An* ***algebraic group*** *G is an algebraic variety together with a group structure such that the maps*

$$\begin{aligned} \mu : G \times G &\longrightarrow G \\ (g,h) &\longrightarrow gh \end{aligned}$$

and

$$\begin{aligned} \iota : G &\longrightarrow G \\ g &\longrightarrow g^{-1} \end{aligned}$$

are morphisms of varieties.

If an algebraic group G is an affine variety (that is, the set of zeros of finitely many polynomials in k^n), then G is called an **affine algebraic group**.

Remark 2.2. Let $G = GL(n,k)$ denote the set of all $n \times n$ matrices over an algebraically closed field k with non-zero determinant. Clearly, for every $A \in G$, the function $\det A$ is a polynomial over k in n^2 variables.

$$GL_n(k) = \{(a_{ij}) \in k^{n\times n} : \det(a_{ij}) \neq 0\}.$$

To show that G is an affine variety, we need a polynomial over k whose zero set is exactly G. Consider $f(t, a_{11}, a_{12} \ldots, a_{nn}) = t. \det A - 1$. Clearly, f is a polynomial in $n^2 + 1$ variables over k. Now, $G = GL_n(k) = \{(a_{ij}) \in k^{n \times n} : f(t, a_{11}, a_{12} \ldots, a_{nn}) = t. \det A - 1 = 0\}$ defines $GL_n(k)$ as the zero set of a polynomial $n^2 + 1$ variables over k. Hence, G is a closed subset of $\mathbb{A}^{n^2+1}$, that is, it is an affine variety. Also, the usual group operations on G are morphisms of this variety, that is, G is a linear algebraic group.

An algebraic group G is called a **linear algebraic group** if it is a closed subgroup of $GL_n(k)$ for some n. A closed subset of an affine variety is also an affine variety. By Remark 2.2, $GL_n(k)$ is an affine algebraic group, so linear algebraic groups, that is, the closed subgroups of $GL_n(k)$ are affine algebraic groups. Conversely, by [31, Theorem 2.3.7], every affine algebraic group can be embedded in $GL_n(k)$ as a closed subgroup for some n, that is, every affine algebraic group is a linear algebraic group.

Definition 2.3. *Let G be a linear algebraic group. The irreducible component of G containing 1_G is called the identity component and it is denoted by G°. A linear algebraic group G is called **connected** if $G = G^\circ$.*

Proposition 2.4. *[17, Section 7.3, Proposition] Let G be an algebraic group.*

1. *G° is a normal subgroup of finite index in G and its cosets are the connected irreducible components of G.*

2. *If H is a closed subgroup of finite index in G then H contains G°.*

Definition 2.5. *An algebraic group G is **simple** if it has no proper non-trivial closed connected normal subgroup.*

Example 2.6. The group of $n \times n$ matrices over k with determinant 1, which is called the Special Linear Group, denoted by $SL_n(k)$, is a simple algebraic group. One can see that the abstract group $SL_n(k)$ has a proper non-trivial normal subgroup, namely the center $Z(SL_n(k))$. But, $Z(SL_n(k))$ is not connected.

Theorem 2.7. *[17, Section 29.5, Corollary] If G is a simple algebraic group with finite center Z then G/Z is simple as an abstract group.*

By [30, Section 4.2], an algebraic group G is called an abelian variety if the algebraic variety G is projective and irreducible. The following result of Chevalley shows that, simple algebraic groups are affine, that is, by Remark 2.2 simple algebraic groups are linear algebraic groups:

Theorem 2.8. *[30, Section 4.2, Theorem C] Let G be an algebraic group over a perfect field k. Then G has a unique normal closed subgroup N such that N is an affine algebraic group and G/N is an abelian variety.*

Hence, the study of simple algebraic groups reduces to the study of simple linear algebraic groups.

Remark 2.9. By Proposition 2.4, if G is a simple algebraic group then G has no closed connected proper non-trivial normal subgroups. By Corollary in [17], (Section 29.4, page 182) if G is simple as an algebraic group then every proper normal subgroup of the abstract group is contained in the center. So, if G is a simple algebraic group over an algebraically closed field any proper normal subgroup of the abstract group has infinite index. Then G° is necessarily equal to G, since it has finite index. Hence, simple algebraic groups are connected.

By [17, Section 19.5, page 125], every algebraic group G has a unique largest solvable closed subgroup K. The identity component K° of K, is the largest connected normal solvable subgroup of G, and it is called the **(solvable) radical** of G and denoted by $R(G)$.

Definition 2.10. *A non-trivial connected algebraic group is called **semisimple** if its (solvable) radical is trivial.*

By [17, Section 19.5, page 125], the subgroup of $R(G)$ consisting of unipotent elements is a normal subgroup of G, which is called the **unipotent radical** of G (and denoted by $R_u(G)$). The unipotent radical of G is the largest connected normal unipotent subgroup of G.

Definition 2.11. *A non-trivial connected algebraic group is called **reductive** if its unipotent radical is trivial.*

Example 2.12. We know that a proper normal subgroup of $SL_n(k)$ is contained in its center (See Page 168, [17]). So, if it is connected, then it must be trivial. Hence, $SL(n,k)$ is semisimple. In fact, every simple algebraic group is necessarily semisimple. Similarly, semisimple algebraic groups are reductive.

Let T be a torus in $GL(n,k)$ where k is an algebraically closed field, that is, T is a subgroup which is isomorphic to a direct product of copies of $k^\times$. It is a linear algebraic group. Now, every element of T is conjugate to a diagonal matrix, that is, T consist of semisimple elements. Hence, $R_u(T) = 1$, that is, T is a reductive group, but it is not semisimple since $R(T) = T$.

We know that every linear algebraic group is isomorphic to a closed subgroup of $GL(n,k)$ for some n. Examples of simple linear algebraic groups include the classical groups, $SL_{l+1}(k), Sp_{2l}(k), SO_{2l+1}(k), SO_{2l}(k)$. The parameter l denotes the dimension of the subgroup of diagonal matrices (maximal torus) in the corresponding group, and called the rank of the group.

Now, we will define the fundamental group of a simple linear algebraic group (for details, see [28, Page 53]).

Definition 2.13. *[28, Page 53] Let T be a maximal torus of the simple linear algebraic group G over the algebraically closed field of characteristic p. Consider the character group $X(T) = Hom(T, k^\times) \cong \mathbb{Z}^n$ where n is the dimension of T. Then $X(T)$ is a lattice in $X(T)_\mathbb{Q} = X(T) \bigotimes_\mathbb{Z} \mathbb{Q}$, containing the root lattice,*

Choose a positive definite inner product $X(T)_\mathbb{Q}$ which is invariant under the Weyl group.

*An element $\lambda \in X(T)_\mathbb{Q}$ such that $\langle \lambda, \alpha \rangle = 2\langle \lambda, \alpha \rangle \langle \alpha, \alpha \rangle^{-1}$ for all roots α is called a **weight**.*

*Let Λ be the set of weights, which forms another lattice in $X(T)_\mathbb{Q}$. So, $\mathbb{Z}\Sigma \leq X(T) \leq \Lambda$. Here, the finite group $\Lambda/\mathbb{Z}\Sigma$ is called the **fundamental group** of G.*

Now, if $X(T) = \Lambda$, then G is called **simply connected**, if $X(T) = \mathbb{Z}\Sigma$, then G is called **adjoint type**.

Simple linear algebraic groups are classified by Dynkin diagrams. A connected semisimple group is simple iff its Dynkin diagram is connected. The possible connected Dynkin diagrams define possible types of simple linear algebraic groups over algebraically closed fields: $A_l, B_l, C_l, D_l, E_6, E_7, E_8, G_2, F_4$.

In Table 2.1, the corresponding adjoint and simply connected groups, obtained as fixed points of some Frobenius maps (see Section 2.3) in classical simple linear algebraic groups are given. We will discuss how to obtain finite simple groups of Lie type from simple linear algebraic groups in Section 2.3.

We will end the discussion about linear algebraic groups with definition of semisimple and unipotent elements and the Jordan decomposition:

Let G be a simple linear algebraic group over an algebraically closed field of characteristic p.

Definition 2.14. *[5, Section 1.4] An element $x \in G \leq GL_n(k)$ is called semisimple if it is diagonalizable. An element x is called unipotent if all of its eigenvalues are* 1.

Remark 2.15. Here, since the group G is defined over a field of characteristic p, we can further say that an element u is unipotent iff $|u| = p^m$ for some m and an element s is semisimple iff $(|s|, p) = 1$.

Theorem 2.16. *[17, Lemma B, page 96] Let $x \in GL_n(k)$.*

1. *There exists unique $x_s, x_u \in GL_n(k)$ satisfying $x = x_s x_u = x_u x_s$ where x_s is semisimple, x_u is unipotent.*

2. *If $y \in C_G(x)$, then $y \in C_G(x_s)$ and $y \in C_G(x_u)$.*

The unique expression $x = x_s x_u$ or $x = x_u x_s$ is called the Jordan decomposition.

2.2 Simple Locally Finite Groups

In this section, we collect background information on simple locally finite groups.

First, we consider some examples of simple locally finite groups:

Example 2.17. Let Ω be an infinite set. The group of all even permutations on the set Ω, which is denoted by $Alt(\Omega)$, is a simple locally finite group with cardinality $|\Omega|$.

Example 2.18. A field is called locally finite if every finitely generated subfield is finite. Let $\mathbb{F}$ be an infinite locally finite field. The group $PSL_n(\mathbb{F})$ is a simple locally finite group.

Example 2.19. Let $\mathbb{F}$ be a finite field. Observe that the map

$$\begin{aligned} \phi_n : & SL_n(\mathbb{F}) \longrightarrow SL_{n+1}(\mathbb{F}) \\ & A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

embeds $SL_n(\mathbb{F})$ into $SL_{n+1}(\mathbb{F})$.
The direct limit of the directed system $(SL_n(\mathbb{F}), \phi_n)$, is a simple locally finite group denoted by $\mathbf{SL^0}(\mathbb{F})$ and called the **Stable Special Linear Group**.

Definition 2.20. *[19, Page 8] A set Σ of subgroups of a group G is called a local system of G if*

1. $G = \bigcup_{S \in \Sigma} S$
2. *if $S, T \in \Sigma$ then there exists $U \in \Sigma$ such that $S, T \subset U$.*

The following result is very useful to understand the structure of infinite simple groups:

Theorem 2.21. *([19, Theorem 4.4]) An infinite group G is simple iff it has a local system consisting of countably infinite simple subgroups of G.*

Remark 2.22. Our aim is to prove theorems about centralizers of finite subgroups in simple locally finite groups. But, by Theorem 2.21, we deduce that any finite subgroup of a simple locally finite group is contained in a countable simple group (which is clearly locally finite). Hence, for us, it is enough to find the centralizer of a finite subgroup in a countable simple locally finite group.

Kegel-Wehrfritz asked the following question:

Question 2.23. *Does every simple locally finite group has a local system consisting of finite simple groups?*

Serezhkin and Zalesskii answered this question negatively in [39]. They proved the following result:

Theorem 2.24. *[13, Proposition 1.7] If k is a finite field of odd order then the Stable Symplectic Group is an infinite simple locally finite group which can not be written as a union of finite simple groups.*

So, we can not write every simple locally finite group as a union of finite simple groups. But still, we have a concept that connects the theory of finite simple groups and the theory of locally finite simple groups:

Definition 2.25. *[13, Definition 2.1, 2.2] Let G be a locally finite group and I be an index set. A set $\{(G_i, N_i) \mid i \in I\}$ consisting of pairs of finite subgroups of G is called a* ***Kegel cover*** *of G if for all i, the subgroup N_i is a maximal normal subgroup of G_i and for every finite subgroup $F \leq G$ there exists $i \in I$ with $F \leq G_i$ and $F \cap N_i = 1$.*

The following form of Definition 2.25 for countable locally finite groups give us more information about the structure of countable simple locally finite groups:

Definition 2.26. *[13, Definition 2.2] Let G be a countable locally finite group. A set $\{(G_i, N_i) \mid i \in \mathbb{N}\}$ consisting of pairs of finite subgroups of G satisfying $N_i \trianglelefteq G_i$, is called a Kegel cover of G if*

$$G = \bigcup_{i \in I} G_i$$

the factors G_i/N_i are finite simple groups and $G_i \cap N_{i+1} = 1$.

By Theorem 2.21, an infinite group is simple iff it has a local system of countably infinite simple groups. Hence every finite subset of an infinite simple group is contained in a countably infinite simple group. Hence, to answer Question 1.5, it is enough to consider the centralizers of finite subgroups in countable simple locally finite groups.

By [13, Lemma 2.4], every simple locally finite group has a Kegel cover. Indeed, Theorem 2.27 shows this result for countable simple locally finite groups.

Theorem 2.27. *[19, Lemma 4.5] Every countable simple locally finite groups has a Kegel cover* $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbb{N}\}$.

We will use the following result to see that for every infinite simple locally finite group we can choose a Kegel cover whose all factors are non-abelian finite simple groups.

Theorem 2.28. *[13, Corollary 2.5] Let G be an infinite, simple locally finite group. Then*

1. *G has a Kegel cover $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbb{N}\}$ where G_i's are perfect.*

2. *G_i/N_i's form a set of finite simple groups of unbounded orders.*

Let G be a countably infinite simple locally finite group with a Kegel cover $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbb{N}\}$ where G_i's are perfect. Since G_i's are perfect, the factors G_i/N_i's are non-abelian finite simple groups. By the classification of finite simple groups, we know that each factor is either an alternating group, or a simple group of Lie type, or a sporadic group. Since there are only finitely many sporadic groups, for any locally finite group G there exist a Kegel cover whose factors are either alternating groups or simple groups of Lie type.

Remark 2.29. For a simple locally finite group G, there are only 4 possible cases:

1. G has a Kegel cover with all G_i/N_i's are alternating groups, or,

2. G has a Kegel cover with all G_i/N_i's are are classical groups of the same type with unbounded rank, or,

3. G has a Kegel cover with all G_i/N_i's are are classical groups of the same type with bounded rank, or,

4. G has a Kegel cover with all G_i/N_i's are exceptional groups of the same type.

By [13, Theorem 2.6], in cases (3) and (4), the group is linear. So, if we have a non-linear simple locally finite group, then the Kegel cover is either alternating type or a fixed classical type with unbounded rank parameters.

2.3 Construction of a Simple Group of Lie type over a Locally Finite Field

We will construct finite and locally finite simple groups of Lie type from the corresponding type linear algebraic groups. First, we need the definition of a Frobenius map, which is the major key of this construction.

Definition 2.30. *Let G be a linear algebraic group over an algebraically closed field k of characteristic p where $p > 0$. Let $q = p^m$ with $k \geq 1$ and F_q be the map given by*

$$F_q : GL(n,k) \longrightarrow GL(n,k)$$
$$(a_{ij}) \longrightarrow (a_{ij}^q).$$

*Now, F_q is a group automorphism of $GL(n,k)$. A homomorphism $F : G \to G$ is called a **standard Frobenius map** if for some n the embedding $i : G \to GL(n,K)$ satisfies $i(F(g)) = F_q(i(g))$ for some $q = p^k$ and for all $g \in G$.*

*A homomorphism is called a **Frobenius map** if some power of F is a standard Frobenius map.*

Now, by [5, page 31], Frobenius maps are algebraic endomorphisms with finite fixed point group. Let $\overline{G}$ be a simple linear algebraic group of adjoint type over an algebraically closed field k of characteristic p. Let σ be a Frobenius map on $\overline{G}$ and $C_{\overline{G}}(\sigma)$ its fixed point group. By [10, Section 3], the subgroup $H = O^{p'}(C_{\overline{G}}(\sigma))$ is a finite simple group of Lie type of the same type with $\overline{G}$ and all finite simple groups of Lie type can be obtained in this way.

Now, we will see a result of Turau which enables us to see the structure of locally finite simple groups of Lie type.

Theorem 2.31. *[14, Lemma 4.3] Let G be a Chevalley group (or a twisted Chevalley group) over an infinite locally finite field k of characteristic p and let $\overline{G}$ be the simple algebraic group over the algebraic closure $\overline{k}$ of k constructed from the same Lie algebra as G. Then there are a Frobenius map σ and a sequence $n_1, n_2, \ldots$ of positive integers such that n_i divides n_{i+1} and $G = \cup_{i=1}^{\infty} G_i$ where $G_i = O^{p'}(C_{\overline{G}}(\sigma^{n_i}))$.*

Theorem 2.31 enables us to express a linear simple locally finite group as a union of finite simple groups of the same Lie type. In fact, for any simple locally finite group G of Lie type, the groups G_i constructed as in Theorem 2.31 form a Kegel cover with $N_i = 1$ for all i.

Remark 2.32. The following Table 2.1, which is given in [5, p.40], shows the identifications with the groups $\overline{G}_\sigma$ with classical groups over finite fields.

Table 2.1: Identifications adjoint and simply connected types of finite simple groups of Lie type with classical groups

$(A_l)_{sc}(q)$	$SL_{l+1}(q)$
$(A_l)_{ad}(q)$	$PGL_{l+1}(q)$
$(^2A_l)_{sc}(q^2)$	$SU_{l+1}(q^2)$
$(^2A_l)_{ad}(q^2)$	$PU_{l+1}(q^2)$
$(B_l)_{sc}(q)$	$Spin_{2l+1}(q)$
$(B_l)_{ad}(q)$	$SO_{2l+1}(q)$
$(C_l)_{sc}(q)$	$Sp_{2l}(q)$
$(C_l)_{ad}(q)$	$PCSp_{2l}(q)$
$(D_l)_{sc}(q)$	$Spin_{2l}(q)$
$(D_l)_{ad}(q)$	$P(CO_{2l}(q)^0)$
$(^2D_l)_{sc}(q^2)$	$Spin_{2l}^-(q)$
$(^2D_l)_{ad}(q^2)$	$P(CO_{2l}^-(q)^0)$

2.4 Regular Unipotent and Semisimple Elements

Let $\overline{G}$ be a connected reductive group. Recall that the dimension of a maximal torus in $\overline{G}$ is called the **rank** of $\overline{G}$. An element $x \in \overline{G}$ is called **regular** if $\dim C_G(x) = rank(\overline{G})$. Steinberg proved in [33] that for every $x \in \overline{G}$ the dimension of the centralizer of x in $\overline{G}$ is greater than or equal to the rank of $\overline{G}$. Moreover, Steinberg also proved that in every connected reductive group, there exists regular elements.

In this section, first we will consider regular unipotent elements.

Proposition 2.33. *[5, **Proposition 5.1.2**] Let $\overline{G}$ be a connected reductive linear algebraic group. There exist regular unipotent elements in $\overline{G}$ and any two of them are conjugate. Moreover, the set $\mathcal{U}$ of regular unipotent elements of $\overline{G}$ is a dense open subset.*

Proposition 2.34. *[5, **Proposition 5.1.3**] Let $\overline{G}$ be a connected reductive linear algebraic group and u be a unipotent element of $\overline{G}$. Then the following are equivalent:*

1. *u is regular.*
2. *u lies in a unique Borel subgroup of $\overline{G}$.*
3. *u is conjugate to an element of the form $\prod_{\alpha\in\Phi^+} x_\alpha(\lambda_\alpha)$ with $\lambda_{\alpha_i} \neq 0$ for all fundamental roots α_i.*

Example 2.35. Let $\overline{G} = SL_3(\overline{k})$ where $char\overline{k} = p$ and $u_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Here,

$$C_{\overline{G}}(u_1) = \{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} : \ a, b, c \in \overline{k} \ \ a^3 = 1\}.$$

Now, $\dim C_{\overline{G}}(u_1)$ is equal to the transcendence degree of the coordinate ring $\overline{k}[a, b, c]/(a^3 - 1)$. Hence, $\dim C_{\overline{G}}(u_1) = 2$. A maximal torus in is conjugate to $\{\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & (\alpha\beta)^{-1} \end{pmatrix} \mid \alpha, \beta, \gamma \in \overline{k}\}$, which has dimension 2. Therefore, any maximal torus in $SL_3(\overline{k})$ has dimension 2, that is, the rank of $\overline{G}$ is 2. We have $SL_3(\overline{k}) = A_2(\overline{k})$.

Since $\dim C_{\overline{G}}(u_1) = 2$, the element u_1 is a regular unipotent element.

Indeed, $u_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = x_r(1)x_s(1)$ where r and s are the fundamental roots for the root system A_2.

Observe that $C_{\overline{G}}(u_1) = Z(\overline{G}).U$ where $U = \{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \overline{k}\}$. Since U is a p-group and $Z(\overline{G})$ is finite, $C_{\overline{G}}(u)$ can not contain infinitely many elements of distinct prime orders. In fact, by Proposition 2.36, if u is a regular unipotent element in $\overline{G}$ then every semisimple element in $C_{\overline{G}}(u)$ is contained in $Z(\overline{G})$ which is a finite group.

Proposition 2.36. ***[5, Proposition 5.1.5]*** *Let $\overline{G}$ be a connected reductive linear algebraic group and u be a regular unipotent element of $\overline{G}$. Then every semisimple element of $C_{\overline{G}}(u)$ belongs to the center of $\overline{G}$.*

Next, we will study the centralizers of regular semisimple elements in semisimple linear algebraic groups.

Theorem 2.37. ***[32, Corollary III.1.7]*** *Let $\overline{G}$ be a semisimple linear algebraic group and s be a semisimple element of $\overline{G}$. The following are equivalent:*

1. *s is a regular semisimple element.*
2. *$C_{\overline{G}}(s)^\circ$ is a maximal torus.*
3. *s is contained in a unique maximal torus.*
4. *$C_{\overline{G}}(s)$ consists of semisimple elements.*
5. *$\alpha(s) \neq 1$ for every root α relative to any maximal torus containing s.*

The following easy lemma (which is an exercise in [32, Example 1.5.a]) will be useful to construct examples of centralizers of regular semisimple elements:

Lemma 2.38. *[32, Example 1.5.a] Let $\overline{G} = SL_n(k)$ and s be a semisimple element of $\overline{G}$. The following are equivalent:*

1. *s is a regular semisimple element.*

2. *The eigenvalues of s are all distinct.*

Proof. **(2 $\Rightarrow$ 1)** Let s be a semisimple element of $\overline{G} = SL_n(k)$ such that all eigenvalues of s are distinct. Then s is conjugate to the diagonal element $s_0 = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ in $\overline{G}$ where $\lambda_i \neq \lambda_j$ for every i, j. Since s and s_0 are conjugate in G, the subgroups $C_{\overline{G}}(s)$ and $C_{\overline{G}}(s_0)$ are conjugate. Hence, they are isomorphic.

