Surface Groups in Doubles of F_n

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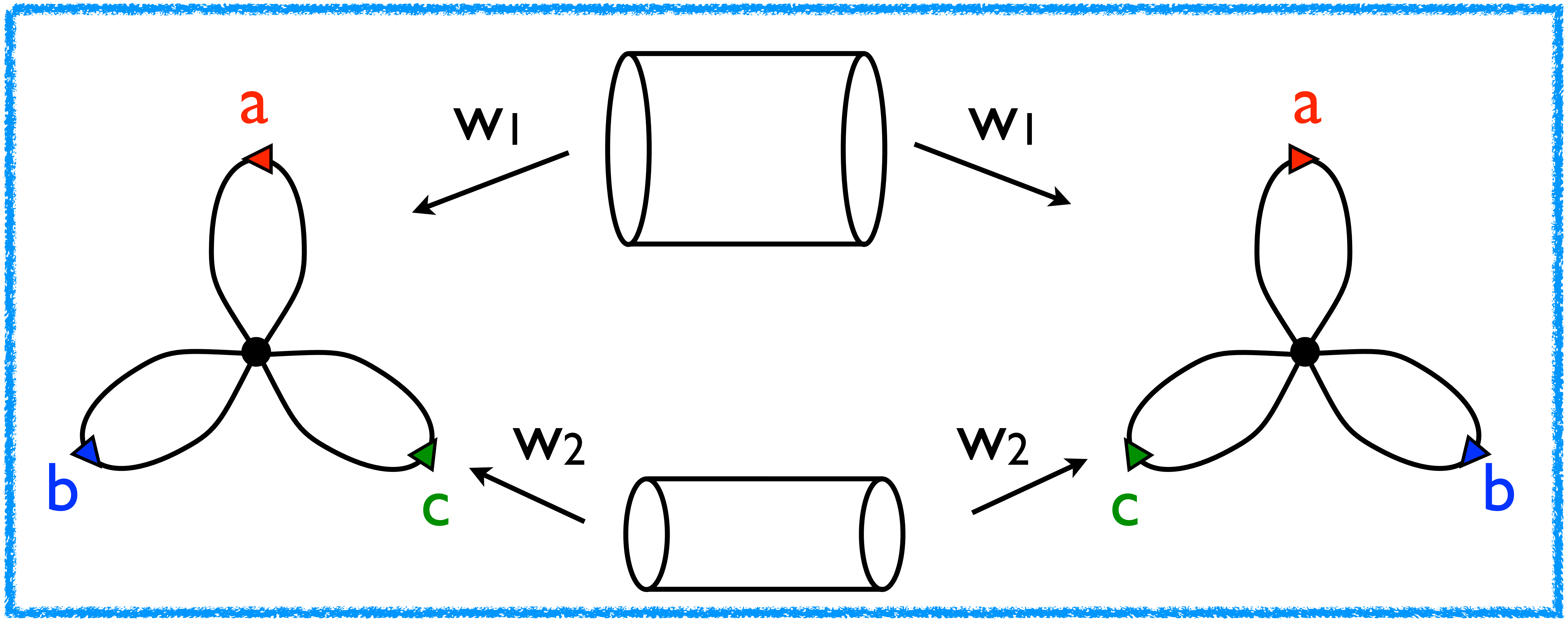
based on a joint work with Henry Wilton (UCL)



Motivation

A *hyperbolic surface group* means $\pi_1(S_g)$, $g>1$.

Q Does every one-ended word-hyperbolic group contain a hyperbolic surface group?

Gromov, *Asymptotic Invariants of Infinite Groups*, p.276:

of Thurston's narrow simplices.) Does a generic finitely presented group contain a non-free infinite subgroup of infinite index? Our discussion on groups presented by $A \subset C_\ell$ with dens $A = d < \frac{1}{2}$ indicate that these contain no surface groups of genus $\leq g$ for $g \to \infty$ with $\ell \to \infty$, but we do not know if some surface group of high genus is always present in a random (or every non-virtually free) hyperbolic group. Do generic groups with $q \gg p$ satisfy

Known Partial Results

- (Gordon–Long–Reid ’04) Coxeter groups.
- (K.’12) Graph products (nontrivial & directly indecomposable)
- (Calegari ’08) Graphs of free gps with **Z**–edges and $\beta_2 \neq 0$.
- (Kahn–Markovic ’09) Closed hyperbolic 3–mfd groups.
- (Gordon–Wilton ’09) Homological / geometric sufficient conditions for doubles $D(w) = F *_{<w>} F$.
- (Baumslag–Rosenberger ’11) D(w) contains $\pi_1(S_2)$ iff $w \in [F_n, F_n]$.

