The Congruence Subgroup Property for Groups of Tree Automorphisms

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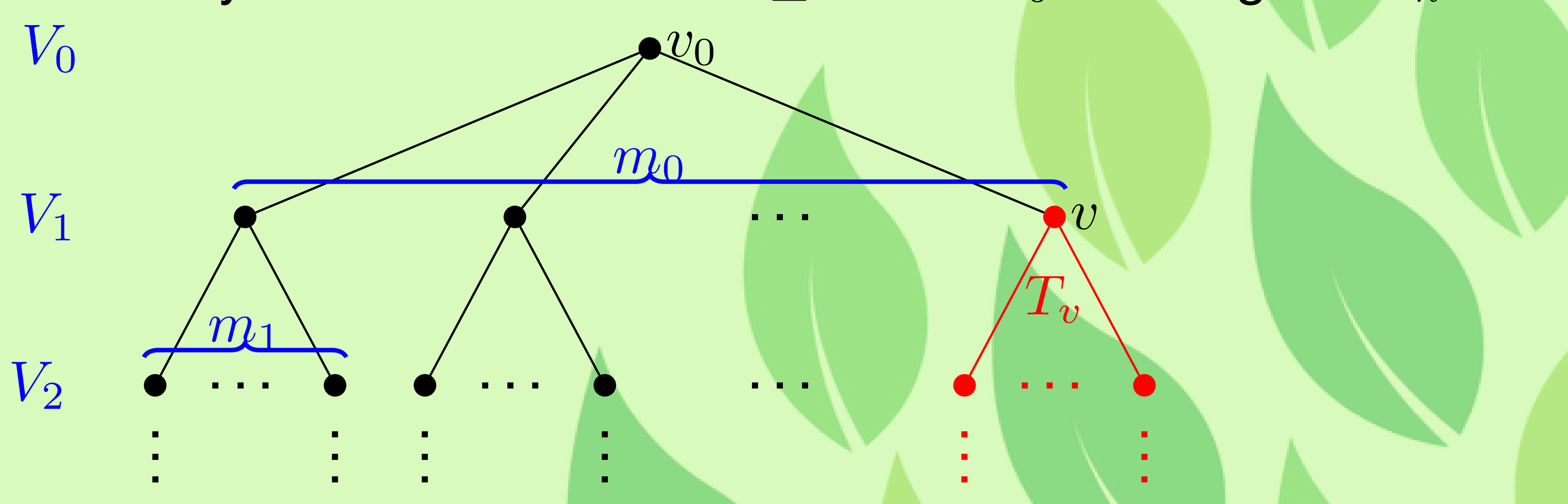
Rooted Trees

Let $(m_n)_{n\geq 0}$ be a sequence of integers with $m_n \geq 2$. A **rooted tree of type** $(m_n)_n$ is a tree T with root v_0 of degree m_0 such that every vertex at distance $n \geq 1$ from v_0 has degree $m_n + 1$.



(Weakly) branch actions

Let G act faithfully on T. Define

- $\mathrm{St}_G(v) := \{g \in G \mid vg = v\}$, the **stabilizer** of $v \in T$
- $\mathrm{St}_G(n) := \bigcap_{v\in V_n} \mathrm{St}_G(v)$, the nth **level stabilizer**
- $\mathrm{rist}_G(v) := \{g \in G \mid g$ fixes $T \setminus T_v$ pointwise$\}$
- $\mathrm{rist}_G(n) := \prod_{v\in V_n} \mathrm{rist}_G(v)$, the nth **level rigid stabilizer**.

This faithful action is a **(weakly) branch action** if for all n

(i) the action is transitive on V_n

(ii) ($\mathrm{rist}_G(n)$ is infinite) $|G : \mathrm{rist}_G(n)|$ is finite.

A group is a **(weakly) branch group** if it has a (weakly) branch action on some T.

Congruence Subgroup Property (CSP)

Let G act faithfully on T. Write $\widehat{G}$ for the profinite completion of G.

- $\overline{G}$ is the **congruence completion** of G, obtained by taking $\{\mathrm{St}_G(n)\}_n$ as a basis of neighbourhoods of the identity. It is also the closure of G in $\mathrm{Aut}\, T$. There is a natural homomorphism $\psi\colon \widehat{G} \twoheadrightarrow \overline{G}$ whose kernel C is called the **congruence kernel**.
- G has the **congruence subgroup property (CSP)** if $C = 1$; i.e., each finite index subgroup contains some $\mathrm{St}_G(n)$.