Let $g = \begin{pmatrix} a_{11} & a_{12} & \ldots & & a_{1n} \\ a_{21} & a_{22} & \ldots & & a_{2n} \\ \vdots & \vdots & . & & \vdots \\ \vdots & \vdots & & . & \vdots \\ a_{n1} & a_{n2} & \ldots & & a_{nn} \end{pmatrix}$ be an element of $C_{\overline{G}}(s_0)$. Then,

$$s_0 g = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & . & & \\ & & & . & \\ & & & & \lambda_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \ldots & & a_{1n} \\ a_{21} & a_{22} & \ldots & & a_{2n} \\ \vdots & \vdots & . & & \vdots \\ \vdots & \vdots & & . & \vdots \\ a_{n1} & a_{n2} & \ldots & & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \ldots & & a_{1n} \\ a_{21} & a_{22} & \ldots & & a_{2n} \\ \vdots & \vdots & . & & \vdots \\ \vdots & \vdots & & . & \vdots \\ a_{n1} & a_{n2} & \ldots & & a_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & . & & \\ & & & . & \\ & & & & \lambda_n \end{pmatrix} = g s_0$$

We obtain,

$$\begin{pmatrix} \lambda_1 a_{11} & \lambda_1 a_{12} & \ldots & & \lambda_1 a_{1n} \\ \lambda_2 a_{21} & \lambda_2 a_{22} & \ldots & & \lambda_2 a_{2n} \\ \vdots & \vdots & . & & \vdots \\ \vdots & \vdots & & . & \vdots \\ \lambda_n a_{n1} & \lambda_n a_{n2} & \ldots & & \lambda_n a_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_1 a_{11} & \lambda_2 a_{12} & \ldots & & \lambda_n a_{1n} \\ \lambda_1 a_{21} & \lambda_2 a_{22} & \ldots & & \lambda_n a_{2n} \\ \vdots & \vdots & . & & \vdots \\ \vdots & \vdots & & . & \vdots \\ \lambda_1 a_{n1} & \lambda_2 a_{n2} & \ldots & & \lambda_n a_{nn} \end{pmatrix}.$$

So, for every $i, j \in \{1, 2, \ldots n\}$ we have $\lambda_i a_{ij} = \lambda_j a_{ij}$. Since all eigenvalues of

s are distinct, $\lambda_i \neq \lambda_j$ for every $i \neq j$. Then if $i \neq j$ we have $a_{ij} = 0$. Hence, $C_{\overline{G}}(s_0) = \{diag(a_{11}, a_{22}, \dots a_{nn}) : \ \prod_{i=1}^n a_{ii} = 1\}$ which is a maximal torus of $\overline{G}$. Hence, $C_{\overline{G}}(s)$ is equal to a maximal torus T of $\overline{G}$. Hence,

$$\dim C_{\overline{G}}(s) = \dim T = rank(\overline{G}).$$

So, s is a regular semisimple element of $SL_n(k)$.

(1 $\Rightarrow$ 2) Assume that s is a semisimple element such that at least two of the eigenvalues of s are equal. Then s is conjugate to

$$s_1 = diag(\alpha, \alpha, \beta_1, \dots, \beta_{n-2}).$$

Hence $C_G(s)$ is isomorphic to $C_G(s_1)$.

Observe that

$$H = \{ \begin{pmatrix} a & b & 0 & \dots & & 0 \\ c & d & 0 & \dots & & 0 \\ 0 & 0 & \lambda_1 & 0 & \dots & 0 \\ \vdots & & & . & & \vdots \\ & & & & . & \\ 0 & 0 & \dots & & & \lambda_{n-2} \end{pmatrix} : \ (ad - bc)\prod_{i=1}^{n-2} \lambda_i = 1\} \leq C_G(s_1).$$

Then $\dim H \leq \dim(C_{\overline{G}}(s_1))$. Now, $\dim H = n + 1$, so $\dim C_{\overline{G}}(s_1) \geq n + 1 \neq n - 1 = rank(\overline{G})$. Therefore, s_1 and s are not regular. $\square$

Lemma 2.39. ***([32, III.2.1])*** *Let $\overline{G} = SL_n(k)$ and x be a regular element of G. Then the normal form of x is*

$$\begin{pmatrix} 0 & 0 & \dots & & 0 & 1 \\ -1 & 0 & \dots & & & c_1 \\ 0 & -1 & & & & c_2 \\ & & . & & & \vdots \\ & & & . & & \\ & & & & -1 & c_{n-1} \end{pmatrix}.$$

The following result is a direct consequence of the definition of a regular element in a simple linear algebraic group.

Lemma 2.40. *Let $\overline{G}$ be a simply connected simple linear algebraic group and g be a regular element of $\overline{G}$. Then gZ is a regular element of $\overline{G}/Z$.*

Proof. Assume that g is a regular element of $\overline{G}$. Then $\dim(C_{\overline{G}}(g)) = rank\overline{G}$. Consider

$$C_{\overline{G}/Z}(gZ) = \{xZ \in \overline{G}/Z \ : \ [g, x] \in Z\}.$$

As Z is finite, we can write $Z = \{z_0, z_1, z_2, \ldots z_k\}$ where $z_0 = 1_{\overline{G}}$. Now, $C_{\overline{G}/Z}(gZ) = \bigcup_{i=0}^{k} C_i$ where $C_i = \{xZ \in \overline{G}/Z \mid [g, x] = z_i\}$. Here C_i's are the connected components of $C_{\overline{G}/Z}(gZ)$ and C_0 is the identity component. By [17, 7.3 Proposition], C_i's are irreducible. Here $C_0 \cong C_{\overline{G}}(g)$, so $\dim C_0 = rank\overline{G}$. But since all C_i's are distinct cosets of C_0, they all have the same dimension, so $\dim(C_{\overline{G}/Z}(gZ)) = \dim(\bigcup_{i=0}^{k} C_i) = max_{i=1}^{k}(\dim C_i) = \dim(C_0) = rank(\overline{G}) = rank(\overline{G}/Z)$. Hence, gZ is regular in $\overline{G}/Z$. □

2.5 Torsion Primes of Simple Linear Algebraic Groups

We need the definition of a torsion prime and the list of torsion primes for simple linear algebraic groups.

Definition 2.41. *Let $\{a_1, a_2, \ldots a_r\}$ be a simple system of roots for the root system Σ. Let $h^* = \sum m_i^* a_i^*$ be the co-root of the highest root expressed in terms of the co-roots of the simple roots. If a prime p divides one of the coefficients m_i^*, then p is called a* ***torsion prime*** *of the root system Σ.*

The following examples may be useful to understand the definition.

Example 2.42. Consider the root system A_l.

Let $\{e_i : \ 1 \leq i \leq l + 1\}$ be the standard basis for the Euclidean space of dimension $l+1$. It is well-known that the set $\{a_i : \ a_i = e_i - e_{i+1}$ where $1 \leq i \leq l\}$ is a simple system of roots for the root system of type A_l. The complete set of positive roots of the root system A_l is $\{e_i - e_j : \ 1 \le i \le j \le l\}$. Then, it is easy to see that

$$e_i - e_j = (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \ldots + (e_{j-1} - e_j) = \sum_{k=i}^{j-1} a_k.$$

Then the highest root is $r = \sum_{k=1}^{l} a_k = e_1 - e_{l+1}$. The co-root of the highest root is $\frac{2r}{(r,r)}$. But since $(r,r) = (a_k, a_k)$ for every $1 \le k \le l$, we have

$$h_r = \frac{2r}{(r,r)} = \sum_{k=1}^{l} \frac{2a_k}{(a_k, a_k)} = \sum_{k=1}^{l} h_{a_k}.$$

Now, we wrote the co-root of the highest root as a sum of co-roots of simple roots and the coefficient of every co-root of a simple root is 1. Hence, there are no torsion primes of the root system A_l.

Example 2.43. Consider the root system of type C_2. Here, we have two simple roots, r and s with $|r| = 1$ and $|s| = \sqrt{2}$. Then $(r,r) = 1$ and $(s,s) = 2$.

Here, the set of positive roots is $\{r, 2r + s, r + s, s\}$. The highest root is $2r + s$. The co-root of the highest root is $h_{2r+s} = \frac{2(2r+s)}{(2r+s,2r+s)} = \frac{2(2r+s)}{2} = 2r + s$ since $|2r + s| = \sqrt{2}$. The co-roots the simple roots are $h_r = \frac{2r}{(r,r)} = 2r$ and $h_s = \frac{2s}{(s,s)} = \frac{2s}{2} = s$. Now, $h_{2r+s} = 2r + s = h_r + h_s$.

So, when we write the co-root of the highest root as a sum of co-roots of simple roots, the coefficients of h_r and h_s are all 1. Therefore, there are no torsion primes of the root system C_2.

The following table gives the list of torsion primes for various types of root systems: (For further information see [32, 4.3]).

Table 2.2: Torsion primes for various type of irreducible root systems

A_l, C_l	none
B_l, D_l, G_2	2
E_6, E_7, F_4	$2, 3$
E_8	$2, 3, 5$

For a reductive linear algebraic group, there are two types of torsion primes, namely, torsion primes of the root system and the torsion primes of the fundamental group. For this thesis, we will not need the definition of a torsion prime of the fundamental group. For the torsion primes of a simple linear algebraic group $\overline{G}$, we will use the information in Corollary 2.45. For details, see [34, Section 2].

By the following two results of Steinberg, we obtain the complete list of the torsion primes of a simple linear algebraic group:

Lemma 2.44. *[34, Lemma 2.5] If $\overline{G}$ is a reductive linear algebraic group, the torsion primes of $\overline{G}$ are the torsion primes of the root system Σ of $\overline{G}$ and the primes dividing the order of the fundamental group of $\overline{G}$.*

Corollary 2.45. *[34, Corollary 2.7] If $\overline{G}$ is a simple linear algebraic group of adjoint type, beyond that the torsion primes of the root system, $\overline{G}$ has torsion primes only in the following cases: for type A_l, the primes $p|(l+1)$ and for type C_l the prime 2.*

2.6 Zsigmondy's Theorem

We will use the following result of Zsigmondy:

Theorem 2.46. ***(Zsigmondy, 1892)*** *Let a, b be two relatively prime natural numbers with $a > b \geq 1$ and $n \geq 1$. Then:*

1. *There exists a prime p such that p divides $a^n - b^n$ and p does not divide $a^k - b^k$ for any $1 \leq k < n$, except the following cases:*
   - *$n = 1$ and $a - b = 1$,*
   - *$n = 2$ and $a - b$ is a power of 2,*
   - *$n = 6$, $a = 2$, $b = 1$.*
2. *There exists a prime p which divides $a^n + b^n$ and p does not divide $a^k + b^k$ for every $1 \leq k < n$, except the case $n = 3$, $a = 2$, $b = 1$.*

Proof. See [36, P1.7]. □

This result was first proved by Bang in 1886 for the particular case $b = 1$. Zsigmondy proved this stronger version in 1892. For details see [24, Section 2.5, p.88].

2.7 Orders of Maximal Tori in Finite Simple Groups of Lie type

In this section, we collect information about orders of maximal tori in finite simple groups of Lie type.

Definition 2.47. *Let G be a finite simple group of Lie type given by*

$$G = O^{p'}(\overline{G}_\sigma)$$

where $\overline{G}$ is an adjoint type simple linear algebraic group and σ is a Frobenius map on $\overline{G}$. Let $\overline{T}$ be a maximal torus of $\overline{G}$. A subgroup $T = \overline{T} \cap G$ is called a maximal torus of the finite group G.

For a simple linear algebraic group over an algebraically closed field, all maximal tori are conjugate. However, for a simple group of Lie type over a finite field, there even exists maximal tori with different orders.

Example 2.48. Let $G = PSL_2(5)$. We will construct two non-isomorphic maximal tori with orders $\frac{q-1}{2} = 2$ and $\frac{q+1}{2} = 3$.

Let $\overline{G}$ be $PGL_2(k)$ where k is the algebraic closure of $\mathbb{F}_5$. Consider the Frobenius map $\sigma : (x_{ij}) \longrightarrow (x_{ij}^5)$. Here, $G = O^{5'}(\overline{G}_\sigma)$. First consider the subgroup $T_1 = \{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} Z : \quad \lambda \in \mathbb{F}_5^*\}$ of $PSL_2(5)$. Clearly, T_1 is a torus. Next, we prove that T_1 is a maximal torus of G.

Now, $\overline{T} = \{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} Z : \quad \alpha, \beta \in k^*\}$ is a maximal torus of $PGL_2(k)$. Since k is algebraically closed, we can write $\overline{T} = \{\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Z : \quad a, b, c \in k^*\}$

where $ca = \alpha, cb = \beta, c^2 = \alpha\beta$ and $ab = 1$. Hence we can write

$$\overline{T} = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Z : \quad a, b \in k^*, ab = 1\}.$$

(So, over an algebraically closed field k, $PGL_n(k)$ and $PSL_n(k)$ are the same group).

Then $\overline{T} \cap G = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Z : \quad a^5 = a, \quad b^5 = b, \quad ab = 1\} = T_1$ since $b = a^{-1}$. By this observation, T_1 is a maximal torus of G.

The map

$$\begin{aligned} \psi : \mathbb{F}_5^* &\longrightarrow \quad T_1 \\ \lambda &\longrightarrow \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} Z \end{aligned}$$

is a group homomorphism with kernel $\{1, -1\}$. Hence, $|T_1| = \frac{5-1}{2} = 2$.

But $|PSL_2(q)| = \frac{(5^2-1)(5^2-5)}{2(5-1)} = \frac{5(5^2-1)}{2} = 60 = 2^2.3.5$.

Now, a Sylow 2-subgroup of $PSL_2(5)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Every involution in $PSL_2(5)$ is contained in a maximal torus of order 2, and the elements of order 5 are unipotent. There are also elements of order 3 in G, which are also semisimple since $(3, 5) = 1$. So, an element of order 3 in G must be contained in a maximal torus of $PSL_2(k)$. In particular, let $s = \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix} Z$. Now, $s^3 = Z$, that is, s is an element of order 3. The eigenvalues of s are λ and λ^2 where $\lambda^2 + \lambda + 1 = 0$.

By basic linear algebra, we can compute that

$$s = \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix} Z = \begin{pmatrix} 3\lambda + 2 & 2\lambda \\ 2\lambda + 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{pmatrix} \begin{pmatrix} 1 & 3\lambda \\ 3\lambda + 4 & 3\lambda + 2 \end{pmatrix} Z = P^{-1}AP$$

where A is the diagonal matrix consisting of eigenvalues of s. Now, $AZ \in \overline{T} = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} Z : \quad a, b \in k^*, ab = 1\}$. Hence, $s \in P^{-1}\overline{T}P$, which is another maximal torus of $PGL_n(k)$. By the result Lemma 4.6 of Steinberg, s is contained in a σ-

invariant maximal torus T' of $PGL_n(k)$. But $s \in T'_\sigma$, so, $T_2 := T' \cap G$ has order divisible by 3. By elementary computations, we observe that $|T_2| = 3$. Indeed, we know s, which is an element of order 3, is contained in T_2 and T_2 is abelian. Then $T_2 \leq C_{PSL_2(5)}(s)$. But $PSL_2(5) \cong A_5$ and we know by Theorem 3.6 that the centralizer of an element of order 3 in A_5, has order 3. Hence, $|T_2| = 3$.

Here, in Theorem 2.49, orders of maximal tori in simply connected finite groups of Lie type are given. To obtain the orders of finite simple groups of Lie type, these orders must be divided by the order of the center, which is bounded by $l+1$. We would like to show that centralizers of some finite subgroups consisting of semisimple elements contain an infinite abelian subgroup isomorphic to $Dr_{p_i}\mathbb{Z}_{p_i}$ for infinitely many primes p_i, hence the orders of maximal tori in simply connected case will give us enough information for our purposes.

Theorem 2.49. *[6, Proposition 7, 8, 9] The orders of the maximal tori of the universal central extensions of the finite simple groups of classical Lie type are as follows:*

1. *If $G = A_l(q)$ and T is a maximal torus of G, then*

$$|T| = (\prod_{i=1}^{k}(q^{\mu_i} - 1))/(q-1)$$

   where $\sum_{i=1}^{k} \mu_i = l+1$.

2. *If $G =^2 A_l(q^2)$ and T is a maximal torus of G, then*

$$|T| = (\prod_{\mu_i \ even}(q^{\mu_i} - 1))(\prod_{\lambda_i \ odd}(q^{\lambda_i} + 1))/(q+1)$$

   where $\sum_{i=1}^{k} \mu_i + \lambda_i = l+1$.

3. *If $G = C_l(q)$ and T is a maximal torus of G, then*

$$|T| = (\prod_{i}(q^{\epsilon_i} - 1))(\prod_{j}(q^{\eta_j} + 1))$$

   where $\sum_i \epsilon_i + \sum_j \eta_j = l$.

4. *If $G = D_l(q)$ and T is a maximal torus of G, then*

$$|T| = (\prod_i (q^{\epsilon_i} - 1))(\prod_j (q^{\eta_j} + 1))$$

where $\sum_i \epsilon_i + \sum_j \eta_j = l$.

5. *If $G =^2 D_l(q)$ and T is a maximal torus of G, then*

$$|T| = (\prod_i (q^{\epsilon_i} - 1))(\prod_j (q^{\eta_j} + 1))$$

where $\sum_i \epsilon_i + \sum_j \eta_j = l$.

6. *If $G = B_l(q)$ with q odd and T is a maximal torus of G, then*

$$|T| = (\prod_i (q^{\epsilon_i} - 1))(\prod_j (q^{\eta_j} + 1))$$

where $\sum_i \epsilon_i + \sum_j \eta_j = l$.

7. *If $G = B_l(q)$ with q even and T is a maximal torus of G, then*

$$|T| = (\prod_i (q^{\epsilon_i} - 1))(\prod_j (q^{\eta_j} + 1))$$

where $\sum_i \epsilon_i + \sum_j \eta_j = l$.

For the orders of possible maximal tori in exceptional groups, we have the following Tables 2.3, 2.4, 2.5, 2.6. In E_6 and E_7, the structures of the maximal tori in the universal covering group is given, that is, $(3, q-1)E_6$ denotes the central extension of E_6 with its Schur multiplier. The information in this table is given in [18]:

By [18, Section 2.8], the list of orders (and the cyclic structure) of maximal tori in $(3, q+1).^2E_6(q)$ is obtained by writing $-q$ instead of q in the list of orders of maximal tori of $(3, q-1).E_6(q)$, given in Table 2.4.

Remark 2.50. Over an algebraically closed field all maximal tori are conjugate. A maximal torus over an algebraically closed field $\overline{k}$ is isomorphic to direct product of finitely many $\overline{k}^*$'s. But over finite fields, we saw that there are maximal tori with even different orders. If a maximal torus T defined over a field k is isomorphic to direct product of finitely many copies k^*, then T is called a maximally split torus. By [25, Proposition 1.2.2] and [26, page 18], if a maximal torus T is defined over k, it splits over a finite Galois extension K of k. Now, $[K : k] < \infty$ and $dimT = dimT'$ is fixed. So, T is contained in a maximally split torus T' over K with $[T' : T] < \infty$. Hence, for each maximal torus T over a finite field $\mathbb{F}_q$, there exists a maximally split torus T' defined over a finite Galois extension of k with $[T' : T] < \infty$ and $|T'| = (q^l - 1)^r$ for some l where r is the dimension of T'.