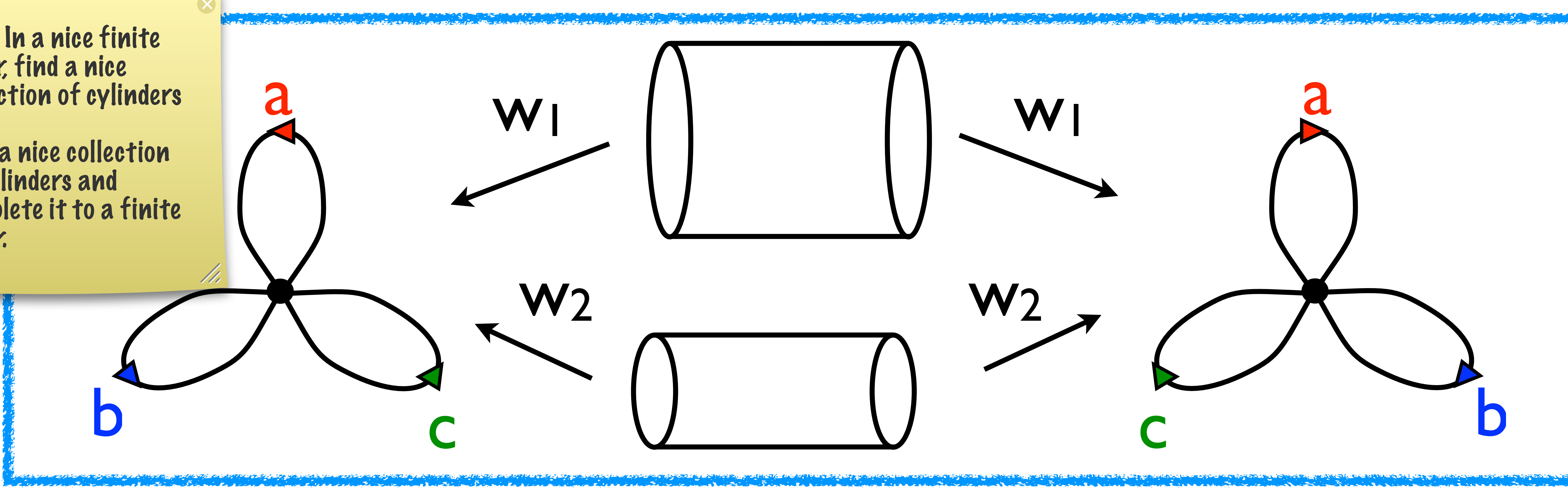
Doubles of Free Groups $D(w) = F *_{<w>} F$

Def (Canary) $U=\{w_1, w_2, \ldots\} \subseteq F$ is *diskbusting* if F cannot be written as $A * B$ such that each w_i is conjugate into A or B.

Fact D(U) is one-ended iff U is diskbusting.

So, how do we find surface subgroups?