If the action of G on T is a **branch action**,

- $\widetilde{G}$ is the completion of G obtained by taking $\{\mathrm{rist}_G(n)\}_n$ as a basis of neighbourhoods of the identity.
- The **branch kernel** B is the kernel of the natural homomorphism $\widehat{G} \twoheadrightarrow \widetilde{G}$. We also have $\widetilde{G} \twoheadrightarrow \overline{G}$.

CSP is independent of weakly branch action

If G has two weakly branch actions then it has the CSP with respect to one iff it has the CSP with respect to the other (this answers Question 2 from [1]):

Theorem 1 ([2]). *Let $\rho\colon G \hookrightarrow \mathrm{Aut}(T_\rho)$ and $\sigma\colon G \hookrightarrow \mathrm{Aut}(T_\sigma)$ be two weakly branch actions of G. Then $\{\mathrm{St}_\rho(n)\}_n$ and $\{\mathrm{St}_\sigma(n)\}_n$ define the same topology on G, so $\overline{G_\rho} = \overline{G_\sigma}$ and $C_\rho = C_\sigma$.*

Similarly,

Theorem 2 ([2]). *Let $\rho\colon G \hookrightarrow \mathrm{Aut}(T_\rho)$ and $\sigma\colon G \hookrightarrow \mathrm{Aut}(T_\sigma)$ be two branch actions of G. Then $\widetilde{G_\rho} = \widetilde{G_\sigma}$ and $B_\rho = B_\sigma$.*

Proof sketch

For every weakly branch action of G on T, and every $v \in T$:

- $\mathrm{rist}_G(vg) = g^{-1}\,\mathrm{rist}_G(v)g$ for every $g \in G$,
- if $\mathrm{rist}_G(vg) \cap \mathrm{rist}_G(v) \neq 1$ then $\mathrm{rist}_G(vg) = \mathrm{rist}_G(v)$.

Proposition ([3]). *For every $u \in T_\rho$ there exists $v \in T_\sigma$ such that $\mathrm{rist}_\rho(u) \geq \mathrm{rist}_\sigma(v)' \neq 1$.*

Claim. For every n there exists m such that $\mathrm{St}_\rho(n) \geq \mathrm{St}_\sigma(m)$.
Proof. Let $u \in V_n \subset T_\rho$. By the Proposition, there exists $v \in V_m \subset T_\sigma$ with $\mathrm{rist}_\rho(u) \geq \mathrm{rist}_\sigma(v)' \neq 1$. For any $g \in \mathrm{St}_\sigma(m)$, we have $1 \neq \mathrm{rist}_\sigma(vg)' = \mathrm{rist}_\sigma(v)' \leq \mathrm{rist}_\rho(u^g) \cap \mathrm{rist}_\rho(u)$, so $\mathrm{rist}_\rho(u^g) = \mathrm{rist}_\rho(u)$ and $g \in \mathrm{St}_\rho(u)$. The claim follows using the transitive action of G on V_n and V_m. □

Hence $\{\mathrm{St}_\rho(n)\}_n$ and $\{\mathrm{St}_\sigma(n)\}_n$ define the same topology on G, so the completions are the same.
A similar argument holds for branch actions and $\{\mathrm{rist}_G(n)\}_n$.

Profinite vs Pro-*p* Completions

Any GGS group with constant $\mathbf{e}$ does *not* have the CSP. However, its pro-p completion is the same as its closure in $\mathrm{Aut}\, T$:

Theorem 4 ([4]). *Let G be a GGS-group. Then*

$$\gamma_n(G) \geq \mathrm{St}_G(n) \text{ for all } n \in \mathbb{N}$$

where $\gamma_n(G)$ is the nth term of the lower central series.

This theorem also holds for the wider class of **spinal groups**.

Examples: GGS groups

GGS-groups: Subgroups of $\mathrm{Aut}\, T$ where T is p-regular tree, p =odd prime.

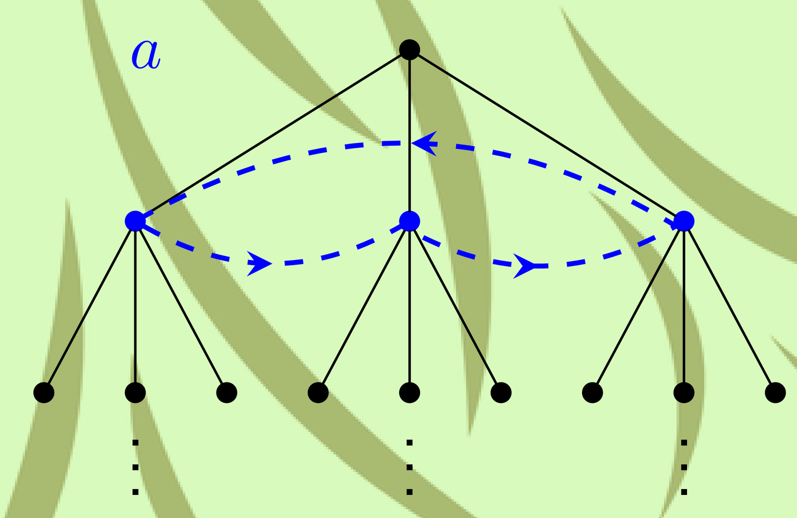
$G := \langle a, b\rangle$ where a acts as the p-cycle $(1\,2\,\ldots\,p)$ on V_1 and