Table 2.3: Orders of maximal tori in ${}^2B_2(q), G_2(q), {}^2G_2(q), {}^3D_4(q), F_4(q)$

G	**Cyclic structure of maximal tori**
${}^2B_2(q),\ q = 2^{2m+1}$	$q-1$
	$q+\sqrt{2q}+1$
	$q-\sqrt{2q}+1$
${}^2G_2(q),\ q = 3^{2m+1}$	$q-1$
	$(q+1/2)\times 2$
	$q+\sqrt{3q}+1$
	$q-\sqrt{3q}+1$
$G_2(q),\ q \geq 3$	$(q-1)\times(q-1)$
	q^2-1
	$(q+1)\times(q+1)$
	q^2-q+1
	q^2+q+1
${}^3D_4(q)$	$(q-1)\times(q^3-1)$
	$(q+1)\times(q^3+1)$
	$(q^3+1)\times(q-1)$
	$(q^3-1)\times(q+1)$
	$(q^2+q+1)\times(q^2+q+1)$
	$(q^2-q+1)\times(q^2-q+1)$
	(q^4-q^2+1)
$F_4(q),\ q$ odd	$(q-1)\times(q-1)\times(q-1)\times(q-1)$
	$(q-1)\times(q-1)\times(q^2-1)$
	$(q-1)\times(q+1)\times(q^2-1)$
	$(q-1)^2\times(q^2-1)$
	$(q-1)\times(q^3-1)$
	$(q+1)\times(q+1)(q^2-1)$
	$(q^4-1)/(2,q-1)\times(2,q-1)$
	$(q^3+1)\times(q-1)$
	$(q^3-1)\times(q+1)$
	$(q+1)\times(q+1)\times(q+1)\times(q+1)$
	$(q^2+q+1)\times(q^2+q+1)$
	$(q+1)\times(q^2+1)(q+1)$
	$(q+1)\times(q^3+1)$
	$(q^2+1)\times(q^2+1)$
	(q^4+1)
	(q^4-q^2+1)
	$(q^2-q+1)\times(q^2-q+1)$

Table 2.4: Orders of maximal tori in ${}^2F_4(q)$, $(3, q-1).E_6(q)$

G	**Cyclic structure of maximal tori**
${}^2F_4(q),\ q = 2^{2m+1}$	$(q-1) \times (q-1)$
	q^2-1
	$(q-1) \times (q-\sqrt{2q}+1)$
	$(q-1) \times (q+\sqrt{2q}+1)$
	(q^2+1)
	$(q-\sqrt{2q}+1) \times (q-\sqrt{2q}+1)$
	$(q+\sqrt{2q}+1) \times (q+\sqrt{2q}+1)$
	$(q+1) \times (q+1)$
	q^2-q+1
	$q^2-\sqrt{2q^3}+q-\sqrt{2q}+1$
	$q^2+\sqrt{2q^3}+q+\sqrt{2q}+1$
$(3, q-1).E_6(q)$	$(q-1) \times (q-1) \times (q-1) \times (q-1) \times (q-1) \times (q-1)$
	$(q-1) \times (q-1) \times (q-1) \times (q-1) \times (q^2-1)$
	$(q-1) \times (q-1) \times (q^2-1) \times (q^2-1)$
	$(q-1) \times (q-1) \times (q-1) \times (q^3-1)$
	$(q^2-1) \times (q^2-1) \times (q^2-1)$
	$(q-1) \times (q^2-1) \times (q^3-1)$
	$(q-1) \times (q-1) \times (q^4-1)$
	$(q+1) \times (q+1) \times (q^2-1) \times (q^2-1)$
	$(q^2-1) \times (q+1)(q^3-1)$
	$(q-1) \times (q^2+q+1) \times (q^3-1)$
	$(q^2-1) \times (q^4-1)$
	$(q-1) \times (q^5-1)$
	$(q^2-1) \times (q-1)(q^3+1)$
	$(q-1)(q^2+1) \times (q-1)(q^2+1)$
	$(q^2+q+1) \times (q+1)(q^3-1)$
	$(q+1) \times (q+1) \times (q^4-1)$
	$(q+1) \times (q^5-1)$
	$(q^2+q+1) \times (q-1)(q^3+1)$
	$(q^2-1) \times (q^4+1)$
	$(q-1)(q^2+1)(q^3+1)$
	$(q^2+q+1) \times (q^2+q+1) \times (q^2+q+1)$
	$(q+1) \times (q^5+q^4+q^3+q^2+q+1)$
	$(q^2+q+1)(q^4-q^2+1)$
	q^6+q^3+1
	$(q^2-q+1) \times (q^4+q^2+1)$

Table 2.5: Orders of maximal tori in $(2, q-1).E_7(q)$

G	**Cyclic structure of maximal tori**
	$(q-1)\times$ order of a torus of $(3, q-1)E_6(q)$ listed in Table 2.4
$(2, q-1).E_7(q)$	$(q-1)\times(q+1)\times(q+1)\times(q^2-1)\times(q^2-1)$
	$(q-1)\times(q^3-1)\times(q^3-1)$
	$(q-1)\times(q^2-1)\times(q^4-1)$
	$(q^3-1)\times(q+1)(q^3-1)$
	$(q-1)\times(q+1)\times(q+1)\times(q^4-1)$
	$(q-1)\times(q+1)(q^5-1)$
	$(q-1)\times(q^6-1)$
	$(q-1)\times(q^2-1)(q^4+1)$
	$(q^2+q+11)\times(q^2+q+1)\times(q^3-1)$
	$(q^3+1)\times(q^3-1)\times(q+1)$
	$(q^3-1)(q^4-q^2+1)$
	$(q-1)(q^6+q^3+1)$
	$(q^2-q+1)\times(q-1)\times(q^4+q^2+1)$
	$(q^3-1)\times(q^4-1)$
	$(q^5-1)(q^2+q+1)$
	$(q-1)(q^2+1)\times(q^2-1)\times(q^2+1)$
	q^7-1
	$(q^4+1)\times(q-1)(q^2+1)$
	also, the orders obtained by writing $-q$ instead of q in this list

Table 2.6: Orders of maximal tori in $E_8(q)$

G	Cyclic structure of maximal tori
$E_8(q)$	$(q-1)\times$ order of a torus of $(2,q-1)E_7(q)$ listed in Table 2.5
	$(q-1)\times(q^3-1)\times(q^4-1)$
	$(q-1)\times(q^5-1)(q^2+q+1)$
	$(q^2-1)\times(q^2+1)(q-1)\times(q^2+1)(q-1)$
	$(q-1)\times(q^7-1)$
	$(q-1)(q^4+1)\times(q-1)(q^2+1)$
	$(q^2-1)\times(q^2-1)\times(q^2-1)\times(q^2-1)$
	$(q^2-1)\times(q^2-1)\times(q+1)(q^3-1)$
	$(q^2-1)\times(q^2-1)\times(q^4-1)$
	$(q+1)(q^3-1)\times(q+1)(q^3-1)$
	$(q+1)(q^3-1)\times(q^4-1)$
	$(q^4-1)\times(q^4-1)$
	$(q^2-1)\times(q^2-1)\times(q^2+1)\times(q^2+1)$
	$(q^2-1)\times(q+1)(q^5-1)$
	$(q^2-1)\times(q^6-1)$ (two conjugacy classes)
	$(q-1)(q^2+1)\times(q^2+1)(q^3-1)$
	$(q^2-1)\times(q^2-1)(q^4+1)$
	$(q^2+q+1)\times(q^2+q+1)\times(q+1)(q^3-1)$
	$(q+1)(q^2+q+1)(q^5-1)$
	$(q+1)(q^2+1)(q^5-1)$
	$(q+1)(q^7-1)$
	(q^8-1)
	$(q^2-1)\times(q^2+1)\times(q^4+1)$
	$(q^2+1)\times(q^2+1)\times(q^4-1)$
	$(q+1)(q^3-1)(q^4+1)$
	$(q^2+1)(q^6-1)$
	$(q^2-1)(q^2+q+1)(q^4-q^2+1)$
	$(q^2-1)(q^6+q^3+1)$
	$(q^2-q+1)\times(q^2-q+1)\times(q+1)(q^3-1)$
	$(q^2-1)(q^6+1)$
	$(q^2+q+1)\times(q^2+q+1)\times(q^2+q+1)\times(q^2+q+1)$
	$(q^4+q^3+q^2+q+1)\times(q^4+q^3+q^2+q+1)$
	$(q^2+q+1)\times(q^6+q^3+1)$
	$(q^2+1)\times(q^2+1)\times(q^2+1)\times(q^2+1)$
	$(q^2+1)\times(q^6+1)$
	$(q^4+1)\times(q^4+1)$
	$(q^4-q^2+1)(q^2+q+1)\times(q^2+q+1)$
	$(q^4+q^2+1)\times(q^2+q+1)\times(q^2-q+1)$
	$q^8+q^7-q^5-q^4-q^3+q+1$
	q^8-q^4+1
	$q^8-q^6+q^4-q^2+1$
	$(q^4-q^2+1)\times(q^4-q^2+1)$
	also, the orders obtained by writing $-q$ instead of q in this list

CHAPTER 3

FIXED POINTS OF AUTOMORPHISMS IN INFINITE ALTERNATING GROUPS

The structure of centralizers of elements in $Alt(\Omega)$ where Ω is a finite set is well known (See [14, Lemma 2.4], or [21, 3.7]). In this chapter, we investigate the structure of fixed points of automorphisms of $Sym(\Omega)$ and $Alt(\Omega)$ where Ω is an infinite set. We will prove that if α is a periodic element of $Aut(G)$ when $G = Sym(\Omega)$, then $C_G(\alpha)$ contains infinite finitary symmetric group and hence it contains infinite alternating group. Moreover, we will describe the centralizers of all possible type of elements in $Sym(\Omega)$ and show that if Ω is uncountable, the fixed point group of any automorphism of $Sym(\Omega)$ have the same cardinality with $Sym(\Omega)$. By [7, Theorem 8.2.A], if $|\Omega| \neq 6$, every automorphism of $Sym(\Omega)$ is inner and $Aut(Alt(\Omega)) = Sym(\Omega)$. Hence, to find the fixed points of automorphisms of $Alt(\Omega)$, we consider the centralizers of elements in $Sym(\Omega)$.

First we need to summarize the background results which we will use.

3.1 Automorphisms of $Alt(\Omega)$

We will consider the fixed points of automorphisms of alternating groups. But first of all, we will state the following result of Baer:

Theorem 3.1. ***[7, Theorem 8.1A]*** *Let Ω be any set with $|\Omega| > 4$. Then the normal subgroups of $Sym(\Omega)$ are precisely $1, Alt(\Omega), Sym(\Omega)$ and the subgroups of the form $Sym(\Omega, c)$ with $\aleph_0 \leq c \leq |\Omega|$ where $Sym(\Omega, c) = \{x \in Sym(\Omega) : |supp(x)| < c\}$.*

Now, we will show that if $|\Omega| > 6$ every automorphism of $Sym(\Omega)$ is inner. This result remains true for $|\Omega| < 6$ but fails for $|\Omega| = 6$. The proof is based on the following lemmas:

Lemma 3.2. ***[7, Lemma 8.2A]*** *Let $|\Omega| > 6$ and $G = Sym(\Omega)$. Every automorphism ϕ of $Sym(\Omega)$ maps $Alt(\Omega)$ onto itself, and so its restriction to $Alt(\Omega)$ is an automorphism of $Alt(\Omega)$. Moreover, if C is the conjugacy class consisting of all three cycles in $Alt(\Omega)$ then $C^{\phi} = C$.*

Proof. Let $Alt(\Omega) = A$. Since $|\Omega| > 4$, we know that A is simple. Let ϕ be an automorphism of $Sym(\Omega)$. Since A^{ϕ} is isomorphic to A, it is also simple. Now, since both A and A^{ϕ} are normal in G, we have $A \cap A^{\phi} \lhd A$ and $A \cap A^{\phi} \lhd A^{\phi}$. Since both A and A^{ϕ} are simple, either $A = A^{\phi}$ or $A \cap A^{\phi} = 1$. But if $A \cap A^{\phi} = 1$ then $A^{\phi} \in C_{Sym(\Omega)}(A) = 1$ which is not the case. So $A = A^{\phi}$, that is, $\phi \in Aut(A)$.

It remains to show that $C^{\phi} = C$.

Recall that C is the conjugacy class consisting of all 3-cycles. We claim that C is the unique conjugacy class of A consisting of elements of order 3 such that for all $x, y \in C$ we have $|xy| = 1, 2, 3, 5$. First we need to show that product of any 3-cycles necessarily order $1, 2, 3$ or 5. Let $(a_1\ b_1\ c_1), (a_2\ b_2\ c_2)$ be two arbitrary 3-cycles in $Alt(\Omega)$. Then the product

$$(a_1\ b_1\ c_1)(a_2\ b_2\ c_2) = \begin{cases} (a_1\ b_2\ c_2\ b_1\ c_1) & \text{if } a_1 = a_2, b_1 \neq b_2, c_1 \neq c_2 \\ (a_1\ c_1)(b_1\ c_2) & \text{if } a_1 = a_2, b_1 = b_2, c_1 \neq c_2 \\ (a_1\ c_2\ c_1) & \text{if } a_1 = b_2, b_1 = a_2, c_1 \neq c_2 \end{cases}$$

Clearly, $(a_1\ b_2\ c_2\ b_1\ c_1)$ has order 5, $(a_1\ c_1)(b_1\ c_2)$ has order 2 and $(a_1\ c_2\ c_1)$ has order 3.

Let C' be another conjugacy class of A containing elements of order 3 with $C' \neq C$. In symmetric groups each conjugacy class is uniquely determined by the cycle type. Since $C' \neq C$ and C' also contains elements of order 3, each element of C' contains at least two 3-cycles, that is,

$$C' = \{(a_1 b_1 c_1) \dots (a_k b_k c_k) : \ \ a_i, b_i, c_i \in \Omega, \ \ \text{for some } k \geq 2\}.$$

Since $|\Omega| \geq 7$ we can find elements whose product is of the following form:

$$(a_2\ a_5\ a_3)(a_4\ a_6\ a_7)\ldots(a_1\ a_3\ a_7)(a_2\ a_5\ a_4) = (a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ldots).$$

So, there are two elements $x, y \in C'$ such that xy has order greater than or equal to 6. So C is the unique conjugacy class of A consisting of elements of order 3 such that for all $x, y \in C$ we have $|xy| = 1, 2, 3, 5$. Now, since ϕ is an automorphism, it preserves the orders of the products. Then C^ϕ is a conjugacy class conjugacy class of A consisting of elements of order 3 such that for all $x, y \in C$ we have $|xy| = 1, 2, 3, 5$. By the uniqueness of C, we have $C^\phi = C$. Hence, C is fixed under any automorphism of $Sym(\Omega)$. □

Lemma 3.3. ***[7, Lemma 8.2B]*** *Let G be a subgroup of symmetric group on Ω which contains $Alt(\Omega)$ where Ω is a set whose cardinality is greater than 6. If $\psi \in Aut(G)$ fixes each element of $Alt(\Omega)$ then $\psi = id_G$*

Proof. Since $Alt(\Omega)$ is normal in $Sym(\Omega)$, it is necessarily normal in G. Then for every $y \in G$, for every $x \in Alt(\Omega)$ we have $y^{-1}xy \in Alt(\Omega)$. So;

$$y^{-1}xy = (y^{-1}xy)^\psi = (y^\psi)^{-1}x(y)^\psi.$$

Therefore, $y^\psi y^{-1} \in C_G(Alt(\Omega))$. But as $C_{Sym(\Omega)}(Alt(\Omega)) = 1$, we obtain $C_G(Alt(\Omega)) = 1$. So, $y^\psi = y$ for every $y \in G$. □

Now, we need to prove that every automorphism of $Sym(\Omega)$ is inner. In fact we will give the following result due to Schreier and Ulam:

Theorem 3.4. ***[7, Theorem 8.2A]*** *Let $|\Omega| > 6$. Suppose that G satisfies $Alt(\Omega) \leq G \leq Sym(\Omega)$ and let $N = N_{Sym(\Omega)}(G)$. Then for each automorphism ϕ of G there exists $y \in N$ such that*

$$x^\phi = y^{-1}xy \ \textit{ for all } \ x \in G.$$

In particular, every automorphism of $Sym(\Omega)$ is inner.

Proof. For each $\alpha, \beta \in \Omega$ consider the set of 3-cycles including α and β and define $L(\alpha, \beta) = \{(\alpha\beta\gamma) \in C : \gamma \in \Omega - \{\alpha, \beta\}\}$. We saw in the proof of Lemma 3.2 that product of any 3-cycle have order 1,2,3 or 5. Now, unless the intersection of the sets of elements occuring in these cycles have 2 element, the product have order $1, 3$ or 5. Therefore $S := L(\alpha, \beta)$ is a maximal subset of C satisfying

$$\text{if } x, y \in S \text{ and } x \neq y \text{ then } xy \text{ has order 2.}$$

The same calculations show that if $S \subseteq C$ satisfies the above property and contains $(\alpha\beta\gamma)$ then since in any two element of C two points must be common by the same reason, S is a subset of $L(\alpha, \beta), L(\beta, \gamma)$ or $L(\gamma, \alpha)$. Therefore $L(\alpha, \beta)$ are the unique maximal subsets of C satisfying the above property. Therefore, since this property is invariant under automorphisms of $Alt(\Omega)$, Lemma 3.2 shows that $(L(\alpha, \beta))^{\phi} = L(\alpha', \beta')$ for some $\alpha', \beta' \in \Omega$.

Now define $y \in Sym(\Omega)$ by $\alpha^y = \alpha', \quad \beta^y = \beta', \quad \gamma^y = \gamma'$ such that $(\alpha\beta\gamma)^{\phi} = (\alpha'\beta'\gamma')$ for all $\gamma \neq \alpha$ or β. Then

$$\begin{aligned} \psi : G &\longrightarrow Sym(\Omega) \\ x &\longrightarrow yx^{\phi}y^{-1} \end{aligned}$$

is a homomorphism which fixes each element of $L(\alpha, \beta)$. Since $\langle L(\alpha, \beta) \rangle = Alt(\Omega)$ we say ψ acts trivially on $Alt(\Omega)$, so $\psi = id_G$ by Lemma 3.3.

$$x^{\phi} = y^{-1}xy \text{ for every } x \in G \text{ for some } y \in N.$$

In particular, if we take $G = Sym(\Omega)$ we can conclude that every automorphism of $Sym(\Omega)$ is inner. $\square$

3.2 A result on fixed points of automorphisms of finite alternating groups

Our aim is to prove the following result:

Theorem 3.5. *Let $G = Alt\,(\Omega)$ where Ω is a finite set and A be a group of automorphisms of G. If $|G : C_G(A)| \leq n$, then $|G| \leq f(n)$ for some function f*

of n.

In this setting, let α be any element of A, that is, let α be a single automorphism of G. Since $|G : C_G(\alpha)| \leq |G : C_G(A)|$, it is enough to show the result when $|G : C_G(\alpha)| = n$.

We know by [7, Theorem 8.2.A] that the automorphism group of $Alt(\Omega)$ is isomorphic to $Sym(\Omega)$ except the case $|\Omega| = 6$. When $|\Omega| = 6$ the automorphism group of $Alt(\Omega)$ contain $Sym(\Omega)$ and $|Aut(Alt(\Omega)) : Sym(\Omega)| = 2$. Clearly, in this case $|Alt(\Omega)|$ is finite, so, we can assume $|\Omega| \neq 6$.

We will use the following result from [21] to calculate the index of the centralizer of an element in $Sym(\Omega)$ where $|\Omega|$ is a finite set. In fact we will prove a more general result including infinite permutation groups when we prove Theorem 3.13.

Theorem 3.6. *[21, 3.7] Let Ω be a finite set with $|\Omega| = m$. Let $G = Sym(\Omega)$ and α be an element of order n in G. Then $C_G(\alpha) = L_1 \times L_2 \times \ldots \times L_t$ where L_j is isomorphic to $\mathbb{Z}_k \wr S_i$ such that k is the lenght of a cycle and i is the number of cycles of lenght k in α.*

Theorem 3.7. *Let G be a symmetric group of degree n and α be an element of G of order r. If $|G : C_G(\alpha)| \leq \lambda$, then $|G|$ is bounded by a function of λ.*

Proof. We know the structure of centralizers of elements by Theorem 3.6. So, we observe that the order $|C_G(\alpha)| = \Pi_{k=1}^{t} k.i!$ where k is the length of a cycle in α and i is the number of cycles of length k in α. Now, if $|\Omega| = n \geq 3$ this order is largest whenever the number of 0-cycles is largest, that is, $|\Omega \backslash supp(\alpha)|$ must be largest.

Hence, the order of $C_G(\alpha)$ is maximum if α is of the form (ab) for some a and b. Then $|C_G(\alpha)| = 2.(n-2)!$. Then $\frac{n!}{2.(n-2)!} = |S_n : C_G(\alpha)| \leq \lambda$. It follows that $\frac{n(n-1)}{2} \leq \lambda$. Then we obtain $n^2 - n - 2\lambda \leq 0$, so $n_{1,2} = \frac{1 \mp \sqrt{1-4(-2\lambda)}}{2}$. Therefore n is bounded. It follows that $|G| = n!$ is bounded by a function of λ.

□

When Ω is a finite set with $|\Omega| \neq 6$, we have $Aut(Alt\Omega) = Sym(\Omega)$ and $C_{Alt(\Omega)}(\alpha) = C_{Sym(\Omega)}(\alpha) \cap Alt(\Omega)$. It follows that $|Alt(\Omega) : C_{Alt(\Omega)}(\alpha)|$ is bounded by n, then $|Alt(\Omega)|$ is also bounded by a function of n.

3.3 Fixed points of automorphisms of infinite alternating groups

A result similar to Theorem 3.5 can not be true for infinite alternating groups since if a group contains a subgroup of finite index then it must contain a normal subgroup of finite index, but alternating groups are simple. Instead, we will first investigate the structure of the fixed point group of a periodic automorphism of the alternating group.

Theorem 3.8. *Let Ω be an infinite set of cardinality κ. Let $G = Sym(\Omega)$ and α be a periodic element of $Aut(G)$. Then $C_G(\alpha)$ contains an infinite symmetric group isomorphic to $Sym(\kappa)$.*

Proof. We know by Theorem 3.4 that $Aut(G) = Sym(\Omega)$. If $\alpha \in FSym(\Omega)$ then $|supp(\alpha)| < \infty$. Then $|\Omega \backslash supp(\alpha)| = |\Omega| = \kappa$. Now every element of $Sym(\Omega \backslash supp(\alpha))$ is fixed by α, which is an infinite symmetric group isomorphic to $Sym(\kappa)$. Similarly if $|supp(\alpha)| < \kappa$ then $|\Omega \backslash supp(\alpha)| = \kappa$. Hence, if $|supp(\alpha)| < \kappa$, the set $\Omega \backslash supp(\alpha)$ is a subset of Ω with cardinality κ which is fixed by α, that is, $C_G(\alpha)$ contains $Sym(\Omega \backslash supp(\alpha))$, which is an infinite symmetric group isomorphic to $Sym(\kappa)$.

Now, assume α is a periodic element of $Aut(G)$ with $|supp(\alpha)| = \kappa$. We know that α can be written as a product of disjoint cycles. First we need to show that lengths of these cycles are bounded. Now, the least common multiple of the lengths of these cycles is equal to $|\alpha|$ which is finite. So, there are only finitely many numbers $n_1, \dots, n_k$ such that any cycle in α has length n_i for some $i \in \{1, \dots k\}$. Define K_i as the set of cycles of length n_i which occur in the disjoint cycle decomposition of α. As $|\bigcup_{i=1}^{k} K_i| = \kappa$, at least one of the K_i's have cardinality κ. Denote this set by K. Now,

$$K = \{\beta = (a_1 \dots a_k) : \beta \text{ is a cycle of length } k \text{ occuring in } \alpha\}.$$

Observe that for any $\beta \in K$ since β is a cycle occuring in α and α is written as a product of disjoint cycles, we have $\beta^\alpha = \beta$.

