Minimal Model Program: Contract Your Bad Curves!

Imperial College London

Andrea Fanelli, Diletta Martinelli

Supervisor: Dr Paolo Cascini

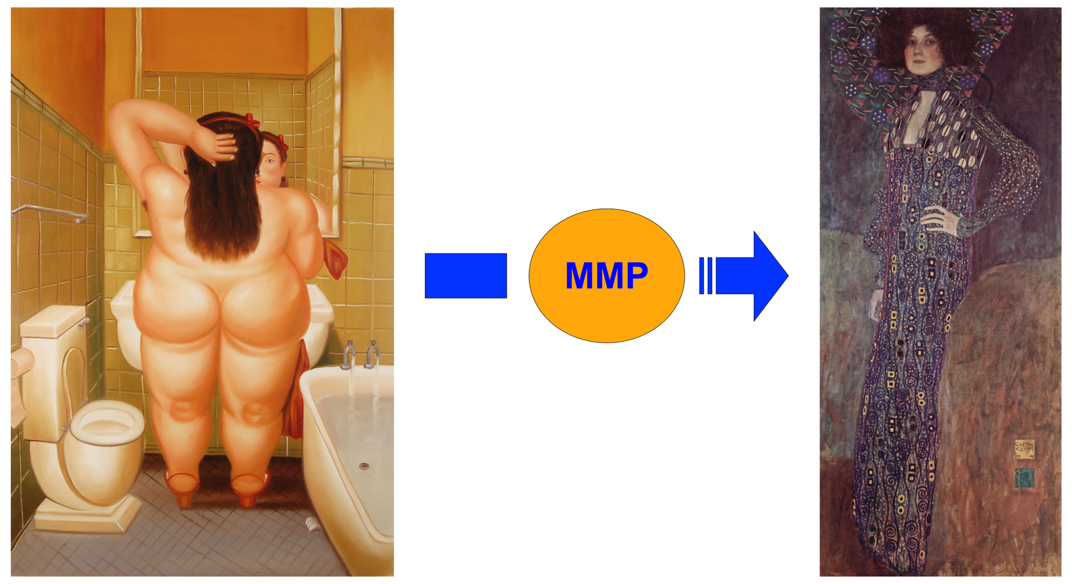
Introduction: Our Gym

Costumers: PROJECTIVE VARIETIES over $\mathbb{C}$ (geometric objects locally defined as zero loci of polynomials, embedded in some projective space).

Our goal: classify them up to birational equivalence (we are allowed to cut off some useless and adipose subvarieties, keeping essential information on them, such us their **function fields**).

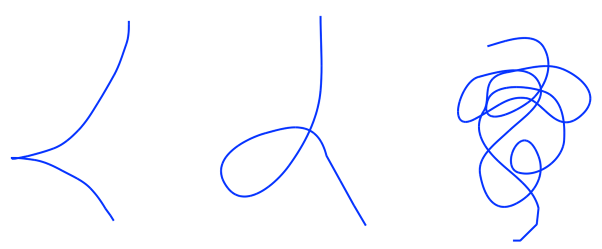
Main Tool: the canonical bundle K_X, which is intrinsic of our variety and can be useful to detect how the training is going. It is our fat detector!

Our Training Program (MMP): start with a variety X, make it sweat and come up with a MINIMAL MODEL X_{min}.



Curves: Too Skinny For Us!

Even curves can easily get very complicated!

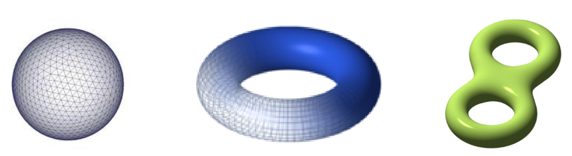


But our work is rather easy for curves: just need to remove the bad points!

Theorem 1. *Every curve is birationally equivalent to a unique nonsingular projective curve.*

Algebraic curves (= Riemann Surfaces) are classified by their genus **g** (number of holes) into 3 classes:

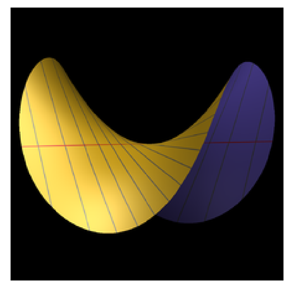
RATIONAL ($g = 0$), ELLIPTIC ($g = 1$) AND CURVES OF GENERAL TYPE ($g \geq 2$)

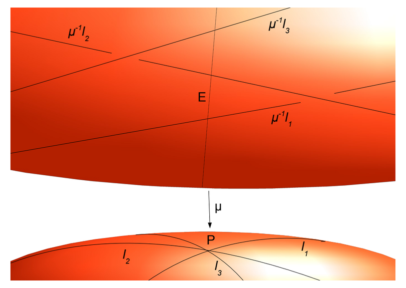


Surfaces: Italian Vintage Training

Finally someone who needs an effective workout! In early 20th century, many Italian surfaces put on some weight, due to fat meals and their mammas' tiramisu'! They ended up with tons of new curves!

Here's the new phenomenon in dimension 2: we can **blow up** points and obtain infinitely many new surfaces, all birational to the one we started with!