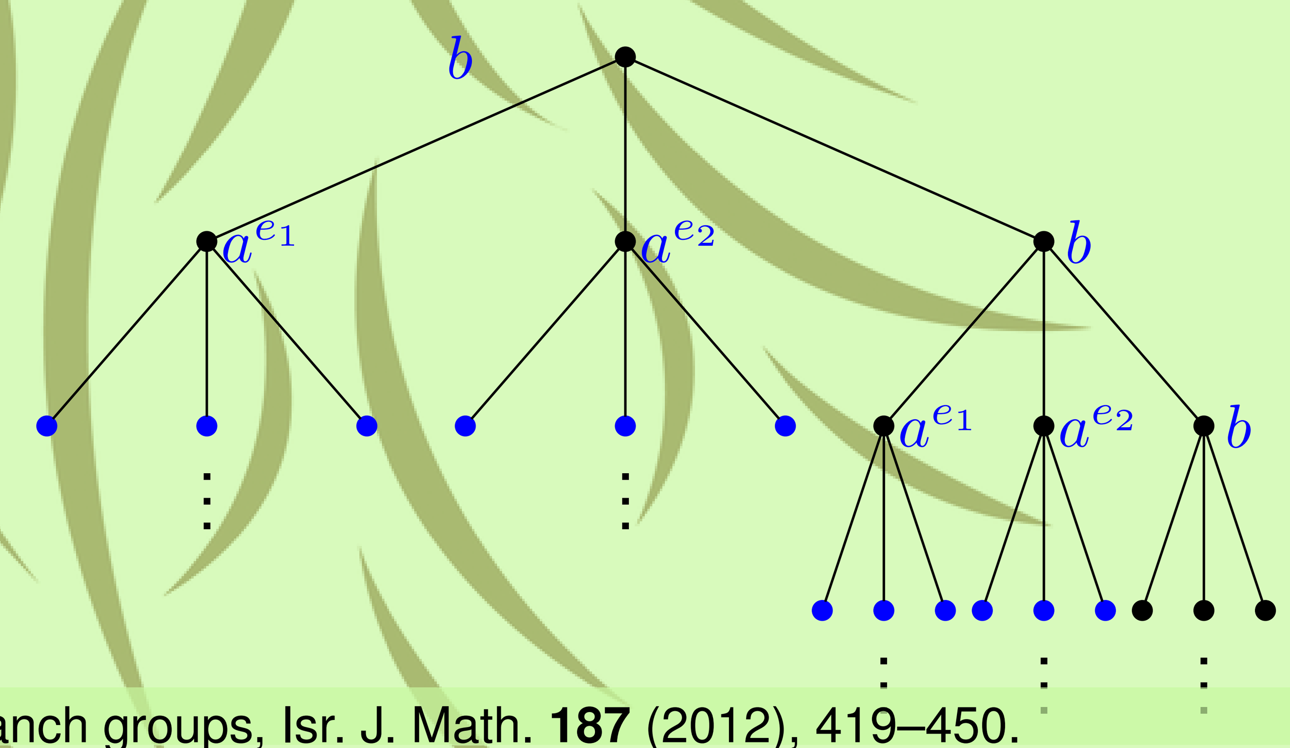
$$b := (a^{e_1}, a^{e_2}, \ldots, a^{e_{p-1}}, b) \in \mathrm{St}_G(1)$$

with $\mathbf{e} := (e_1, e_2, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$. (See figures for $p = 3$).

If $\mathbf{e}$ is not constant then G is a branch group; otherwise it is weakly branch.

Theorem 3 ([4]). *A GGS-group G has CSP iff $\mathbf{e}$ is not constant.*





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[2] A. Garrido. On the congruence subgroup problem for branch groups. *To appear in Isr. J. Math.* arXiv:1405.3237.
[3] A. Garrido and J.S. Wilson. On subgroups of finite index in branch groups, J. Algebra **397** (2014), 32–38.
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