Now, the elements of K have all the same length and they commute with

α. Every element of K commutes with α we have $K \leq C_G(\alpha)$. We claim that $Sym(K)$ is contained in $C_G(\alpha)$. Let f be a permutation on K. We have $\alpha^f = \alpha$ since f only changes the place of two disjoint commuting cycles in the decomposition of α. So, all the elements of $Sym(K)$ commute with α, so $C_G(\alpha)$ contains a subgroup isomorphic to $Sym(K)$ where $|K| = \kappa$. □

Corollary 3.9. *Let $G = Alt(\Omega)$ and α be a periodic automorphism of G. Then $C_G(\alpha)$ contains a subgroup which is isomorphic to an infinite alternating group.*

Proof. By Theorem 3.8 we know that $C_{Sym(\Omega)}(\alpha)$ contains an infinite symmetric group $Sym(K)$ where $|K| = |\Omega|$. Observe that $C_{Alt(\Omega)}(\alpha) = C_{Sym(\Omega)}(\alpha) \cap Alt(\Omega)$. So all of the even permutations in $C_{Sym(\Omega)}(\alpha)$ are contained in $C_{Alt(\Omega)}(\alpha)$. Therefore, the alternating group contained in $Sym(K)$ is also contained in $C_{Alt(\Omega)}(\alpha)$, hence $C_{Alt(\Omega)}(\alpha)$ contains an infinite simple group. □

Example 3.10. Corollary 3.9 is not true in general for any torsion-free element of $Aut(Alt(\Omega))$. An automorphism of this form can even be fixed point free. Now $f : n \longrightarrow n + 1$ is a permutation of $\mathbb{Z}$, hence, by Theorem 3.4 f is an automorphism of $Alt(\mathbb{Z})$. Now, for any element in $x \in Alt(\Omega)$ we know that x can be written as a product of finitely many disjoint cycles, that is, say $x = (a_{11} \ldots a_{1n}) \ldots (a_{k1} \ldots a_{km})$. We have $(f^{-1}xf).(a_{ij} - 1) = (f^{-1}x).a_{ij} = (f^{-1}).a_{i,j+1} = a_{i,j+1} - 1$ for $1 \leq i, j \leq n - 1$. Similarly, $(f^{-1}xf).(a_{in} - 1) = (f^{-1}x).a_{in} = (f^{-1}).a_{i,1} = a_{i,1} - 1$.

Consider the image

$$f^{-1}(a_{11} \ldots a_{1n}) \ldots (a_{k1} \ldots a_{km})f = (a_{11} - 1 \ldots a_{1n} - 1) \ldots (a_{k1} - 1 \ldots a_{km} - 1).$$

Denote the sum $a_{11} + \ldots + a_{1n} + \ldots + a_{km} = S$. Now, if $x \in C_G(f)$, that is, if $f^{-1}xf = x$ then the sums of the points occuring in x and $f^{-1}xf$ must be equal.

Observe that the sum of the points occuring in $f^{-1}xf$ is decreased by one for each point in $supp(x)$, hence if $f^{-1}xf = x$ then S must be equal to $S - |supp(x)|$, that is, $supp(x) = \emptyset$. Therefore f only fixes the identity element, so, f is a fixed-point-free automorphism in $Aut(Alt(\Omega))$.

3.4 Main result on automorphisms of infinite alternating groups

We know that any two cycles $(a_1a_2 \dots a_k)$ and $(b_1b_2 \dots b_k)$ of the same lenght k in $Sym(\Omega)$ are conjugate. In fact the element $g = (a_1b_1)(a_2b_2) \dots (a_kb_k)$ satisfy $(a_1a_2 \dots a_k)^g = (b_1b_2 \dots b_k)$. We will observe that this result is not true for cycles of infinite lenght.

Remark 3.11. Observe that every element in $Sym(\Omega)$ can be written as a product of disjoint cycles of at most countable lenght. Although any two cycle of lenght k are conjugate in $Sym(\Omega)$, this is not true for any two infinite cycles of $Sym(\Omega)$.

Consider $f, g \in Sym(\mathbb{Z})$ such that $f : n \longrightarrow n+1$ and

$$g(n) = \begin{cases} 2k+2 & \text{if } n = 2k. \\ 2k+1 & \text{if } n = 2k+1. \end{cases}$$

Now, f and g both can be written as infinite cycles, that is, $f = (\dots -3\ \ -2\ \ -1\ \ 0\ \ 1\ \ 2\ \dots)$ and $g = (\dots -6\ \ -4\ \ -2\ \ 0\ \ 2\ \ 4\ \dots)$.

Assume that they are conjugate, that is, there exists $h \in Sym(\Omega)$ such that $f^h = g$.

Then $fh = hg$. So $hg(n) = fh(n)$ for every $n \in \mathbb{Z}$. Now, if n is odd, we have $hg(n) = h(n) = fh(n) = h(n) + 1$ which implies $1 = 0$. Hence, any two arbitrary infinite cycle in $Sym(\mathbb{Z})$ need not be conjugate.

Theorem 3.12. *Any two infinite cycles $\alpha, \beta \in Sym(\Omega)$ are conjugate iff*

$$card(\Omega \backslash supp(\alpha)) = card(\Omega \backslash supp(\beta)).$$

Proof. Assume that $\alpha = (\dots a_{-2}\ a_{-1}\ a_0\ a_1\ a_2\ \dots)$ and $\beta = (\dots b_{-2}\ b_{-1}\ b_0\ b_1\ b_2\ \dots)$ are two infinite cycles in $Sym(\Omega)$ such that $card(\Omega \backslash supp(\alpha)) = card(\Omega \backslash supp(\beta))$. Then, since the cardinalities are equal, there exists a bijection $\psi : \Omega \backslash supp(\alpha) \longrightarrow \Omega \backslash supp(\beta)$.

Now, construct

$$\begin{aligned} \phi : supp(\alpha) &\longrightarrow supp(\beta) \\ a_i &\longrightarrow b_i \end{aligned}$$

which is a bijection from $supp(\alpha)$ to $supp(\beta)$. Define $\delta : \Omega \longrightarrow \Omega$ such that

$$\delta(x) = \begin{cases} \phi(x) & \text{if } x \in supp(\alpha). \\ \psi(x) & \text{if } x \in \Omega \backslash supp(\alpha). \end{cases}$$

Clearly δ is a bijection from Ω to Ω, that is $\delta \in Sym(\Omega)$. Now,

$$\delta^{-1}\beta\delta(x) = \begin{cases} a_{i+1} & \text{if } x \in supp(\alpha), \text{ that is, } x = a_i \text{ for some } i \in \mathbb{Z} \\ x & \text{if } x \in \Omega \backslash supp(\alpha). \end{cases}$$

Therefore $\delta^{-1}\beta\delta(x) = \alpha(x)$ for every $x \in \Omega$, that is, $\delta^{-1}\beta\delta = \alpha$. Hence, α and β are conjugate.

Conversely, assume that α, β are two conjugate infinite cycles in $Sym(\Omega)$. Then there exists $\gamma \in Sym(\Omega)$ such that $\gamma^{-1}\alpha\gamma = \beta$. To show that $card(\Omega \backslash supp(\alpha)) = card(\Omega \backslash supp(\beta))$, we need to construct a bijection between $\Omega \backslash supp(\beta)$ and $\Omega \backslash supp(\alpha)$. Now, for every $x \in \Omega \backslash supp(\beta)$ we have $\gamma^{-1}\alpha\gamma(x) = \beta(x) = x$ since x is not an element of $supp(\beta)$. Then, we have $\alpha\gamma(x) = \gamma(x)$, that is, $\gamma(x)$ is fixed by α. Now, define

$$\begin{aligned} \Phi : \Omega \backslash supp(\beta) &\longrightarrow \Omega \backslash supp(\alpha) \\ x &\longrightarrow \gamma(x). \end{aligned}$$

We need to prove that Φ is a bijection. Since γ is one-to-one, Φ is necessarily one-to-one. Now, let $y \in \Omega \backslash supp(\alpha)$. Since γ is a bijection of Ω, there exists $z \in \Omega$ such that $y = \gamma(z)$. Consider $\beta(z) = \gamma^{-1}\alpha\gamma(z) = \gamma^{-1}\alpha(y)$ since $y = \gamma(z)$. Since $y \in \Omega \backslash supp(\alpha)$ we have $\alpha(y) = y$, so $\beta(z) = \gamma^{-1}\alpha(y) = \gamma^{-1}(y) = z$. Therefore, Φ is onto, hence $card(\Omega \backslash supp(\alpha)) = card(\Omega \backslash supp(\beta))$. $\square$

So, we proved that any two infinite cycles are not necessarily conjugate. But, if α, β are two disjoint infinite cycles, then since $supp(\alpha) \cap supp(\beta) = \emptyset$ and $card(supp(\alpha)) = card(supp(\beta)) = \aleph_0$. Now, since $supp(\beta) \subseteq \Omega \backslash supp(\alpha)$ and $supp(\alpha) \subseteq \Omega \backslash supp(\beta)$ we have $card(\Omega \backslash supp(\alpha)) = card(\Omega \backslash supp(\beta))$, that is, α and β are conjugate. Therefore disjoint cycles of same lenght in $Sym(\Omega)$ are conjugate.

We have not found a reference or information about the fixed points of automorphisms in infinite alternating groups. The following result follows from [7, Exercise 4.2.4, 4.2.5], and it might be well-known but we will write the proof for convenience. In the proof, we use the argument in [21, 3.7] with allowing the cycle lengths to be infinite.

Proposition 3.13. *Let $G = Sym(\Omega)$ and α be a possibly torsion-free element in G then $C_G(\alpha) = Dr_{k \in \mathbb{N} \cup \{\infty\}} L_k$ where L_k is isomorphic to $H_k \wr Sym(\Omega_k)$ where H_k is either isomorphic to $\mathbb{Z}_k$ if k is the length of one cycle occurring in α or $\mathbb{Z}$ if the length of the cycles are infinite ($k = \infty$) and Γ_k is the set of cycles of length k (or the set of infinite cycles for $k = \infty$) in α.*

Proof. Let $G = Sym(\Omega)$ and α be an automorphism of G. Since Theorem 3.7 in [21] gives the result for the finite case of $|\Omega|$, we can assume that Ω is infinite.

Let $\alpha = \Pi_{k \in \mathbb{N} \cup \{\infty\}} \Pi_{\lambda_{i \in \Gamma_k}} \lambda_{k_i}$ be the cycle decomposition of α in G where for each k the cycles λ_{k_i} denotes cycles of length k occurring in α. Here $k \in \mathbb{N} \cup \{\infty\}$. Let Y_k be the set of points in Ω occuring in a k cycle in α where $k \in \mathbb{N} \cup \{\infty\}$. So Y_k's are a partition of $supp(\alpha)$ into disjoint sets. Let $x \in C_G(\alpha)$. Then since Y_i's consist of points in cycles occurring in α, they are α-invariant, and they are x-invariant also. Now, x can be written as $x = \Pi_{k \in \mathbb{N} \cup \{\infty\}} x_k$ where x_k is the restriction of x to the action on Y_k. Since Y_k's are disjoint, x_i and x_j commute. Now, $x \in C_G(\alpha)$ iff x_i commutes with α_i where α_i is the restriction of α to the action on Y_i for every i. So $C_G(\alpha) = Dr_{k \in \mathbb{N} \cup \{\infty\}} C_{Sym(Y_k)}(\alpha_k)$. So, it is enough to find $C_{Sym(Y_k)}(\alpha)$ where α is an element written as a product of cycles of length $k \in \mathbb{N} \cup \{\infty\}$ for some fixed k.

Let α be the product of $|\Gamma_k|$ cycles of length k for $k \in \mathbb{N} \cup \{\infty\}$. We need to show that $C_G(\alpha) = H \wr Sym(Y_k)$ where H is isomorphic to $\mathbb{Z}_k$ if k is finite and isomorphic to $\mathbb{Z}$ if $k = \infty$.

Let $\alpha = \Pi_{i \in \Gamma_k \, j \leq k}(a_{ij})$.

Now, if $\tau \in Sym(Y_k)$ define τ' by $a_{ij}\tau' = a_{i.\tau,j}$. Then $f : \tau \longrightarrow \tau'$ is a homomorphism from $Sym(Y_k)$ to $C_{Sym(Y'_k)}(\alpha)$ where Y'_k is the set of all elements in $supp(Y_k)$.

Let $\theta_l : a_{li} \longrightarrow a_{li+1}$. Clearly $\theta_l \in C_{Sym(Y'_k)}(\alpha)$.

Now, $W_k = \langle f(\tau), \theta_l : l \in H, \tau \in Sym(Y_k)\rangle$ is isomorphic to $H \wr Sym(Y_k)$.

Conversely if $g \in C_{Y'_k}(\alpha)$, then g permutes the cycles occurring in α. Then there exists $\tau \in Sym(Y_k)$ such that $g\tau'$ fixes every cycle of α. Since the centralizer in $Sym(supp(x))$ of a cycle x is a cyclic group generated by x, we have $(g\tau')\Pi_{l\in\Gamma}\ \theta_l^{k_l} = 1$ where $k_l < |x|$. So, $g = (\tau'\Pi_{l\in\mathbb{N}\cup\{\infty\}}\ \theta_l^{k_l})^{-1}$, that is $g \in W_k$. Therefore $C_{Y'_k}(\alpha) = W_k$ is isomorphic to $H \wr Sym(Y_k)$.

□

In the next two examples we will construct some automorphisms of the symmetric group whose centralizer is of the form $\mathbb{Z}_n \wr Sym(\Omega_\gamma)$ or $\mathbb{Z} \wr Sym(\Omega_\gamma)$ respectively.

Example 3.14. Let $G = Sym(\mathbb{Z})$ and α_n be the automorphism given by $\alpha_n = \Pi_{j\in\mathbb{Z}}(jn\, jn+1 \ldots jn+n-1))$. So, α is written as a product of countably many disjoint cycles of length n, hence by Theorem 3.13 we know that $C_G(\alpha)$ is isomorphic to $\mathbb{Z}_n \wr Sym(\mathbb{Z})$.

By this way, for each $n \in \mathbb{N}$ we can construct a subgroup of symmetric group $Sym(\mathbb{Z})$ which is isomorphic to $\mathbb{Z}_n \wr Sym(\mathbb{Z})$.

Example 3.15. Let $\Omega = \{\pm p^k : p \text{ prime},\ k \in \mathbb{N}\backslash\{0\}\}$ and let Γ be the set of all prime numbers. Now, let $G = Sym(\Omega)$ and α be the automorphism of G given by

$$\alpha = \Pi_{p\in\Gamma}(\ldots - p^2\ \ -p\ \ p\ \ p^2\ \ p^3 \ldots)$$

So, α is written as a product of countably many disjoint infinite cycles, hence by Theorem 3.13 we know that $C_G(\alpha)$ is isomorphic to $\mathbb{Z} \wr Sym(\Gamma)$. Since Γ is countable, $Sym(\Gamma)$ is isomorphic to $Sym(\mathbb{Z})$.

We will show that $card(C_{Sym(\Omega)}(\alpha)) = card(Sym(\Omega))$ for every uncountable set Ω.

Theorem 3.16. *Let Ω be an uncountable set and α be an automorphism of $Sym(\Omega)$. Then $card(C_{Sym(\Omega)}(\alpha)) = card(Sym(\Omega))$.*

Proof. Let Ω be a set with $card(\Omega) = \kappa$ where κ is an uncountable cardinal number. Let $G = Sym(\Omega)$ and α be an automorphism of G. Let Y_k be the set of cycles of lenght k occuring in α and Y_0 be the set of infinite cycles in α. Let

$$S = \{k \in \mathbb{N} : \alpha \text{ contains a cycle of lenght } k\}.$$

By Theorem 3.13 we know that

$$C_G(\alpha) = (Dr_{k\in S}\, \mathbb{Z}_k \wr Sym(Y_k)) \times (\mathbb{Z} \wr Sym(Y_0)).$$

Here, for the elements fixed by α, we write cycles of length 1, we assume the corresponding $\mathbb{Z}_k = 1$.

Now, if $card(supp(\alpha))$ is less than κ, then the cardinality of the set 0-cycles in α is κ, so $C_G(\alpha)$ involves $Sym(\kappa)$ as a direct factor, that is $card(C_G(\alpha)) = 2^\kappa$. If $card(supp(\alpha)) = \kappa$, then since S is countable and $card(supp(\alpha)) = \aleph_0.card(Y_0) + \Sigma_{k\in S}\, k.card_{Y_k}$ at least one of Y_k's have cardinality κ, hence $C_G(\alpha)$ contains $Sym(\kappa)$. Therefore $card(C_{Sym(\Omega)}(\alpha)) = card(Sym(\Omega)) = 2^\kappa$ for every uncountable set Ω. $\square$

Remark 3.17. This is not true when Ω is countable. For example, let $G = Sym(\mathbb{Z})$ and let α be the map from $\mathbb{Z}$ to $\mathbb{Z}$ sending each element to its successor. Clearly α is the infinite cycle $(\ldots -3 \ \ -2 \ \ -1 \ 0 \ 1 \ 2 \ldots)$, that is, $C_G(\alpha)$ is isomorphic to $\mathbb{Z}$ which is countable, but $Sym(\mathbb{Z})$ is uncountable.

Finally we will prove the following result:

Theorem 3.18. *Let Ω be an infinite set with cardinality κ. Denote $G = Sym(\Omega)$ and α be an automorphism of G. Then $C_G(\alpha)$ has a normal series involving infinite simple factors.*

Proof. By Theorem 3.13 we know that $C_G(\alpha) = Dr_{\gamma\in\Gamma} L_\gamma$ where L_γ is isomorphic to $H_\gamma \wr Sym(\Omega_\gamma)$ where H_γ is either isomorphic to $\mathbb{Z}_k$ if k is the lenght of one cycle occuring in α or $\mathbb{Z}$ if the lenght of the cycles are infinite and Ω_γ is the set of cycles of lenght n (or the set of infinite cycles). Now, it is enough to construct such a normal series for one of the direct factors, that is, consider $L_\gamma = H_\gamma \wr Sym(\Omega_\gamma)$.

Denote $B = Dr_{\beta \in Sym(\Omega_\gamma)} H$, that is, the base group B of the wreath product is a direct sum of copies of $\mathbb{Z}$ or $\mathbb{Z}_n$.

By Theorem 3.1 we know that the normal subgroups of $Sym(\Omega_\gamma)$ are exactly $1, Alt(\Omega), Sym(\Omega)$ and the subgroups of the form $Sym(\Omega_\gamma, c)$ with $\aleph_0 \leq c \leq |\Omega|$ where $Sym(\Omega, c) = \{x \in Sym(\Omega) : |supp(x)| < c\}$. We know that under Axiom of Choice the proper class of cardinal numbers are totally ordered (see [22, 2.21]), hence the set of cardinal numbers less than $card(\Omega_\gamma)$ is totally ordered. Now,

$$1 \lhd B \lhd B \leftthreetimes Alt(\Omega_\gamma) \lhd B \leftthreetimes Sym(\Omega_\gamma, \aleph_0) \lhd B \leftthreetimes Sym(\Omega_\gamma, \aleph_1) \ldots \lhd B \leftthreetimes Sym(\Omega_\gamma) = L_\gamma$$

is a normal series of length equal to order type of $2^{card(\Omega_\gamma)}$.

When $c' < c < card(\Omega_\gamma)$, if there are no cardinals between c and c', that is, when c is the successor cardinal of c', then by Theorem 3.1 $Sym(\Omega_\gamma, c')$ is the largest normal subgroup of $Sym(\Omega_\gamma, c)$. Now, we need to show that the factor group $BSym(\Omega_\gamma, c)/BSym(\Omega_\gamma, c')$ is an infinite simple group. Consider

$$\begin{aligned} BSym(\Omega_\gamma, c)/BSym(\Omega_\gamma, c') &\cong BSym(\Omega_\gamma, c)Sym(\Omega_\gamma, c')/BSym(\Omega_\gamma, c') \\ &\cong Sym(\Omega_\gamma, c)/(Sym(\Omega_\gamma, c) \cap BSym(\Omega_\gamma, c')) \\ &\cong Sym(\Omega_\gamma, c)/Sym(\Omega_\gamma, c')(B \cap Sym(\Omega_\gamma, c)). \end{aligned}$$

by the Third Isomorphism Theorem and the Dedekind Modular Law. But since $B \cap Sym(\Omega_\gamma, c)$ is identity, we have

$$BSym(\Omega_\gamma, c)/BSym(\Omega_\gamma, c') \simeq Sym(\Omega_\gamma, c)/Sym(\Omega_\gamma, c')$$

which is isomorphic to an infinite simple group.

Hence, $C_{Sym(\Omega)}(\alpha)$ has a normal series involving infinite simple factors. $\square$

Remark 3.19. By Theorem 3.1 we know that for any infinite successor cardinal $\aleph_{\alpha+1}$ the group $Sym(\Omega, \aleph_{\alpha+1})/Sym(\Omega, \aleph_\alpha)$ is simple. This is not the case when c is a limit cardinal, that is, $c = \aleph_\beta$ where β is a limit ordinal. In this case, for any cardinal $a < c$ there exist b greater than a and less than c. Hence the factor group $Sym(\Omega, c/Sym(\Omega, a)$ contains a normal subgroup of the form $Sym(\Omega, b/Sym(\Omega, a)$ by Theorem 3.1. So, in the normal series, the factor groups $Sym(\Omega, c/Sym(\Omega, a)$ where c is a limit cardinal can never be simple.