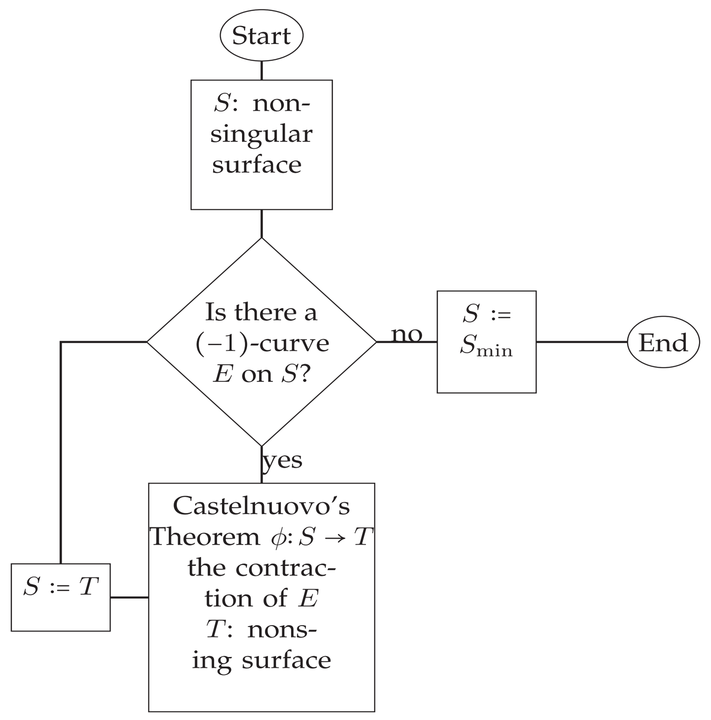




But every blow-up leaves a (fat and superfluous) trace: a rigid **(-1)-curve**.

Theorem 2 (Castelnuovo's Contractibility Criterion). *If a non-singular surface S contains a (-1)-curve, we can contract it via a birational morphism $\phi: S \to T$ to obtain another non-singular surface T.*

And here's our personalised program for surfaces:



After a **finite** number of steps, the routine ends, since the contraction ϕ actually simplifies the structure of S and we get a MINIMAL surface in perfect shape.
One can also see that the canonical bundle of S_{min} has a very interesting property: its intersection with curves is positive ($K_{S_{min}}$ is NEF).

References: The Coaching Staff

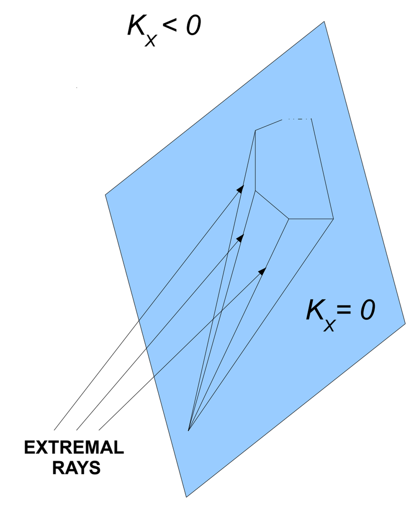
- Debarre, *Higher-Dimensional Algebraic Geometry*, Springer-Verlag;
- Hacon, Kovács, *Classification of Higher Dimensional Algebraic Varieties*, Birkhäuser;
- Kollár, Mori, *Birational Geometry of Algebraic Varieties*, Cambridge University Press;
- Matsuki, *Introduction to the Mori Program*, Springer-Verlag, Berlin.

Threefolds: The 80's, Flipping Plastic Surgery

Everyone knows that: **the bigger you are, the harder is to contract bad curves!**

A fat 3-fold

It took many decades to understand how to attach the problem for varieties of dimension 3 or bigger!
Problem 1. (-1)-curves mean nothing on a 3-fold.
Solution: "rigid" curves are the ones which intersect negatively the canonical bundle K_X (extremal rays) $\rightsquigarrow$ **Mori's Cone theorem**, which describes the cone of effective curves on our variety:



So now we know our enemy! We start contracting them as for surfaces, BUT...
Problem 2. When you contract the rays, your variety can become singular!
Solution: We need to enlarge the category we work with $\rightsquigarrow$ **terminal singularities** (the mildest singularities you can imagine!)

Well... Fair enough... Even with some stretch marks our variety looks great! OH NO... What is that?!?!?
Problem 3. If we contract a subvariety in high codimension, we come up with a new variety where neither intersection theory makes sense!
Solution: There is only one thing to do: **FLIP THE CURVE**. This is a totally different step of the MMP: starting with X we need to construct a new variety X^+ (HOW?!?!) and a birational map

$$f: X \dashrightarrow X^+$$

cutting off the curve C which negatively intersects K_X and replace it with a new curve C^+ such that $(K_{X^+} \cdot C^+) > 0$: this is the plastic surgery we need!
We expect to end up with a MINIMAL model (with **NEF** canonical bundle) also in this case, but there is a priori no reason why the flipped variety should be "skinnier" than X! So: "Do flips terminate?"

MMP Today: BCHM's Liposuction

In 2010, **B**irkar, **C**ascini, **H**acon and **M**^c^Kernan developed a new revolutionary surgical technique: cooking up the canonical bundle they can
- construct **flips**;
- come up with a **minimal model** for a large class of varieties, called of GENERAL TYPE.

It requires a lot of work and many techniques, but the result is outstanding!

Future: Fusion Workout

What's left? Tons of varieties still waiting for your help: they really need to loose weight!
Here's some open problems:
- minimal models for many **varieties of special type**;
- **termination of flips** in dimension higher than 4;
- **Abundance Conjecture** (minimal models are even better than expected... they got ripped!);
- MMP in **positive characteristic** (this is hardcore!);
- **moduli spaces** of varieties of general type (yes, they are in good shape now... what about a contest?)

Flowchart: The Training Table

This is the (conjectural) training program for varieties in arbitrary dimension! $\mathfrak{C}$ is the category of varieties with terminal singularities.