CHAPTER 4

CENTRALIZERS OF FINITE SUBGROUPS IN LINEAR SIMPLE LOCALLY FINITE GROUPS

In this chapter we study centralizers of finite subgroups consisting of semisimple elements in linear simple locally finite groups. Hartley asked Question 1.1 for non-linear simple locally finite groups. Naturally one wonders about linear groups, that is, it is natural to ask the following:

Question 4.1. *Is the centralizer of a finite subgroup in a linear simple locally finite group necessarily infinite?*

It is easy to see that the linear version of Hartley's question has a negative answer. In fact, the following observation shows that in a linear simple locally finite group, we can always find finite subgroups with trivial centralizer:

Remark 4.2. By the result of Belyaev, Borovik, Hartley-Shute and Thomas, we know that a linear simple locally finite group is a Chevalley or twisted Chevalley group over a locally finite field. By Theorem 2.31, a linear simple locally finite group is a subset of the fixed points of powers of a Frobenius map in a simple linear algebraic group. Here, we can first find the centralizers in the linear algebraic group, and then intersect with the fixed points of the Frobenius maps. A linear algebraic group is an affine variety and the centralizers of elements are closed subsets. By [30, Section 1.1, p.90], the closed subsets of an algebraic variety satisfy descending chain condition. Now, let G be an adjoint type simple linear algebraic group (it has trivial center), let $g_1 \in G$ and $C_1 = C_G(g_1)$. Since G has trivial center, $G \neq C_1$. Choose $g_2 \in G \backslash C_1$ and let $C_2 = C_G(g_1, g_2)$. Now, since

$g_1 \notin C_2$, we have $C_1 \neq C_2$. Since $C_2 \neq Z(G)$, there exists an element $g_3 \in G$ such that $C_G(g_3) \not\geq C_2$. Therefore, $C_G(g_1, g_2, g_3) = C_G(g_1, g_2) \cap C_G(g_3)$ is a proper subgroup of $C_G(g_1, g_2)$. Denote $C_3 = C_G(g_1, g_2, g_3)$. Assume C_n is constructed and is non-trivial. Since C_n is not equal to the center, there exists $g_{n+1} \in G$ such that $C_G(g_{n+1}) \not\geq C_n$. Now, denote $C_{n+1} = C_G(g_{n+1}) \cap C_n$. Clearly, C_{n+1} is a proper subgroup of C_n. Here, all C_i's are closed subsets of G. Then the chain $G > C_1 > C_2 > \ldots$ must terminate at finitely many steps. By the constructions of C_i's, if the last element C_m of the chain is not equal to the center, we can always construct a proper closed subgroup C_{m+1}. Hence the last element of the chain is necessarily 1. Then, if G is an adjoint type simple linear algebraic group, we can always construct a finite subgroup with trivial center, that is, a linear simple locally finite group has a finite subgroup with trivial center. So, there exists a finite subgroup with trivial center, that is, Question 4.1 is answered negatively.

Another way of seeing this is the following: It is easy to show that, if $G = \mathcal{B}(k)$ is a locally finite, simple group of Lie type $\mathcal{B}$ over an infinite locally finite field k of characteristic p, and $F = \mathcal{B}(\mathbb{F}_p)$, then $C_G(F) = 1$. Indeed, F contains elements $\chi_r(1)$ and $\chi_{-r}(1)$ for every positive negative root r in the root system of G. Then $C_G(F)$ consists of elements commuting with $\chi_r(1)$ and $\chi_{-r}(1)$ for every positive root r. Then $C_G(F) = Z(G) = 1$.

But if the subgroup A itself is abelian, clearly $C_{\overline{G}}(A) \geq A$. In linear case, we study the centralizers of finite abelian subgroups. In Section 4.3 we will see that centralizer of even a single unipotent element can easily fail to contain infinitely many elements of distinct prime orders. So, we consider centralizers of finite abelian subgroups consisting of semisimple elements.

4.1 Centralizers of d-abelian subgroups in simple locally finite groups of Lie type

We start with the definition of a d-abelian subgroup of a simple linear algebraic group:

Definition 4.3. *Let $\overline{G}$ be a simple linear algebraic group. A finite abelian subgroup A consisting of semisimple elements of $\overline{G}$ is called a* **d-*abelian*** *subgroup if it satisfies one of the following:*

1. *The root system associated with $\overline{G}$ has type A_l and Hall-π-subgroup of A is cyclic where π is the set of primes dividing $l+1$*

2. *The root system associated with $\overline{G}$ has type B_l, C_l, D_l or G_2 and the Sylow 2-subgroup of A is cyclic.*

3. *The root system associated with $\overline{G}$ has type E_6, E_7 or F_4 and the Hall-$\{2,3\}$-subgroup of A is cyclic.*

4. *The root system associated with $\overline{G}$ has type E_8 and the Hall-$\{2,3,5\}$-subgroup of A is cyclic.*

Here, since A is a finite abelian group, it has Hall-$\{\pi\}$-subgroups for every set of primes π.

In this section, our aim is to prove the following result:

Theorem 4.4. *Let G be a locally finite simple group of Lie type defined over an infinite locally finite field of characteristic p. Let A be a d-abelian subgroup of G. Then $C_G(A)$ contains an abelian subgroup isomorphic to $Dr_{p_i}\mathbb{Z}_{p_i}$ for infinitely many distinct primes p_i.*

We will use the following result to see whether an abelian subgroup A of $\overline{G}$ is contained in a maximal torus of $\overline{G}$ or not.

Theorem 4.5. ***(Steinberg, [34, Corollary 2.25])*** *Let $\overline{G}$ be a connected reductive linear algebraic group over an algebraically closed field of characteristic p and A a commutative subgroup consisting of semisimple elements. Write $A/A^\circ.(A\cap Z(\overline{G}))$ as*

$$A/A^\circ.(A\cap Z(\overline{G}))\cong \mathbb{Z}_{n_1}\times\mathbb{Z}_{n_2}\times\ldots\times\mathbb{Z}_{n_k}$$

where $n_i|n_{i+1}$.

Let ρ be the number of n_i's which are divisible by the torsion primes of $\overline{G}$.

1. *If $\rho \leq 1$ then A is a contained in a maximal torus of $\overline{G}$.*

2. *If $\rho \leq 0$ then $C_G(A)$ is connected and simply connected in $\overline{G}$.*

3. *If $\overline{G}$ is simply connected, then the values of ρ in (1) and the first part of (2) may be increased by 1.*

By this result, we will see that a d-abelian subgroup A of $\overline{G}$ is always contained in a maximal torus. The following result of Steinberg shows that, A is contained in a σ-invariant maximal torus also.

Lemma 4.6. ***(Steinberg, [32, Lemma 5.9])*** *Let $\overline{G}$ be a connected linear algebraic group and σ be a Frobenius map on $\overline{G}$ and A be a subset of $\overline{G}$ such that $a^\sigma = a$ for every $a \in A$ and contained in a maximal torus. Then A is contained in a maximal torus T which is invariant under σ, that is, $T^\sigma = T$.*

Now, we present an example of a non-d-abelian subgroup A such that $C_{\overline{G}}(A) = A$.

Example 4.7. Let $\overline{G}$ be the adjoint group $A_1(K) = PSL_2(K)$ defined over an algebraically closed field K of odd characteristic. For $\lambda^2 = -1$, consider the subgroup A of $PSL(2, K)$ generated by the elements

$$x = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} Z \text{ and } y = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix} Z.$$

The subgroup $A = \langle x, y \rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. $\overline{G}$ has Lie rank $l = 1$. The order of A is 4 and $l + 1 = 2$. Therefore A is not d-abelian in $PSL_2(K)$ as A (the Sylow-2-subgroup of A) is not cyclic. In particular, A is not contained in a maximal torus of $PSL_2(K)$.

Here, one can easily observe that $C_G(A) = A$. In this case $|C_G(A)| = 4$, hence $C_G(A)$ does not contain infinitely many elements of distinct prime orders. In fact, in Section 4.2, we will observe that for every n, there exists a non-d-abelian subgroup A of $PSL_n(k)$ with $C_{PSL_n(k)}(A) = A$.

We will use the following consequence of Theorem 2.46. It is proved by Hartley in [12, Lemma 2.5], but we will give a different proof.

Lemma 4.8. *Let $\overline{G}$ be an adjoint type simple linear algebraic group, σ be a Frobenius map on $\overline{G}$and $\overline{T}$ be a σ-invariant maximal torus of $\overline{G}$. Let*

$$G = \bigcup_{i=1}^{\infty} O^{p'}(\overline{G}_{\sigma^{n_i}})$$

where $n_i | n_{i+1}$, and $T = \bigcup_{i=1}^{\infty}(\overline{T} \cap O^{p'}(\overline{G}_{\sigma^{n_i}}))$. Then T contains infinitely many elements of distinct prime orders.

Proof. Let $\overline{G}$ be an adjoint type simple linear algebraic group, σ be a Frobenius map on $\overline{G}$and $\overline{T}$ be a σ-invariant maximal torus of $\overline{G}$. Let

$$G = \bigcup_{i=1}^{\infty} O^{p'}(\overline{G}_{\sigma^{n_i}})$$

where $n_i | n_{i+1}$, and $T = \bigcup_{i=1}^{\infty}(\overline{T} \cap O^{p'}(\overline{G}_{\sigma^{n_i}}))$. We first need to consider the orders of $T_i = \overline{T} \cap O^{p'}(\overline{G}_{\sigma^{n_i}})$. Observe that, since $n_i | n_{i+1}$, we have $T_i \leq T_{i+1}$. Here, by Definition 2.47 of a maximal torus of a finite simple group of Lie type, T_i is a maximal torus of $G_i = O^{p'}(\overline{G}_{\sigma^{n_i}})$. Recall that G_i is a simple group of Lie type over a finite field of size q^{n_i}.

Now, by the results in Section 2.7, we know the cyclic structures of possible maximal tori in finite simple groups of Lie type. We observe that for a maximal torus T_i of a finite simple group of Lie type G_i has order $f(q^{n_i})$ where f is one of the polynomials given in these results.

We observe from Theorem 2.49 and Tables 2.3, 2.4, 2.5 that there are 4 possibilities for $f(q)$:

1. **$f(q)$ is divisible by $q^k - 1$ for some $k \in \mathbb{N}$:** In this case, for each $i \in \mathbb{N}$ we have $f_{q^{n_i}}$ is divisible by $(q^{kn_i} - 1)$. By Theorem 2.46, for each power m of q, there exists a prime p such that p divides $q^m - 1$ and p does not divide $q^s - 1$ for any $1 \leq s < m$. Hence, for each i, there exists a prime $p_i | q^{kn_i} - 1$ such that p_i divides $q^{kn_i} - 1$ and p does not divide $q^s - 1$ for any $1 \leq s < kn_i$. Since for each i there exists such p_i dividing $|T_i|$, the union $T = \bigcup_i^{\infty} T_i$ contains infinitely many elements of distinct prime orders.

2. **$f(q)$ is divisible by $q^k + 1$ for some $k \in \mathbb{N}$:** In this case, we will apply

Theorem 2.46 for $q^{k.n_i} + 1$ and obtain the same result.

3. $f(q)$ **is divisible by** $\frac{q^{mk}-1}{q^m-1}$ **for some** $m, k \in \mathbb{N}$**:** In particular, m may be 1. In this case we use Theorem 2.46 for $q^{mkn_i} - 1$. Then we conclude that there exists a prime $p_i | q^{mkn_i} - 1$ such that p_i divides $q^{mkn_i} - 1$ and p does not divide $q^s - 1$ for any $1 \leq s < mkn_i$. Hence, p_i can not divide $q^m - 1$, therefore, $p_i | \frac{q^{mk}-1}{q^m-1}$.

4. **The cases where we can not use Theorem 2.46 directly:** These are one of the non-split maximal tori in $G =^2 B_2(q)$, $^2F_4(q)$ where $q = 2^{2m+1}$, and $f(q) = q + \sqrt{2q} + 1$ or $q^2 \pm \sqrt{2q^3} + q \pm \sqrt{2q} + 1$, the non-split maximao tori in $G =^2 G_2(q)$ where $q = 3^{2m+1}$ and $f(q) = q \pm \sqrt{3q} + 1$ and $G = E_8$ and $f_q = q^8 + q^7 - q^5 - q^4 - q^3 + q + 1$. In fact, the proof of the theorem for this case follows in all the cases, but to see the orders explicitly we wrote them separately. Here, by Theorem 2.31, G is a linear simple locally finite group over an infinite locally finite field K of characteristic p. By Remark 2.50 a maximal torus T over the locally finite field K splits over a finite Galois extension L of K, say $|L : K| = l < \infty$. Here, since L is a finite extension of K, it is a locally finite field. We have $L = \bigcup L_i$ where L_i's are finite subfields of L with $[L_i : L_i \cap K] \leq l$. Now, $L_i \cap K$ is a finite field of size q_i and $|L_i^*| = q_i^l - 1$. Since , $L^* = \bigcup L_i^*$, by Theorem 2.46, for each i the polynomial $q_i^l - 1$ has a primitive prime factor, so L^* contains infinitely many elements of distinct prime orders. Since T' is isomorphic to a direct product of finitely many L^*, it contains an infinite abelian subgroup isomorphic to $Dr_{p_i}\mathbb{Z}_{p_i}$ for infinitely many primes p_i. Now, $[T' : T] < \infty$, since $[L : K] < \infty$ and $dim\overline{T} = r \leq \infty$. So, T contains an infinite abelian subgroup A isomorphic to $Dr_{p_i}\mathbb{Z}_{p_i}$ for infinitely many primes p_i.

□

Remark 4.9. Let G be a locally finite simple group of Lie type over an infinite locally finite field K of characteristic p. In fact, for any non-trivial torus T of G, the same result follows by the following: Recall that, by Theorem 2.31, there exists an adjoint type simple linear algebraic group $\overline{G}$, a Frobenius map σ on $\overline{G}$

and a sequence of integers $n_i|n_{i+1}$ such that

$$G = \bigcup_{i=1}^{\infty} O^{p'}(\overline{G}_{\sigma^{n_i}}).$$

Let $\overline{T}$ be a σ-invariant torus of $\overline{G}$. Then a torus T of G is $T = \bigcup_{i=1}^{\infty}(\overline{T} \cap O^{p'}(\overline{G}_{\sigma^{n_i}}))$.

By [26, page 18], T splits over a finite Galois extension L of K, hence, there exists a torus T' containing T such that $|T' : T|$ is finite and T' contains infinitely many elements of distinct prime orders by the same argument in Lemma 4.8.

The following result is an immediate consequence of Remark 4.9:

Corollary 4.10. *Let F be a subgroup of G. If $C_{\overline{G}}(F)$ contains a non-trivial σ-invariant torus T of $\overline{G}$ (which is not necessarily maximal), then $C_G(F)$ contains an infinite abelian subgroup isomorphic to $Dr_{p_i}\mathbb{Z}_{p_i}$ for infinitely many distinct primes p_i.*

Now, we are ready to prove Theorem 4.4:

Proof of Theorem 4.4

Proof. Let G be an infinite simple locally finite group of Lie type over an infinite locally finite field K of characteristic p. Then, by Theorem 2.31, there exists a simple linear algebraic group $\overline{G}$ of adjoint type, a Frobenius map σ on $\overline{G}$ and an infinite sequence of integers $n_i|n_{i+1}$ for $i = 1, 2, 3, \ldots$ such that $G = \bigcup_{i=1}^{\infty} G_i$ where $G_i = O^{p'}(\overline{G}_{\sigma^{n_i}})$. Denote $\overline{G}_{\sigma^{n_i}} = H_i$. If $x \in H_i$, that is, $x^{\sigma^{n_i}} = x$, then $x^{\sigma^{n_{i+1}}} = x$ as $n_i|n_{i+1}$. Therefore, $H_i \leq H_{i+1}$ and the union $H = \bigcup_{i=1}^{\infty} H_i$ of H_is form an ascending chain of subgroups of $\overline{G}$. Hence H is a subgroup of $\overline{G}$. By [4], Section 7.1, we know that $|H_i/G_i|$ is bounded by $l + 1$.

Claim $O^{p'}(H) = G = \bigcup_{i=1}^{\infty} G_i$.

Recall that $O^{p'}(H_i)$ is the subgroup generated by all p-elements of H_i. Let $x \in O^{p'}(H)$. There exist p-elements $g_1, ..., g_k \in H$ such that $x = g_1 \ldots g_k$. Since the elements $g_j \in H = \bigcup_{i=1}^{\infty} H_i$, there exists some i such that $g_j \in H_i$ for all $1 \leq j \leq k$. Since g_js are p-elements and there exists some i such that $g_j \in H_i$ for all $1 \leq j \leq k$, the elements g_j's are contained in $O^{p'}(H_i) = G_i \leq G$ for all $1 \leq j \leq k$. Hence $x \in G$.

Conversely, if $x \in G = \bigcup_{i=1}^{\infty} G_i$ then $x \in G_j$ for some j, so $x \in O^{p'}(H_j) = G_j$. Then x can be written as a product of p-elements in H_j, hence in H. So, $x \in O^{p'}(H) = G$.

Claim $|H/G| \leq l+1$.

Now $H = \bigcup_{i=1}^{\infty} H_i$ and $G = \bigcup_{i=1}^{\infty} G_i$. Denote $|H_i/G_i| = k_i$. By [4], Section 7.1, we know that k_i depends on the type of the associated Lie algebra and it is bounded by $l+1$. Since all G_j's are constructed from $\overline{G}$, the Lie algebras associated with them are also the same for all j. So, the indices $k_i = |H_i/G_i|$ are all equal for $i = 1, 2, \ldots$, that is, $|H_j/G_j| = k \leq l+1$ for all j. We prove the statement by contradiction. Assume that $|H/G| > l+1$ and let $x_1, \ldots, x_{l+2}$ be $l+2$ distinct coset representatives of G in H. Then there exists some m such that $x_i \in H_m$ for all $i = 1, 2, \ldots l+2$. Without loss of generality, we may assume $x_1 \in G$. Now, $\{x_2, \ldots, x_{l+2}\} \subseteq H - G$, so $x_i \notin G_m$ for all $i = 2, 2, \ldots l+2$. It follows that $|H_m/G_m| > l+1$ which is a contradiction. So, the index $|H/G| \leq l+1$.

We want to show that $C_G(A)$ contains infinitely many elements of distinct prime order. Since

$$|\frac{C_H(A)}{C_G(A)}| = |C_H(A)/(G \cap C_H(A))| = |C_H(A)G/G| \leq |H/G| \leq l+1$$

it is enough to show that $C_H(A)$ contains infinitely many elements of distinct prime order, then it follows that $C_G(A)$ contains infinitely many elements of distinct prime order.

First, we need to calculate ρ for the finite d-abelian subgroup A. The identity component of a linear algebraic group is contained in every closed subgroup of finite index (See [17][7.3]). Since A is a finite subgroup of $\overline{G}$ and 1 is closed, we have $A^\circ = 1$. Since $\overline{G}$ is of adjoint type, $Z(\overline{G}) = 1$. Hence, in our case, $[A/A^\circ.(A \cap Z(\overline{G})) = A$.

Write $A \cong Z_{n_1} \times Z_{n_2} \times \ldots \times Z_{n_k}$ where $n_i | n_{i+1}$.

First case: Let the root system of G have type A_l. Then the root system has no torsion prime. So, the torsion primes of G are the primes which divide the order, $l+1$, of the fundamental group. Denote the set of primes that divide $l+1$ by π. Since the root system of G has type A_l and the Hall π-subgroup of the d-abelian subgroup A is cyclic, the number n_i that are divisible by a torsion prime of $\overline{G}$ is at most 1, that is, $\rho = 1$.

Second case: Let the root system of G have type B_l, C_l, D_l or G_2. If the type of the root system is C_l, the root system has no torsion prime, but 2 is a torsion prime for the fundamental group. If the type of the root system of G is B_l, D_l or G_2 then the only torsion prime of the root system is 2.

Since A is a d-abelian subgroup and the type of the root system of $\overline{G}$ is B_l, C_l, D_l or G_2, the Sylow 2-subgroup of A is cyclic. When A is written as a product of cyclic groups of order n_i, the number of n_is not relatively prime with 2 is at most 1, that is, $\rho \leq 1$.

Third case: Let the root system of G have type E_6, E_7 or F_4. Then the torsion primes of G are 2 and 3. But since A is a d-abelian subgroup, Hall-$\{2,3\}$-subgroup of A is cyclic. Therefore, $\rho \leq 1$. Similarly we can deduce that if $\overline{G} = E_8$ and A is d-abelian, $\rho \leq 1$.

Hence, if A is a d-abelian subgroup, ρ is always less than 1. By Theorem 4.5, A is contained in a maximal torus T' of $\overline{G}$. Now, A is a subset of $\overline{G}$ fixed by σ^{n_1} and contained in a maximal torus T'. By Lemma 4.6, there is a maximal torus $\overline{T}$ of rank $r \geq 1$ containing A which is invariant under σ^{n_1}. Since $n_1 | n_i$ for all i, we have $\overline{T}$ is invariant under σ^{n_i} for all $i = 1, 2, 3, \ldots$. Since $\overline{T}$ is an abelian group containing A, we have $\overline{T} \leq C_{\overline{G}}(A)$.

Now, by Lemma 4.8, since $\overline{G}$ is an adjoint type simple linear algebraic group, σ is a Frobenius map on $\overline{G}$ with

$$G = \bigcup_{i=1}^{\infty} \overline{G}_{\sigma^{n_i}}$$

where $n_i | n_{i+1}$, and $\overline{T}$ is a σ-invariant maximal torus of $\overline{G}$, the subgroup $T = \bigcup_{i=1}^{\infty}(\overline{T} \cap \overline{G}_{\sigma^{n_i}})$ contains infinitely many elements of distinct prime orders. So, $T \leq \bigcup_{i=1}^{\infty}(C_{\overline{G}}(A))_{\sigma^{n_i}}$, that is, $T \leq C_G(A)$, hence $C_G(A)$ contains an infinite abelian subgroup which can be written as $Dr_{i=1}^{\infty}\mathbb{Z}_{p_i}$ for infinitely many primes p_i. $\square$

4.2 Construction of an infinite family of self centralizing finite abelian subgroups in $PSL_n(k)$

In this section we will construct non-d-abelian subgroups of $PSL_n(k)$ for each n.

If G is an infinite locally finite simple group of Lie type, that is, an infinite linear locally finite simple group, then the structure of the centralizer of a finite abelian subgroup necessarily depend on the number of the torsion primes dividing $|A|$. If A is d-abelian, by Theorem 4.4, $C_G(A)$ contains infinitely many elements of distinct prime orders. If A is not d-abelian, it may not be the case. The following result will show that for every n, there exists an abelian subgroup of order n^2 in $PSL_n(k)$ whose centralizer is equal to itself.

Lemma 4.11. *Let $G = PSL_n(k)$ where k is the algebraic closure of the finite field of characteristic p. Assume $(p, n) = 1$. Let*

$$xZ = \begin{pmatrix} 0 & 0 & \dots & & 0 & 1 \\ -1 & 0 & \dots & & 0 & 0 \\ 0 & -1 & \dots & & & \\ & & . & & & \vdots \\ & & & . & & \\ 0 & \dots & & & -1 & 0 \end{pmatrix} Z \in G.$$

If yZ is an element of $C_G(xZ) \backslash C_G(xZ)^\circ$ of order n, then

$$C_G(\langle xZ, yZ \rangle) = \langle xZ, yZ \rangle.$$

Proof. Observe that for every $n \in \mathbb{N}$ the determinant of $x = \begin{pmatrix} 0 & 0 & \dots & & 0 & 1 \\ -1 & 0 & \dots & & 0 & 0 \\ 0 & -1 & \dots & & & \\ & & . & & & \vdots \\ & & & . & & \\ 0 & \dots & & & -1 & 0 \end{pmatrix}$ is equal to 1, hence $x \in SL_n(k)$. So, xZ is an element of $G = PSL_n(k)$.

By Lemma 2.39, x is a regular element of $SL_n(k)$. Then, by Lemma 2.40, xZ is regular in $PSL_n(k)$. Observe that $|xZ| = n$, which is relatively prime with p. So, xZ is semisimple.

Observe that $C_G(xZ) = \{gZ \in G \ : \ [g,x] \in Z\}$ where $Z = Z(SL_n(k)) = \{\alpha.I \ : \ \alpha^n = 1\}$. Then $C_G(xZ) = \bigcup_{k=0}^{n-1} C_k$ where

$$C_k = \{gZ \in G \ : \ [g,x] = \lambda^k.I\}$$

with λ a primitive n-th root of unity, and denote C_k the connected components of the centralizer.

Since xZ is a regular semisimple element of G, the identity component of $C_G(xZ)$ is a maximal torus, that is, $C_G(xZ)^\circ = T$. Since the identity component $C_G(xZ)^\circ$ of $C_G(xZ)$ is a normal subgroup in $C_G(xZ)$, the normalizer $N_G(T)$ contains $C_G(xZ)$. Since the centralizer of a maximal torus in a connected reductive group is itself, we have $C_G(T) = T$.

Observe that $[C_G(x) : \ C_G(x)^\circ] = n$. Let $yZ \in C_1$ where $C_1 = \{gZ \in G \ : [g,x] = \lambda.I\}$. Here $yZ \in C_G(x)/C_G(x)^\circ$.

Claim: $|yZ| = n$.

Assume $(yZ)^k \in C_G(x)^\circ$. Then $[y^k, x] = I$. But since $yZ \in C_1$, we have $[y,x] = \lambda.I$. Here;

$$[y^k, x] = y^{-k}x^{-1}y^k x = y^{-k+1}y^{-1}x^{-1}y(xx^{-1})y^{k-1}x = y^{-k+1}[y,x]x^{-1})y^{k-1}x$$

$$= \lambda.I[y^{k-1}, x].$$

Inductively, we deduce that $[y^k, x] = \lambda^k.I$. But by assumption $[y^k, x] = 1$. Therefore, $\lambda^k = 1$, that is, $k = n$, so $|yZ| = n$.

Hence, there exist an element $yZ \in C_G(x) \backslash C_G(x)^\circ$ such that $|yZ| = n$. Since $C_G(x)^\circ = T$ and $C_G(x) \leq N_G(T)$, the element yZ induces an element w of order n in $N_G(T)/T$, namely the Weyl group.

Recall that since $[xZ, yZ] = Z$, the subgroup $A = \langle xZ, yZ \rangle$ is isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_n$. Here, $C_T(yZ) = C_{C_G(x)^\circ}(yZ)$ is a subgroup of index at most n in $C_G(A)$. Since the primes dividing n are torsion primes, A is not d-abelian ($\rho = 2$).

Our aim is to show that $C_T(yZ)$ has order n. Since yZ induces an element

$w \in W$, we will consider $C_T(w)$. For simplicity, consider the maximal torus

$$T' = \{\begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & . & & \\ & & & . & \\ & & & & \lambda_n \end{pmatrix} Z : \prod_{k=1}^{n} \lambda_k = 1\}.$$

The maximal tori T and T' are conjugate by an element $h \in G$. Now, $w' = w^h \in N_G(T')/T'$ is an element of order n in the Weyl group such that $C_T(w) \cong C_{T'}(w')$.

Now, for $s = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & . & & \\ & & & . & \\ & & & & \lambda_n \end{pmatrix} Z \in C_{T'}(w')$, we have

$$s^{w'} = (\begin{pmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & . & & \\ & & & . & \\ & & & & \lambda_n \end{pmatrix} Z)^{w'} = \begin{pmatrix} \lambda_2 & & & & \\ & \lambda_3 & & & \\ & & . & & \\ & & & \lambda_n & \\ & & & & \lambda_1 \end{pmatrix} Z.$$

Hence, $\lambda_n = \lambda_1 z = \lambda_2 z^2 = \ldots = \lambda_{n-1} z^{n-1}$ for some $z \in Z$. So, for each $z \in Z$, we have a unique element in $C_{T'}(w')$, that is, $|C_T(w)| = |C_{T'}(w')| = |Z| = n$. But $C_T(w)$ has index n in $C_G(A)$, so $|C_G(A)| \leq n^2$. Since $A = n^2$, we have $C_G(A) = A$. $\square$

4.3 Centralizers of unipotent elements in simple locally finite groups of classical Lie type in odd characteristic

In this section, our aim is to obtain information about centralizers of unipotent elements and answer the following question: If G is a simple locally finite group of Lie type over an infinite locally finite field k of characteristic p and $u \in G$ a unipotent element, when does $C_G(u)$ contain infinitely many elements of distinct prime orders?

By Remark 4.9, we deduced that when $C_{\overline{G}}(u)$ contains a σ-invariant maximal torus, then $C_G(u)$ contains infinitely many elements of distinct prime orders. So, we need to analyse when $C_{\overline{G}}(u)$ contains a non-trivial torus.

Recall that an element x of the simple linear algebraic group $\overline{G}$ is called regular if $\dim(C_{\overline{G}}(x)) = rank\overline{G}$. By Proposition 2.36, the centralizers of regular unipotents contain only central semisimple elements, we are interested in the centralizers of irregular unipotent elements in locally finite simple groups of classical Lie type.

First, lets consider an example of an irregular unipotent element in a simple linear algebraic group.

Example 4.12. Let $G = SL(3, \overline{k})$ where $char\overline{k} = p$ and $u_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, $C_G(u_2) = \{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix} : a, b, c, d, e \in \overline{k}, \quad a^2 e = 1\}$. Now, $\dim C_G(u_2) = 4 > 2 = rank(G)$, hence, u_2 is an irregular unipotent.

Observe that $C_G(u_2)$ contains the subgroup

$$T = \{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & e \end{pmatrix} : a, b, c, d, e \in \overline{k}, \quad a^2 e = 1\}$$

which is a torus.

The following theorem indicates that the unipotent classes of $PSL_n(k)$ is in one-to-one correspondence with the unipotent classes of $SL_n(k)$.

Proposition 4.13. ***[5, Proposition 5.1.1]*** *Let $\overline{G}$ be a connected reductive linear algebraic group. The canonical epimorphism $\overline{G} \longrightarrow \overline{G}/Z(\overline{G})$ restricts to a bijective morphism from the unipotent variety of $\overline{G}$ to the unipotent variety of $\overline{G}/Z(\overline{G})$ and induces a bijection between the unipotent classes of $\overline{G}$ and the unipotent classes of $\overline{G}/Z(\overline{G})$.*

Remark 4.14. Observe that for a periodic group G with a finite normal subgroup N, the quotient group G/N contains infinitely many elements of distinct prime orders if and only if G contains infinitely many elements of distinct prime orders.

Let $\pi : SL_n(k) \longrightarrow PSL_n(k)$ be the canonical epimorphism. Let uZ be a unipotent element in $PSL_n(k)$. By Proposition 4.13, there exists unique $u \in SL_n(k)$ such that $\pi(u) = uZ$. Now, if $x \in C_{SL_n(k)}(u)$ then $xZ \in C_{PSL_n(k)}(uZ)$. Hence, $\pi(C_{SL_n(k)}(u))$ is a subgroup of $C_{PSL_n(k)}(uZ)$. But

$$\pi(C_{SL_n(k)}(u)) \cong C_{SL_n(k)}(u)/Z.$$

By Proposition 4.13, the conjugacy classes of unipotent elements in $SL_n(k)$ and $PSL_n(k)$ are in one-to-one correspondence. Then, we may consider our unipotent element as an element of $SL_n(k)$. Now, since the order of $Z(SL_n(k))$ is bounded by n, the centralizer of a unipotent element u in $SL_n(k)$ contains infinitely many elements of distinct prime order if and only if the centralizer of uZ in $PSL_n(k)$ contains infinitely many elements of distinct prime order. So, it is enough to prove the result for $C_{SLn(k)}(u)$.

The following result will give us a useful characterization of irregular unipotent elements in classical groups.

Lemma 4.15. ***[27, Lemma 1.2]*** *Let G be a classical algebraic group over an algebraically closed field of characteristic p and W be the underlying module of G. Let $d = \dim W$.*

1. *If $G = A_l, B_l$ or C_l, then the regular unipotent elements in G have a unique Jordan block on W.*

2. *If $G = D_l$ and p is odd then the regular unipotent elements of G have two Jordan blocks of size 1 and $2l-1$.*

3. *If $G = D_l$ and p is even then the regular unipotent elements of G have two Jordan blocks of size 2 and $2l-2$.*

For the relation between d and l in various types of classical groups, see Table 2.1.

By Lemma 4.15, an irregular unipotent element of G must have at least 2 Jordan blocks.

We will prove the following result:

Lemma 4.16. *Let $G = SL_n(k)$ or $PSL_n(k)$ where k is an infinite locally finite field of characteristic p and u be an irregular unipotent element. Then $C_G(u)$ contains infinitely many elements of distinct prime order.*

Proof. By Remark 4.14, it is enough to prove for $G = SL_n(k)$. Let u be an irregular unipotent element in G. Clearly, u is contained in the corresponding linear algebraic group $\overline{G} = SL_n(\overline{k})$. Recall that, by Theorem 2.31, $G = \bigcup_{i=1}^{\infty} O^{p'}(\overline{G}_{\sigma^{n_i}})$ for some Frobenius map σ on $\overline{G}$ and a sequence of natural numbers $n_i | n_{i+1}$. Since $\overline{G} = SL_n(\overline{k})$, the Frobenius map σ is standard. So, it remains to show that $C_{\overline{G}}(u)$ contains a non-trivial torus T. Since σ is standard, T is σ-invariant. Then, by Corollary 4.10, $C_G(u)$ contains infinitely many elements of distinct prime order.

Let $J_u = \begin{pmatrix} J_1 & & & & & & \\ & J_2 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & J_s \end{pmatrix}$ be the Jordan form of u with $l_i \times l_i$ Jordan blocks J_i.

Consider the torus

$$T_2 = \{\begin{pmatrix} \alpha_1 I_{l_1} & & & & & \\ & \alpha_2 I_{l_2} & & & & \\ & & \cdot & & & \\ & & & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \alpha_s I_{l_s} \end{pmatrix} : \prod_{i=1}^{s} \alpha_i^{l_i} = 1\}.$$

Since each $\alpha_i I_{l_i}$ commutes with the corresponding Jordan block J_i in $GL_{l_i}(\overline{k})$, the subgroup T_2 is contained in $C_{\overline{G}}(u)$. Indeed T_2 is a torus of dimension $s-1$ in $\overline{G}$. Since the number of Jordan blocks of an irregular unipotent element in $SL_n(k)$ is greater than 1, we have $s-1>0$. Hence, $C_{\overline{G}}(u)$ contains a non-trivial torus. So, by Corollary 4.10, we conclude that $C_G(u)$ contains infinitely many elements of distinct prime orders.

□

We will use Steinberg and Springer's results on centralizers of unipotent elements in symplectic, orthogonal and unitary groups in odd characteristic.

Let k denote any field of odd characteristic and $\overline{k}$ be its algebraic closure. We start with a finite dimensional vector space V over k of odd characteristic. Let σ_0 be an automorphism of k with $\sigma_0^2 = id$. Let $\langle\ ,\ \rangle$ be a non-degenerate σ_0-sesquilinear form on $V \times V$. We assume that

$$\langle x, y\rangle = \epsilon\sigma_0\langle y, x\rangle$$

where $\epsilon^2 = 1$.

Theorem 4.17. *[32, **Springer-Steinberg**, 2.19] Let X be a nilpotent element in the Lie algebra $g(k)$.*

1. $\sigma_0 \neq id$. There exist vectors e_i with $1 \le i \le s$ and integers $d_i > 0$ such that

- *$X^{d_i}e_i = 0$ and $X_a e_i$ with $a < d_i$ and $1 \leq i \leq s$ form a k-basis for V, and,*
- *there exist non-zero elements $a_i \in k$ such that*

$$\langle X^a e_i, X^b e_j \rangle = 0$$

if $i \neq j$ or $a + b \neq d_i - 1$,

$$\langle X^a e_i, X^{d_i - a - 1} e_i \rangle = (-1)^a a_i.$$

2. $\sigma_0 = id$. There exist vectors e_i, f_j, g_j for $1 \leq i \leq s$ and $1 \leq j \leq t$ and integers $d_i, \delta_j > 0$ such that

- *$X^{d_i}e_i = X^{\delta_j} f_j = X^{\delta_j} g_j = 0$ and $X^a e_h, X^b f_i, X^c g_j$ for $0 \leq a \leq d_k$, $0 \leq b \leq \delta_i$, $o \leq c \leq \delta_j$, $1 \leq h \leq s$, $1 \leq i, j \leq t$ form a k-basis of V*
- *the value of $\langle\ ,\ \rangle$ on a pair of these vectors is 0, except the following ones:*

$$\langle X^a e_i, X^{d_i - a - 1} e_i \rangle = (-1)^a a_i$$

where $a_i \in k^*$,

$$\langle X^a f_j, X^{\delta_j - a - 1} g_j \rangle = \epsilon \langle X^{\delta_j - a - 1} g_j, X^a f_j \rangle = (-1)^a$$

Remark 4.18. ([32, **Springer-Steinberg**, 2.22, 2.23, 2.25]) The algorithm to find the centralizer of a unipotent element U in a simple linear algebraic group G is as follows: We first consider the corresponding nilpotent element X in the Lie algebra g. The Cayley transform $X \longrightarrow (a - X)(a + X)^{-1}$ sends nilpotents to the corresponding unipotent elements in G, where $a \in k$ and $\sigma_0(a) = a^{-1}$. By using the basis depending on the nilpotent element X in the Lie algebra $g(k)$ which is described in Theorem 4.17, we construct a torus S in G. Now, define a k-homomorphism λ from the multiplicative group G_m to G as follows:

$$\lambda(x) X^a e_i = x^{1 - d_i + 2a} X^a e_i$$

$$\lambda(x)X^b f_j = x^{1-\delta_j+2b}X^b f_j$$

$$\lambda(x)X^b g_j = x^{1-\delta_j+2b}X^b g_j$$

Here, $\lambda(G_m)$ is a 1-dimensional k-torus in G^0. Let $Z = C_G(X)$ and $C = C_Z(S)$. Here C is the reductive part of $C_G(U)$, which has our particular interest. Denote the k-rational points of C by $C(k)$.

Theorem 4.19. *([32,* ***Springer-Steinberg 2.25****]) The structure of $C(k)$ is isomorphic to*

$$\prod_{i=1}^{d} U_{h_i}(k)$$

when G is a unitary group;

$$\prod_{i=1,\ i\ odd}^{d} Sp_{r_i}(k) \times \prod_{i=1,\ i\ even}^{d} O(h_i, k)$$

where G is a symplectic group

$$\prod_{i=1,\ i\ even}^{d} Sp_{r_i}(k) \times \prod_{i=1,\ i\ odd}^{d} O(h_i, k)$$

when G is an orthogonal group where r_j denote the number of d_i and δ_i which are equal to j.

For the details see [32, **Springer-Steinberg**, 2.22, 2.23, 2.25].

Theorem 4.20. *([29,* ***Seitz****, Proposition 3.6]) Let u be a unipotent element of $GL(V)$. Write the decomposition of V under the action of u into Jordan blocks*

$$V = \bigoplus_i V_i = \bigoplus_i (J_i)^{r_i}$$

as each V_i is the sum of r_i Jordan blocks of size i.

(i) *A conjugate of u is contained in $Sp(V)$ iff r_i is even whenever i is odd.*

(ii) *A conjugate of u is contained in $O(V)$ iff r_i is even whenever i is even*

(iii) *Two unipotent elements of $Sp(V)$ or $O(V)$ are conjugate iff they are conjugate in $GL(V)$.*

The numbers d_i, θ_i, h_i in Remark 4.18 correspond to the multiplicities of sizes of Jordan blocks of u.

Definition 4.21. *Let G be a simple locally finite group of classical type. A unipotent element u is called a* **d−*unipotent*** *if:*

(i) *G is isomorphic to $PSL_n(k)$ and u is an irregular unipotent, or,*

(ii) *G is isomorphic to $PSp_{2n}(k), PS\Omega_{2n+1}(k)$ or $PSU_n(k)$ and the Jordan form of u contains a repeated Jordan block of size i.*

Remark 4.22. In particular, a regular unipotent element is necessarily not d-unipotent. By [27, Lemma 1.2] a regular unipotent has a single Jordan block in type A_l, B_l and C_l and the sizes of Jordan blocks of a regular unipotent element in type D_l is 1 and $2l-1$ in odd characteristic, which can not be equal. Hence, in all cases, no size of Jordan blocks can be repeated.

Theorem 4.23. *Let G be a simple locally finite group of classical Lie type and u be a unipotent element in G. The following are equivalent:*

1. *u is d-unipotent*
2. *$C_G(u)$ contains infinitely many elements of distinct prime orders.*

Proof. 1. First consider the case $G = PSL_n(k)$. Since for a periodic group G with a finite normal subgroup N, the quotient group G/N contains infinitely many elements of distinct prime orders if and only if G contains infinitely many elements of distinct prime orders, is enough to prove for $G = SL_n(k)$. Let u be a d-unipotent element of $SL_n(k)$. Then it is an irregular unipotent element. By Lemma 4.16, $C_G(u)$ contains infinitely many elements of distinct prime orders.

Conversely, assume that $C_{PSL_n(k)}(u)$ contains infinitely many elements of distinct prime orders and u is not d-unipotent. But, in this case, since $G = PSL_n(k)$, the element u must be a regular unipotent. Then all the semisimple elements of $C_G(u)$ are central. But $Z(PSL_n(k)) = 1$, so it can not contain infinitely many element, we get a contradiction.

2. Let $G = PSp_{2n}(k), P S\Omega_{2n+1}(k)$ or $PSU_n(k)$ and u be a d-unipotent. Then the Jordan form of u contains a block size repeated at least twice. By Theorem 4.19, the reductive part $C(k)$ of $C_G(u)$ involves $Sp_{2n}(k)$ with $n \geq 1$, or $O_{2n+1}(k)$ with $n \geq 1$ or $U_n(k)$ with $n > 1$. These groups all contain k-tori when k is an infinite locally finite field. By Lemma 4.8, $C_G(u)$ contains infinitely many elements of distinct prime orders.

   Conversely, assume that $C_G(u)$ contains infinitely many elements of distinct prime orders where G is a symplectic or orthogonal type simple group over a locally finite field of odd characteristic. We know by Theorem 4.19 that if u is not d-unipotent, then the reductive part of $C_G(u)$ is isomorphic to the direct product of finitely many $O_1(k)$'s, hence it is an elementary abelian 2-group. Hence, if $C_G(u)$ contains infinitely many elements of distinct prime orders then u must be d-unipotent. For unitary groups, if u is not a d-unipotent, then by Theorem 4.19, the reductive part of the centralizer of u is isomorphic to direct product of finitely many $U_1(k)$. Observe that $U_1(k) = \{x \in k : \ x.x^{\alpha} = 1\}$ for some $\alpha \in Aut(k)$ with $|\alpha| = 2$. Since a quadratically closed field can not have an automorphism of order 2, unitary groups over quadratically closed fields do not exist. Hence, we may regard k as a vector space of dimension 2 over a subfield k_0 of k where, k_0 is the fixed field of α.

   We fix the basis $\{1, a\}$ of k over k_0. Write

   $$k = k_0 \cup ak_0.$$

   Let $x \in U_1(k)$. Then $x \in k$ with $x.x^{\alpha} = 1$. Now, if $x \in k_0$, we have $x^{\alpha} = x$, so $x^2 = 1$. Therefore, in this case $x = \pm 1$.

   If $x \notin k_0$, it is an element of ak_0. Hence, there exists $y \in k_0$ such that $x = ay$. Now, $1 = x.x^{al} = aya^{\alpha}y^{\alpha} = (aa^{\alpha})(yy^{\alpha}) = (aa^{\alpha})y^2$ since $y \in k_0$. Then $y^{-2} = aa^{\alpha}$ where a is the fixed basis element. Hence, y^{-1} is a root of the polynomial $T^2 - aa^\alpha \in k[T]$. Therefore, $|U_1(k)| \leq 4$. So, if the centralizer of a unipotent element in a unitary group contains infinitely many elements of distinct prime orders, then it must be a d-unipotent.

$\square$

CHAPTER 5

CENTRALIZERS OF FINITE SUBGROUPS IN NON-LINEAR SIMPLE LOCALLY FINITE GROUPS

In [14, Theorem A2], it is shown that in an infinite locally finite simple group, the centralizer of every element is infinite. In this work we study the following problem of Brian Hartley.

Question 5.1. *Is the centralizer of every finite subgroup, in a simple non-linear locally finite group infinite?*

The counterpart of this question, whether the centralizer of every finite subgroup in a simple non-linear locally finite group, involve an infinite non-linear simple group is resolved negatively by Meierfrankenfeld in [23]. He proved in [23, Corollary 7] that, for a given non-empty set Π of primes, there exists a non-linear, locally finite simple group G such that

1. The centralizer of every non-trivial Π-element has a locally soluble Π-subgroup of finite index.

2. There exists an element whose centralizer is a locally soluble Π-group.

In particular in the above groups there are elements whose centralizers can not involve even finite non-abelian simple groups.

The stronger version of the Hartley's question is the following:

Question 5.2. *Determine all non-linear simple locally finite groups in which centralizer of a finite subgroup has an abelian subgroup isomorphic to a direct product of cyclic groups of order p_i for infinitely many prime p_i.*

By [19, Theorem 4.4], a non-linear simple locally finite group contains a countable non-linear locally finite simple group. Indeed, recall that by Remark 2.22, we know that any finite subgroup of a simple locally finite group is contained in a countable simple group. Hence, in this work, we may assume that the groups we deal with are all countable.

Recall that, by Remark 2.29, if G is a countable non-linear simple locally finite group then either G has a Kegel cover $\{(G_i, N_i) : \quad i \in \mathbb{N}\}$ and G_i/N_i's are alternating groups or G_i/N_i are a fixed type classical groups with unbounded rank parameters. We will prove our results for these two cases separately.

5.1 Centralizers in simple locally finite groups with an alternating type Kegel cover

The following easy result may give idea about the method we will use in the proofs of main results.

Theorem 5.3. *Let Ω be an infinite set and $G = Alt(\Omega)$. Then, for any $x \in G$, the centralizer $C_G(x)$ has an infinite abelian subgroup isomorphic to $Dr_{i=1}^{\infty}\mathbb{Z}_{p_i}$, where p_i is the i-th prime and $\mathbb{Z}_{p_i}$ is the cyclic group of order p_i.*

Proof. Let $\Delta = supp(x)$. Since $|\Delta|$ is finite, the set $\Omega \backslash \Delta$ is infinite. Hence, $Alt(\Omega\backslash\Delta)$ is an infinite alternating group contained in $C_G(x)$.

Since $\Omega\backslash\Delta$ is infinite, it has a countable subset $T = \{t_i : \quad i \in \mathbb{N}\}$.

Now, let

$$\begin{aligned}
\alpha_1 &= (t_1 t_2)(t_3 t_4) \\
\alpha_2 &= (t_5 t_6 t_7) \\
\alpha_3 &= (t_8 t_9 t_{10} t_{11} t_{12}) \\
&\vdots \\
\alpha_i &= (t_{\lambda_{i-1}+1} \dots t_{\lambda_i})
\end{aligned}$$

where p_i is the i-th prime and α_i is a cycle of length p_i. By construction, all α_j's are mutually disjoint, hence they commute pairwise. Therefore, the group

$A = \langle \alpha_i : \ \ i \in \mathbb{N} \rangle$ is an abelian group and is isomorphic to $Dr_{i=1}^{\infty}\mathbb{Z}_{p_i}$. $\square$

Countable non-linear locally finite groups which have a Kegel cover

$$\{(G_i, N_i) : \ \ i \in \mathbb{N}\}$$

with $G_i \cong Alt(n_i)$ and $N_i = 1$ for all $i \in \mathbb{N}$ need special attention.

Remark 5.4. Observe that the groups described in the next result Theorem 5.5 contains the class of simple groups constructed in [19, Chapter 6]. The direct limits of alternating groups are contained in this class and there are $2^{\aleph_0}$ non-isomorphic simple locally finite groups of this type. For details, see [19, Chapter 6].

Theorem 5.5. *Let G be a simple locally finite group which has a local system consisting of alternating groups, that is,*

$$G = \bigcup_{i=1}^{\infty} Alt(n_i)$$

where $Alt(n_i)$ lies in $Alt(n_{i+1})$.

Then for all finite $F \leq G$, the centralizer $C_G(F)$ has an infinite abelian subgroup containing elements of order p_i for any prime p_i.

Proof. Let $F \leq G = \bigcup_{i=1}^{\infty} Alt(n_i)$. Clearly, this group need not be the finitary alternating group since the embeddings need not be trivial.

We construct an infinite ascending sequence of finite subgroups D_i such that for every prime p_i, D_i has an element ρ_i of order p_i and $\rho_i \in C_G(F)$.

Let $D_0 = F$. Assume that D_i is already constructed. Then there exists $n_i \in \mathbb{N}$ such that $D_i \leq Alt(n_i) \leq G$. The subgroup D_i is contained in $Alt(n_{i+k})$ for every $k \geq 1$ and D_i acts on the set $\Omega_{n_{i+k}}$ where $|\Omega_{n_{i+k}}| = n_{i+k}$. Since the number of inequivalent transitive permutation representations of D_i is equal to the number of conjugacy classes of subgroups of D_i, by choosing k sufficiently large, we may assume that the orbits of D_i on $\Omega_{n_{i+k}}$ can be written as

$$O_1 \cup O_2 \cup \ldots O_{p_{i+1}} \cup O'$$

where each O_i gives equivalent transitive permutation representations of D_i on $\Omega_{n_{i+k}}$. Now, let ρ_{i+1} be the cycle $(1\ 2\ 3\ldots p_{i+1})$ and let $\overline{\rho_{i+1}}$ be the image of ρ_{i+1} in $Alt(n_{i+k})$ in the following way; the elements w_j are fixed element in O_j corresponding to the stabilizer of a point which gives the equivalent representation. Then for any $c \in D_i$, we have

$$w_j.c\overline{\rho_{i+1}} = w_{j.\rho_{i+1}}.c$$

and $\overline{\rho_{i+1}}$ fixes O' elementwise.

Now, for any w_j for $j = 1, \ldots p_{i+1}$ and for any $x \in D_i$ we have $w_j.(x\overline{\rho_{i+1}}) = w_{j.\rho_{i+1}} = w_{j+1}$ and $w_j(\overline{\rho_{i+1}}.x) = w_{j.\rho_{i+1}}.x = w_{j+1}.x = w_{j+1}$. Hence, for any $x \in D_i$ we have $x.\overline{\rho_{i+1}} = \overline{\rho_{i+1}}.x$.

Let $D_{i+1} = \langle F, \overline{\rho_1}, \ldots, \overline{\rho_{i+1}} \rangle$ and $A_i = \langle \overline{\rho_1}, \ldots, \overline{\rho_{i+1}} \rangle$ where $A_0 = A_1 = 1$. Then the union $A = \bigcup A_i$ is the required abelian subgroup which is isomorphic to $Dr_{p_i}\mathbb{Z}_{p_i}$ for any prime p_i.

□

Remark 5.6. This theorem works for embeddings or direct limits of alternating groups. In particular, for Hall universal group and the groups constructed in [19, Chapter 6].

Now, we will be able to prove the same result for a wider class of non-linear simple locally finite groups with alternating Kegel factors.

Definition 5.7. *A Kegel cover $\mathcal{K} = \{(G_i, N_i) : \ i \in I\}$ is called a **split Kegel cover** if $C_{G_i/N_i}(KN_i/N_i) = C_{G_i}(K)N_i/N_i$ for every finite subgroup K of G_i.*

In particular, observe that if $(|G_i/N_i|, |N_i|) = 1$, then the Kegel cover is a split Kegel cover.

Corollary 5.8. *Let G be a non-linear locally finite simple group with a split Kegel cover $\mathcal{K} = \{(G_i, N_i) \mid i \in \mathbb{N}\}$ and G_i/N_i is isomorphic to $Alt(n_i)$ for all $i \in \mathbb{N}$. If F is a finite subgroup of G, then $C_G(F)$ contains an abelian subgroup isomorphic to $Dr_{p_i\ prime}\mathbb{Z}_{p_i}$.*

Proof. We may assume that $F \leq G_1$ and let $A_0 = 1$ and $C_0 = F$. Then by Theorem 5.5 there exists n_1 such that $C_{G_{n_1}/N_{n_1}}(FN_{n_1}/N_{n_1})$ contains an element of order p_1. Now, $\mathcal{K}$ is a split Kegel cover, so, $C_{G_{n_1}/N_{n_1}}(FN_{n_1}/N_{n_1}) = C_{G_{n_1}}(F)N_{n_1}/N_{n_1}$. Hence, $C_{G_{n_1}}(F)$ contains an element of order p_1.

Let $C_1 = \langle F, \rho_1 \rangle$ and $A_1 = \langle \rho_1 \rangle$ where ρ_1 is an element of order p_1 in $C_{G_{n_1}}(F)$. C_1 is a subgroup of G_{n_1}, and so there exists n_2 such that $C_{G_{n_2}/N_{n_2}}(C_1N_{n_2})/N_{n_2}$ contains an element of order p_2. We have

$$C_{G_{n_2}/N_{n_2}}(C_1N_{n_2})/N_{n_2} = C_{G_{n_2}}(C_1)N_{n_2}/N_{n_2}$$

since $\mathcal{K}$ is a split Kegel cover. Hence $C_{G_{n_2}}(C_1)$ contains an element ρ_2 of order p_2. Let $C_2 = \langle C_1, \rho_2 \rangle$ and $A_2 = \langle \rho_1, \rho_2 \rangle$. Then $C_1 \leq C_2 \leq C_3 \leq \ldots$ and $A_1 \leq A_2 \leq A_3 \ldots$. Then the union $A = \bigcup A_i$ is the required abelian subgroup of G which is isomorphic to $Dr_{p_i \text{ prime}} \mathbb{Z}_{p_i}$.

□

5.2 Centralizers in simple locally finite groups with a Kegel cover with Lie type factors

In this section we consider non-linear simple locally finite groups G whose Kegel factors are finite simple groups of Lie type. By Remark 2.29, we know that in this case G has a Kegel cover with all G_i/N_i's are a fixed type classical group with unbounded rank parameters.

We need a general notion of a $\mathcal{K}$-semisimple element in a simple locally finite group:

Definition 5.9. *Let G be a non-linear simple locally finite group and*

$$\mathcal{K} = \{(G_i, N_i) \ : \ i \in I\}$$

*be a Kegel cover for G. An element x in G is called $\mathcal{K}$-**semisimple** if $\mathcal{K}$ is a Kegel cover consisting of alternating groups or G_i/N_i is a finite simple group of Lie type and xN_i is a semisimple element of G_i/N_i for every $i \in I$.*

We will use the main idea of the following two results in the proof of the main theorem:

Theorem 5.10. *Let $G = SL_n(k)$, and*

$$x = \begin{pmatrix} \begin{matrix} a_{11} & \dots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \dots & a_{ss} \end{matrix} & & & & 0 \\ & \begin{matrix} a_{11} & \dots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \dots & a_{ss} \end{matrix} & & & 0 \\ & & \cdot & & \\ & & & \cdot & \vdots \\ & & & \begin{matrix} a_{11} & \dots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \dots & a_{ss} \end{matrix} & \\ 0 & 0 & \dots & 0 & A' \end{pmatrix}$$

be an element in G which contains m repeating blocks of size s. Then $C_G(x)$ contains a subgroup isomorphic to $SL_m(k)$.

Proof. Let V be the natural module for G. Write

$$V = W_1 \bigoplus W_2 \bigoplus \dots \bigoplus W_m \bigoplus W'$$

where each W_i is an $\langle x \rangle$-invariant submodule of dimension s and the action of $\langle x \rangle$ on W_i is equivalent, that is, if $\beta_i = \{v_{i1}, \dots, v_{is}\}$ is a basis for W_i, $\;i = 1, 2, \dots m$ then $v_{ij}x = v_{1j}x$ for all j.

Now, consider the linear transformation $\bar{c}$ on $W_1 \bigoplus W_2 \bigoplus \dots \bigoplus W_m$:

$$\begin{aligned} \bar{c} : W_1 \bigoplus W_2 \bigoplus \dots \bigoplus W_m &\longrightarrow W_1 \bigoplus W_2 \bigoplus \dots \bigoplus W_m \\ v_{ij} &\longrightarrow b_{i1}v_{1j} + b_{i2}v_{2j} + \dots + b_{im}v_{mj} \end{aligned}$$

on the direct sum of W_i's for $b_{ij} \in k$.

Now, $\bar{c} = \begin{pmatrix} b_{11}I_{ss} & b_{12}I_{ss} & & \dots & b_{1m}I_{ss} \\ b_{21}I_{ss} & b_{22}I_{ss} & & & \\ \vdots & & . & & \vdots \\ & & & . & \\ b_{m1}I_{ss} & & \dots & & b_{mm}I_{ss} \end{pmatrix}$

A linear transformation which send v_{ij} to a linear combination of $v_{1j}, \dots v_{mj}$ defines a linear transformation of $W_1 \bigoplus W_2 \bigoplus \dots \bigoplus W_m$.

Consider the invertible linear transformations obtained as $\bar{c}$. Let

$$H = \{ \begin{pmatrix} x_{11}I_{ss} & x_{12}I_{ss} & & \dots & x_{1m}I_{ss} \\ x_{21}I_{ss} & x_{22}I_{ss} & & & \\ \vdots & & . & & \vdots \\ & & & . & \\ x_{m1}I_{ss} & & \dots & & x_{mm}I_{ss} \end{pmatrix} \mid x_{ij} \in k, \;\; \det(x_{ij}) \neq 0 \}.$$

H is a subgroup in $SL_{ms}(k)$.

It is easy to see that the map

$$\begin{array}{lccc} \psi : & H & \longrightarrow & GL_m(k) \\ & \begin{pmatrix} b_{11}I_{ss} & b_{12}I_{ss} & & \dots & b_{1m}I_{ss} \\ b_{21}I_{ss} & b_{22}I_{ss} & & & \\ \vdots & & . & & \vdots \\ & & & . & \\ b_{m1}I_{ss} & & \dots & & b_{mm}I_{ss} \end{pmatrix} & \longrightarrow & \begin{pmatrix} b_{11} & b_{12} & & \dots & b_{1m} \\ b_{21} & b_{22} & & & \\ \vdots & & . & & \vdots \\ & & & . & \\ b_{m1} & & \dots & & b_{mm} \end{pmatrix} \end{array}$$

is a group isomorphism between the group of all invertible linear transformations obtained as $\bar{c}$ and $GL_m(k)$.

Hence, the group of all invertible linear transformations obtained as $\bar{c}$ is isomorphic to $GL_m(k)$. One can see that for each element of $\psi(\bar{c}) \in GL_m(k)$ we can define a linear transformation c of V which acts on $W_1 \bigoplus W_2 \bigoplus \dots \bigoplus W_m$ as linear transformations defined as above. We extend the action of $\bar{c}$ from $W_1 \bigoplus W_2 \bigoplus \dots \bigoplus W_m$ to V by assuming $\bar{c}$ acts on W' trivially. We need to show that $cx = xc$. Here,

$$x = \begin{pmatrix} A & 0 & 0 & \dots & & 0 \\ & A & & & & 0 \\ & & \cdot & & & \\ & & & \cdot & & \vdots \\ & & & & A & 0 \\ 0 & 0 & & \dots & 0 & A' \end{pmatrix}$$

where $A = \begin{pmatrix} a_{11} & \dots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \dots & a_{ss} \end{pmatrix}$. Observe that the linear transformation $\bar{c}$ can be written with respect to the ordered basis v_{ij} as

$$c = \begin{pmatrix} b_{11}I_{ss} & b_{12}I_{ss} & & \dots & b_{1m}I_{ss} & 0 \\ b_{21}I_{ss} & b_{22}I_{ss} & & & & 0 \\ \vdots & & \cdot & & \vdots & \vdots \\ & & & \cdot & & \\ b_{m1}I_{ss} & & \dots & & b_{mm}I_{ss} & 0 \\ 0 & 0 & \dots & & 0 & I_{W'} \end{pmatrix}.$$

where I_{ss} denotes the $s \times s$ identity block and $I_{W'}$ is the identity on W'. Then

$$xc = \begin{pmatrix} A & 0 & 0 & \dots & & 0 \\ & A & & & & 0 \\ & & \cdot & & & \\ & & & \cdot & & \vdots \\ & & & & A & 0 \\ 0 & 0 & \dots & & 0 & A' \end{pmatrix} \begin{pmatrix} b_{11}I_{ss} & b_{12}I_{ss} & & \dots & b_{1m}I_{ss} & 0 \\ b_{21}I_{ss} & b_{22}I_{ss} & & & & 0 \\ \vdots & & \cdot & & \vdots & \vdots \\ & & & \cdot & & \\ b_{m1}I_{ss} & & \dots & & b_{mm}I_{ss} & 0 \\ 0 & 0 & \dots & & 0 & I_{W'} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11}A & b_{12}A & \dots & b_{1m}A & 0 \\ b_{21}A & b_{22}A & & b_{2m}A & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ b_{m1}A & b_{m2}A & \dots & b_{mm}A & 0 \\ 0 & 0 & \dots & 0 & A' \end{pmatrix}$$

Similarly

$$cx = \begin{pmatrix} b_{11}I_{ss} & b_{12}I_{ss} & \dots & b_{1m}I_{ss} & 0 \\ b_{21}I_{ss} & b_{22}I_{ss} & & & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ b_{m1}I_{ss} & & \dots & b_{mm}I_{ss} & 0 \\ 0 & 0 & \dots & 0 & I_{W'} \end{pmatrix} \begin{pmatrix} A & 0 & 0 & \dots & 0 \\ & A & & & 0 \\ & & \ddots & & \vdots \\ & & & A & 0 \\ 0 & 0 & \dots & 0 & A' \end{pmatrix}$$

$$= \begin{pmatrix} b_{11}A & b_{12}A & \dots & b_{1m}A & 0 \\ b_{21}A & b_{22}A & & b_{2m}A & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ b_{m1}A & b_{m2}A & \dots & b_{mm}A & 0 \\ 0 & 0 & \dots & 0 & A' \end{pmatrix}.$$

Hence, $xc = cx$.

The set of all invertible c defined as above give us $GL_m(k)$, if we choose the linear transformations with determinant 1, we obtain $SL_m(k) \leq C_{SL_n}(k)$.

□

Theorem 5.11. *Let $G = A_{n-1}(k)$ for some field k of characteristic p, and F be a finite subgroup consisting of semisimple elements of G. If $n > (r-1)|F|^2 + 1$ then the centralizer of F in G contains a subgroup isomorphic to $PSL_r(k)$.*

Proof. Let F be a finite subgroup consisting of semisimple in $G = PSL_n(k)$ where k is a field of characteristic p. The vector space V of $n \times 1$ column vectors over the field k forms a natural module for $k[SL_n(k)]$. Now, V as a $k[SL_n(k)]$

module is irreducible. Let $v, w \in V$. If $\{v, w\}$ is not linearly independent, we have $w \in \langle v \rangle$, that is, $w = \alpha v$. If $B_1 = \{v, v_1, v_2, \ldots v_{n-1}\}$ be a basis for V, then $v \notin \langle v_1, \ldots v_{n-1} \rangle$. Hence, $w = \alpha v \notin \langle v_1, \ldots v_{n-1} \rangle$. Therefore, $B_2 = \{w, v_1, v_2, \ldots v_{n-1}\}$ is also a basis for V. So, there exists $g \in GL_n(k)$ which sends v to w. In fact, any element of the form $g = \begin{pmatrix} \alpha & 0 \\ & A \end{pmatrix}$ where $A \in GL_{n-1}(k)$ will send v to w, hence we may choose $\det A = \alpha^{-1}$ to have $g \in SL_n(k)$.

Now, we assume v and w are linearly independent. Since $n > r|F|^2 + 1$, we may assume that $n > 2$.

So, there exists $u \in V \backslash \langle v, w \rangle$, that is, $\{u, v\}$ and $\{u, w\}$ are linearly independent. Now, there exists bases

$$\beta_1 = \{v, v_1, v_2, \ldots, v_{k-2}, u\}$$

and

$$\beta_2 = \{w, w_1, w_2, \ldots, w_{k-2}, u\}$$

for V. Since β_1 and β_2 are two bases, there exists an element $g \in GL_n(k)$ such that which transforms β_1 to β_2, in particular $g.v = w$. By taking the last vector $\lambda.u$ instead of u in β_2 we can arrange the determinant of the matrix g as 1, that is, there exists $g' \in SL_n(k)$ with $g'.v = w$. Hence, for any $v \neq 0$, we have $v.SL_n(k) = V$, so, V is an irreducible $k[SL_n(k)]$-module.

Claim: Let L be the inverse image of F in $SL_n(k)$. An $L-$composition series of V contains at most $|F|$ isomorphism types of factors, each of dimension at most $|F|$. This is proved in [11, Theorem B.c]. For the reader's convenience, we will give the proof in detail.

Indeed, since $Z(SL_n(k)) = Z$ consists of semisimple elements, L is a subgroup of $SL_n(k)$ consisting of semisimple elements. We consider V as a $k[L]$-module. Since L consists of semisimple elements, $(p, |L|) = 1$. By Maschke Theorem (see [38, Corollary 1.6]), V is a completely reducible $k[L]$-module.

Since $Z \trianglelefteq SL_n(k)$, by Clifford's Theorem (see [38, Theorem 1.7]), $V|_Z$ is a direct sum of irreducible $k[Z]$-modules each of dimension one, and V can be written as a direct sum of homogeneous components.

Since Z is the center of $SL_n(k)$, there exists only one homogeneous component. Indeed, if $V = W_1 \bigoplus W_2 \bigoplus \ldots \bigoplus W_k$ where W_i's are homogeneous components of V. Let $W_i = X_{i1} \bigoplus \ldots \bigoplus X_{ik}$ where $X_{i1} \cong X_{i2} \cong \ldots \cong X_{ik}$ and each $X_{ij}, \quad j = 1, 2, \ldots, k$ is a one dimensional $k[Z]$-module.

Now, let $g \in SL_n(k)$. Then $W_i^g = X_{i1}^g + X_{i2}^g + \ldots + X_{ik}^g$.

Moreover, $X_{i1} \cong X_{i1}^g$ as $k[Z]$-module.

Consider the map

$$\begin{aligned} \theta : X_{i1} &\longrightarrow X_{i1}^g \\ v &\longrightarrow v^g. \end{aligned}$$

Now,

$$\theta(v_1 + v_2) = (v_1 + v_2)^g = v_1^g + v_2^g = \theta(v_1) + \theta(v_2)$$

and

$$\theta(cv) = cv^g = c\theta(v).$$

For all $z \in Z$, since $zg = gz$, we have $\theta(v^z) = (v^z)^g = (v^g)^z = \theta(v)^z$ Hence, θ is a $k[Z]$-module isomorphism as X_{i1} is irreducible.

Now, $X_{i1}^g \leq W_i$ implies $W_i^g = W_i$. Hence, there exists only one homogeneous component and $V|_Z$ is a direct sum of a unique $k[Z]$-module W.

Let X be an irreducible $k[L]$-module. Then $X|_Z$ is a direct sum of irreducible modules isomorphic to W. Hence, $Hom_Z(X_Z, W) \neq 0$. Here, $Z \leq L$ and W is a $k[Z]$-module. Let W^L be the induced $k[L]$-module.

Now, by Nakayama's Frobenius Reciprocity Theorem (see [16, V.16.6]), we have $Hom_L(X, W^L) \neq 0$. From this, it follows that X can be embedded into W^L. The number of irreducible $k[L]$-modules is less than or equal to the number of irreducible modules in W^L. Since W is unique, the number of irreducible modules is less than or equal to the dimension of W^L. But $\dim(W^L) = |F|$. Hence, the number of distinct irreducible $k[L]$-modules is less than or equal to $|F|$ and the dimension of each irreducible $k[L]$-module is less than or equal to $|F|$. This completes the proof of the claim, that is, an L-composition series of V contains at most $|F|$ isomorphism types of factors, each of dimension at most $|F|$ ([11, Theorem B.c]).

Therefore, if $n > (r-1)|F|^2+1$ then at least one of the irreducible components of L repeats r-times. Choosing the basis of V suitably we may say that at least r blocks repeats in all elements of L and the elements which permute these blocks generate a subgroup isomorphic to $SL_r(k)$ which is contained in $C_{SL_n(k)}(L)$ by the argument in Lemma 5.10. Therefore,

$$PSL_r(k) \leq SL_r(k)Z/Z \leq C_{PSL_n(k)}(F).$$

□

The following consequence of Theorem 5.11 shows that if F is a finite subgroup consisting of semisimple elements in $PSL_n(k)$ where n is sufficiently large, then $C_G(F)$ contains infinitely many elements of distinct prime order.

Corollary 5.12. *Let $G = A_{n-1}(k)$ over an infinite locally finite field k of characteristic p, and F be a finite subgroup consisting of semisimple elements of G. If $n > |F|^2 + 1$ then the centralizer of F in G has an infinite abelian subgroup A isomorphic to $Dr_{p_i}\mathbb{Z}_{p_i}$ for infinitely many prime p_i.*

Remark 5.13. Observe that in Corollary 5.12, F need not be d-abelian. In fact, it can even be non-abelian. We know that centralizers of finite d-abelian subgroups contain a maximal torus of G. But here, even when F is not abelian, but the rank is big enough, we can say that $C_G(F)$ contains infinitely many elements of distinct prime orders, but they are not necessarily contained in an abelian subgroup.

Proof. **(Proof of Corollary 5.12)** Let $G = A_{n-1}(k)$ over an infinite locally finite field k of characteristic p, and F be a finite subgroup consisting of semisimple elements of G with $n > |F|^2 + 1$. Write k as a union of finite fields $\mathbb{F}_{p^{k_i}}$ where $k_i|k_{i+1}$. We know by Theorem 2.31 that $G = \bigcup_{i=1}^{\infty} G_i$ where $G_i \cong PSL_n(p^{k_i})$. Since $n > |F|^2 + 1$, by Theorem 5.11, $C_{PSL_n(p^{k_i})}(F)$ contains a subgroup isomorphic to $H_i \cong PSL_2(p^{k_i})$. Then, since $C_G(F) = \bigcup_{i=1}^{\infty} C_{G_i}(F)$, for every i, the centralizer $C_{G_i}(F)$ contains a subgroup H_i isomorphic to $PSL_2(p^{k_i})$. The order of $PSL_2(q_i)$ is equal to $\frac{q_i(q_i^2-1)}{(2,q_i-1)}$ for $q_i = p^{k_i}$.

So, $|H_i|$ and $|C_{G_i}(F)|$ are divisible by $\frac{p^{2k_i-1}}{2,p-1}$. Since $k_i|k_{i+1}$, we have $(p^{2k_i}-1)$ divides $(p^{2k_{i+1}}-1)$. By Theorem 2.46, for each i there exists a prime q_i which divides $(p^{2k_i}-1)$ but does not divide p^m-1 if $m<2k_i$. Then for each H_i contains an element x_i of prime order q_i, which is not contained in H_{i-1}. Then the subgroup $H=\langle x_i \mid i\in\mathbb{N}\rangle$ is isomorphic to $Dr_{p_i}\mathbb{Z}_{p_i}$ for infinitely many prime p_i and $H\leq C_G(F)$. □

Now, we will prove the analogue of Theorem 5.5 for non-linear simple locally finite groups with a local system consisting of classical groups with unbounded rank.

Theorem 5.14. *Let G be a non-linear simple locally finite group which has a local system $\mathcal{K}=\{G_i: \ i\in\mathbb{N}\}$ consisting of classical groups. Then for any finite subgroup F consisting of $\mathcal{K}$-semisimple elements in G , the centralizer $C_G(F)$ has an infinite abelian subgroup A isomorphic to $Dr_{p_i}\mathbb{Z}_{p_i}$ for infinitely many prime p_i.*

Proof. Since G is non-linear, the rank parameter of the groups in $\mathcal{K}$ is unbounded. So, all the groups in $\mathcal{K}$ are of classical type. By Remark 2.29, G_i's are of the same fixed classical type, and the rank parameter is increasing.

For the characteristic of the fields where G_i is defined we have the following: If the number of primes which appear as characteristic is finite, say $q_1,q_2,\ldots q_n$, then let $J_k=\{G_i\in\mathcal{K}: G_i$ is defined over a field of characteristic $q_k\}$.

Here, $\mathcal{K}=J_1\cup J_2\cup\ldots\cup J_n$, so at least one of the J_k's is infinite. So, we may assume that there exists a prime p such that all the G_i is defined over a fixed prime p.

If infinitely many primes occur as characteristic, we may delete the repeating ones and assume that each G_i is defined over different characteristic.

Hence, we have two cases:

1. All the groups in the local system is defined over a field of characteristic p for a fixed prime p.

2. All the groups are defined over different characteristic.

Case 1 for groups with a local system consisting of groups of type A_l: Here, G_i's are $PSL_{n_i}(k)$, where n_i's are increasing. Let F be a finite subgroup of G consisting of $\mathcal{K}$-semisimple elements. Then, by definition, F consists of semisimple elements of G_i. If necessary, by deleting finitely many terms of the local system, we may assume that $F \leq G_1$ that is, $F \leq G_i = PSL_{n_i}(k_i)$ for all i. We will construct an abelian subgroup $A \leq C_G(F)$ such that A is isomorphic to the direct product of cyclic subgroups of order p_i for infinitely many distinct primes p_i. For this, we start $T_0 = F$. We work as in Theorem 5.11. Since F consists of semisimple elements of $PSL_{n_i}(k_i)$, the inverse image of F in $SL_{n_i}(k_i)$ also consists of semisimple elements where k_i is a field of characteristic p for all i, that is, $(|F|, p) = 1$. Let $\Gamma = \{p_1 = 2, p_2 = 3, p_3 = 5 \ldots\}$ be the set of all primes except p the characteristic of k, ordered by the usual order in $\mathbb{N}$. By Theorem 5.11, by choosing $n_1 > (p_1 - 1)|F|^2 + 1$, we find a subgroup isomorphic to $PSL_{p_1}(k)$ which is contained in $C_{PSL_{n_1}(k)}(F)$. Then, let x_1 be an element in $PSL_{p_1}(k)$ with order p_1. We know that $PSL_n(k)$ contains an element of order n by Lemma 4.11. Then, let g_1 be an element in $PSL_{p_1}(k)$ with order p_1. Now, let $T_1 = \langle g_1, F\rangle$ and $A_1 = \langle g_1\rangle$. Here, since g_1 and the elements of F commute and $\langle g_1\rangle \cap F = 1$, we have $T_1 = \langle g_1\rangle \times F$. Therefore, since p_1 and $|F|$ are relatively prime with p, T_1 consists of semisimple elements in $G_2 = PSL_{n_2}(k)$. Now, apply the same argument, that is, choose $n_2 > (p_2 - 1)|T_1|^2 + 1 = (p_2 - 1)p_1^2|F^2| + 1$, and by Theorem 5.11, we obtain a subgroup isomorphic to $PSL_{p_2}(k)$ in $C_{G_2}(T_1) = C_{PSL_{n_2}(k)}(T_1)$. We take an element g_2 of order p_2 in $PSL_{p_2}(k)$ and $T_2 = \langle g_2, \rangle T_1$. Denote $\langle g_1, g_2\rangle$ by A_2.

Assume T_{i-1} is already constructed. If we choose $n_i > (p_i - 1)|T_{i-1}|^2 + 1$, we can find a subgroup isomorphic to $PSL_{p_i}(k)$ in $C_{G_i}(T_{1-1}) = C_{PSL_{n_i}(k)}(T_{i-1})$. We take an element g_i of order p_i in $PSL_{p_i}(k)$ and $T_i = \langle g_i, \rangle T_{i-1}$ consists of semisimple elements of G_{i+1}.

Continuing like this we obtain the chains of groups

$$T_0 \leq T_1 \leq T_2 \leq \ldots \text{ and } A_1 \leq A_2 \leq \ldots.$$

Here, A_i is the abelian subgroup generated by $\{g_1, \ldots g_i\}$ which commutes with F. Now, $A = \bigcup_{i=1}^{\infty} A_i$ is an infinite abelian subgroup contained in $C_G(F)$ such that $A \cong Dr_{p_i \in \Gamma} \mathbb{Z}_{p_i}$ where Γ is the set of all primes except $p = char k$.

Case 2 for groups with a local system consisting of groups of type A_l: In this case infinitely many distinct primes occur as characteristics of the fields k_i's where $G_i = PSL_{n_i}(k_i)$. In this case, every element is $\mathcal{K}$-semisimple, so every finite subgroup F consists of $\mathcal{K}$-semisimple elements. Indeed, if F is a finite subgroup of G, since the number of primes dividing $|F|$ is finite, we may delete the terms of the local system $\mathcal{K}$ in which the characteristic of the field divides $|F|$ and assume that F consists of semisimple elements of G_i for every $G_i \in \mathcal{K}$. Then we may assume that $G_i \cong PSL_{n_i}(q_i)$ where $q_i \neq q_j$ for all $j \in \mathbb{N}$ and $(|F|, q_i) = 1$.

Let as before $F = T_0$. Assume T_{i-1} is already constructed. Here, if we choose $n_i > |T_{i-1}|^2 + 1$, we can find a subgroup isomorphic to $PSL_2(k_i)$ in $C_{G_i}(T_{i-1}) = C_{PSL_{n_i}(k_i)}(T_{i-1})$ where $char k_i = q_i$. Since q_i divides $|PSL_2(q_i)|$, by Cauchy Theorem, there exists an element g_i of order q_i in $C_{G_{n_i}}(T_{i-1}) \leq C_{G_{n_i}}(F)$.

We take an element g_i of order q_i in $PSL_{q_i}(k)$ and $T_i = \langle g_i, T_{i-1} \rangle$ consists of semisimple elements of G_{i+1}.

Again, continuing like this, we obtain the chains of groups

$$T_0 \leq T_1 \leq T_2 \leq \dots \text{ and } A_1 \leq A_2 \leq \dots .$$

Here, A_i is the abelian subgroup generated by $\{g_1, \dots g_i\}$ which commutes with F. Now, $A = \bigcup_{i=1}^{\infty} A_i$ is an infinite abelian subgroup contained in $C_G(F)$ such that $A \cong Dr_{q_i} \mathbb{Z}_{q_i}$ where the infinitely many distinct primes q_i's are the characteristics of k_i.

Case 1 for groups with a local system consisting of groups of type $B_l, C_l, D_l, {}^2A_l, {}^2D_l$:

In this case, for all G_i in the local system $\mathcal{K}$, the characteristic of the field over which G_i is defined is p. We will use the result [11, Theorem B.e]: Let F be a finite subgroup of G which consists of semisimple elements of G, let L be the inverse image of F in the universal central extension of G, that is, $L/Z = F$ where Z is the center of the universal central extension of G. Let m be the rank of G over the field k. Let r be an arbitrary positive integer. By [11, Theorem B.e], if $m \geq 2r|F|^3 + 4|F|$, the natural module V contains an $k[L]$-module U which is direct sum of r isomorphic simple $k[L]$-modules and is totally isotropic (resp. totally singular).

Let $U = U_1 \bigoplus U_2 \bigoplus \ldots \bigoplus U_r$ where U_i's are copies of a single $k[L]$-module and U is totally isotropic or totally singular subspace of V. Since L consists of semisimple elements of the group G, by Maschke Theorem (see [38, Corollary 1.6]), V is a completely reducible $k[L]$-module.

If $u_1 \in U^{\perp}$, $\ u \in U$ and $g \in L$ then $(u_1.g, u) = (u_1, u.g^{-1}) = 0$ as U is an $k[L]$-module and $u_1 \in U^{\perp}$. Hence, $u_1.g \in U^{\perp}$. Therefore, $U^{\perp}$ is an $k[L]$-module. Then $U^{\perp} = U \bigoplus W$ where W is another $k[L]$-module. Observe that the form on V induces a non-degenerate form on $U^{\perp}/U$. By decomposition on $U^{\perp}$ we see that the form induced on W is non-degenerate.

Since each U_i is an irreducible $k[L]$-module, we have $\dim U_i \leq |F|$, hence $\dim U \leq |F|r$. Then $\dim(U^{\perp}/U) = \dim U^{\perp} - \dim U = \dim V - 2\dim U = m - 2\dim U$. In particular, if m is sufficiently large, we have $\dim(U^{\perp}/U)$ is sufficiently large, that is, $\dim W$ is sufficiently large. Let $W = W_1 \bigoplus W_2 \bigoplus \ldots \bigoplus W_t$.

Then we write $V = U \bigoplus W_1 \bigoplus W_2 \bigoplus \ldots \bigoplus W_t \bigoplus Y$ where W_i's are irreducible $k[L]$-modules and $U^{\perp} = U \bigoplus W_1 \bigoplus W_2 \bigoplus \ldots \bigoplus W_t = U \bigoplus W$. Also, Y is a direct sum of irreducible $k[L]$-modules.

Since U_i's are isomorphic irreducible $k[L]$-modules, for each i we may find a basis $\beta i = \{u_{i1}, \ldots, u_{ik}\}$ for U_i and the action of each element $g \in L$ to the elements of the basis gives the same matrix representation, that is, if $u_{i1}.g = \sum_{s=1}^{k} a_{is}u_{is}$ then $u_{j1}.g = \sum_{s=1}^{k} a_{is}u_{js}$. As before in Theorem 5.11, we obtain for each $g \in L$ the matrix representation is the copies of the same matrix repeated r times in the first r component.

Now, define the linear transformations on U which are induced from the action on $SL_r(k)$ to U in the following way. The elements of $SL_r(k)$ acts on the block as we done before in Theorem 5.11,

$$\begin{aligned} U &\longrightarrow U_1 \bigoplus \ldots \bigoplus U_r \\ \omega_i &\longrightarrow \sum_{i=1}^{r} \lambda_i \omega_i \end{aligned}$$

where $\omega_i = (u_{i1}, \ldots, u_{ik})$.

Extend the action on U and observe that the action of $SL_r(k)$ to U is by isometries of U as U is a totally isotropic (resp. totally singular) subspace of V. We may extend this action on $U^\perp$ by acting trivially on W_i. This action is also an action by isometries: Indeed, if we take an element $g \in SL_r(k)$ and $u_1, u_2 \in U^\perp$ then $u_1 = v_1 + w_1$ and $u_2 = v_2 + w_2$ where $u_1, u_2 \in U$ and $w_1, w_2 \in W_i$'s.

Then

$$\begin{aligned}(u_1, u_2) &= (v_1 + w_1, v_2 + w_2)\\ &= (v_1, v_2) + (v_1, w_2) + (w_1, v_2) + (w_1, w_2) \qquad = (w_1, w_2)\end{aligned}$$

since $(v_1, v_2), (v_1, w_2), (w_1, v_2)$ are 0. Similarly,

$$\begin{aligned}(u_1 g, u_2 g) &= ((v_1 + w_1)g, (v_2 + w_2)g)\\ &= (v_1 g, v_2 g) + (v_1 g, w_2 g) + (w_1 g, v_2 g) + (w_1 g, w_2 g)\\ &= (w_1, w_2).\end{aligned}$$

Hence, $SL_r(k)$ act by isometries of $U^\perp$. Now, by Witt Extension Theorem we may extend the isometries of $U^\perp$ to isometries of V.

As in the proof of [11, Theorem B.f], let $C^* = N_T(U) \cap C_T(W)$. Let $D_1 = C_{C^*}(U)$. Now, D_1 acts trivially on $V/U^\perp = Y$, $U^\perp/U = W$ and U. Hence $D_1 = 1$ and $C = SL_r(k)$ is in the centralizer of L. Hence $C/Z \cong SL_r(k)/Z \leq C_G(F)$.

Now, for Case 1, since the characteristics of the fields where G_i is defined is a fixed prime p, let Γ be the set of all primes except p. Let $F = T_0$. If $n_1 \geq 2(p_1-1)|F|^3+4|F|$, by the above argument, we can find $SL_{p_1}(k)$ in $C_G(F)$. Take an element g_1 of order p_1 and let $T_1 = \langle F, g_1\rangle$. Assume T_{i-1} is already constructed. If we choose $n_i \geq (p_i - 1)|T_{i-1}|^3 + 4|T_{i-1}|$, we can embed $SL_{p_i}(k) \leq C_{G_i}(T_{i-1})$ and take an element g_i of order p_i in $SL_{p_i}(k)$. Then $T_i = \langle g_i, T_{i-1}\rangle$ consists of semisimple elements of G_{i+1} and the proof follows as in type A_l. The required subgroup $A = \cup_{i=1}^{\infty} A_i$ where $A_i = \langle g_1, \ldots g_i\rangle$.

Case 1 for groups with a local system consisting of groups of type $B_l, C_l, D_l, {}^2A_l, {}^2D_l$: For Case 2, that is, if the characteristic of the field is a different prime p_i for each G_i, then we observed that every subgroup is $\mathcal{K}$-semisimple. Let $\Gamma = \{q_i : \ i \in \mathbb{N}\}$ be the set of all primes which does not divide $|F|$.

We take $F = T_0$ and choose $n_1 \geq 2(q_1 - 1)|F|^3 + 4|F|$ to find $SL_{q_1}(k_1)$ in

$C_{G_1}(F)$. By the same argument in Case 1, we construct T_i's and A_i's. Here, $A = \bigcup_{i\in\mathbb{N}} A_i \cong Dr_{q_i\in\Gamma}\mathbb{Z}_{q_i}$ is the required subgroup of $C_G(F)$.

$\square$

Now, we are able to prove the same result with non-linear simple locally finite groups with a split Kegel cover.

Corollary 5.15. *Let G be a non-linear simple locally finite group with a split Kegel cover $\mathbb{K} = \{(G_i, N_i) \mid i \in \mathbb{N}\}$ consisting of simple groups of Lie type. Then for any subgroup F consisting of $\mathcal{K}$-semisimple elements, the centralizer $C_G(F)$ has an infinite abelian subgroup A isomorphic to $Dr_{p_i}\mathbb{Z}_{p_i}$ for infinitely many prime p_i.*

Proof. We may assume that $F \leq G_1$ and let $A_0 = 1$ and $C_0 = F$. Then by Theorem 5.14 there exists n_1 such that $C_{G_{n_1}/N_{n_1}}(FN_{n_1}/N_{n_1})$ contains an element of order p_1. Since $\mathcal{K}$ is a split Kegel cover, $C_{G_{n_1}/N_{n_1}}(FN_{n_1}/N_{n_1}) = C_{G_{n_1}}(F)N_{n_1}/N_{n_1}$. So, $C_{G_{n_1}}(F)$ contains an element of order p_1. Let $C_1 = \langle F, \rho_1\rangle$ and $A_1 = \langle \rho_1\rangle$ where ρ_1 is an element of order p_1 in $C_{G_{n_1}}(F)$. C_1 is a subgroup of G_{n_1}, and so there exists n_2 such that $C_{G_{n_2}/N_{n_2}}(C_1N_{n_2})/N_{n_2}$ contains an element of order p_2. We have

$$C_{G_{n_2}/N_{n_2}}(C_1N_{n_2})/N_{n_2} = C_{G_{n_2}}(C_1)N_{n_2}/N_{n_2}$$

since $\mathcal{K}$ is a split Kegel cover. Hence $C_{G_{n_2}}(C_1)$ contains an element ρ_2 of order p_2. Let $C_1 = \langle F, \rho_1\rangle$ and $A_1 = \langle \rho_1\rangle$ where ρ_1 is an element of order p_1 in $C_G(F)$. Let $C_2 = \langle C_1, \rho_2\rangle$ and $A_2 = \langle \rho_1, \rho_2\rangle$. Then $C_1 \leq C_2 \leq C_3 \leq \ldots$ and $A_1 \leq A_2 \leq A_3 \ldots$. Then the union $A = \bigcup A_i$ is the required abelian subgroup of G which is isomorphic to $Dr_{p_i \text{ prime}}\mathbb{Z}_{p_i}$.

$\square$

Remark 5.16. Observe that the "split Kegel cover" assumption is necessary as the groups constructed by Meierfrankenfeld in [23] are non-linear locally finite simple groups but there are elements whose centralizer is a p-group for some fixed prime p.

By Corollary 5.8 and Corollary 5.15 we reach conclusion of the thesis:

Theorem 5.17. *Let G be a non-linear simple locally finite group with a split Kegel cover $\mathcal{K}$ and F be a finite subgroup consisting of $\mathcal{K}$-semisimple elements. Then $C_G(F)$ has an infinite abelian subgroup A isomorphic to $Dr_{p_i}\mathbb{Z}_{p_i}$ for infinitely many prime p_i.*

Remark 5.18. Hall-Kulatilaka Theorem says that in an infinite locally finite group, there are infinite abelian subgroups. (See [9]). Here we prove that in a non-linear locally finite simple group which has a "nice" Kegel cover, the centralizers of finite subgroups have abelian subgroups with elements of order p_i for infinitely many primes.

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- "Centralizers of finite subgroups in simple locally finite groups" (in Turkish), Ankara Mathematics Days, 4-5 June 2009, METU, Ankara, Turkey.
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- "Polynomial Invariants of Knots", (in Turkish), XIII. National Mathematics Symposium, September 6th-9th, 2000, Istanbul, Turkey.