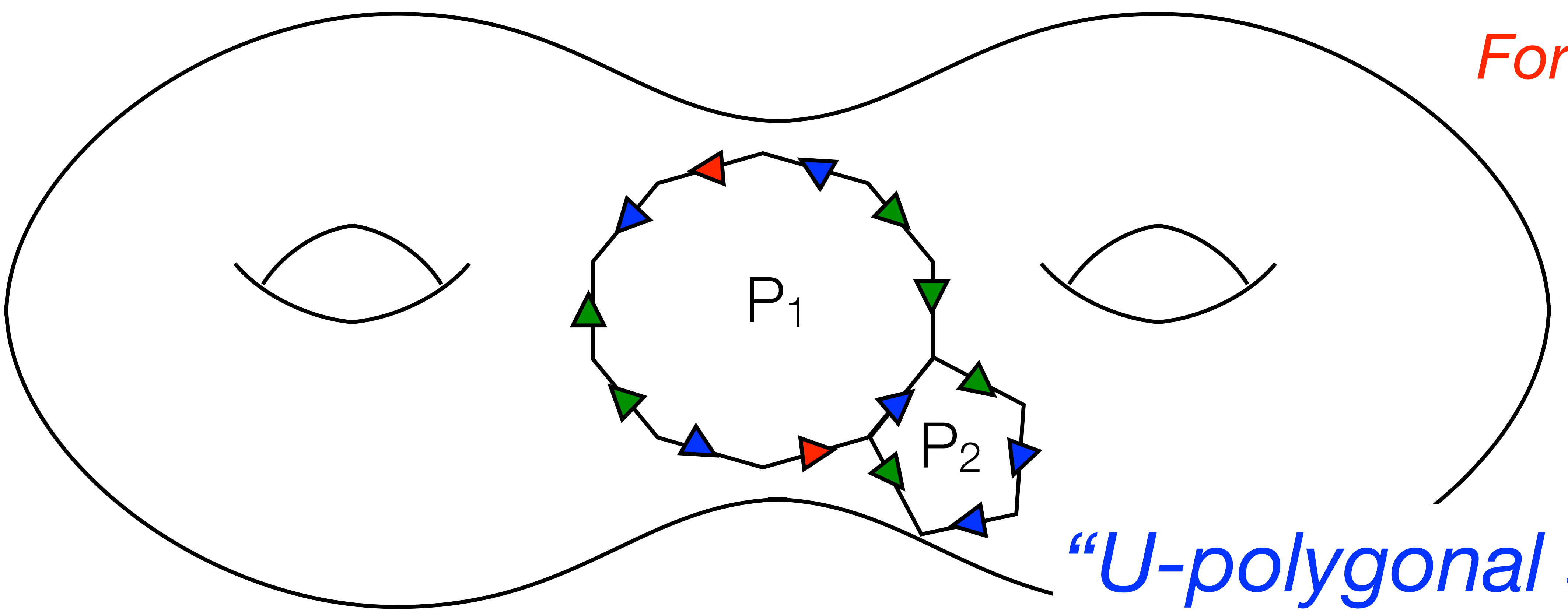
Idea: In a nice finite cover, find a nice collection of cylinders
=
Find a nice collection of cylinders and complete it to a finite cover.

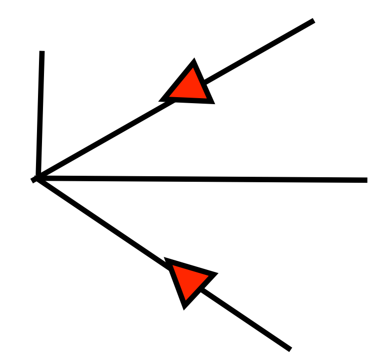


For $U=\{w_1, w_2, \ldots\} \subseteq F_n$, $\pi_1(X(U)) = D(U)$

Polygonality

Def $U \subseteq F$ is *polygonal* if there exists a closed surface $S = \coprod P_i / \sim$ equipped with $S^{(1)} \nrightarrow Cay(F)/F = \vee S^1$ s.t.
(i) $\partial P_i \to S^{(1)} \to Cay(F)/F$ reads a power of an element in U;
(ii) $\chi(S) < m$ = (# of disks)





π_1–injectively Immersed Surfaces

S with $\chi(S)<m(S)$

$\Rightarrow$ Double $\chi(D(S))=2(\chi(S)-m(S))<0$

$\Rightarrow$ Complete to a cover of $X(U)$

Theorem (K.–Wilton ’09)
If $U \subseteq F$ is *polygonal,* then $X(U)$ admits a π_1-injectively immersed hyperbolic surface.

$|U|$:= sum of the lengths of the words in U.

U is *minimal*, if no automorphism of F reduces $|U|$.

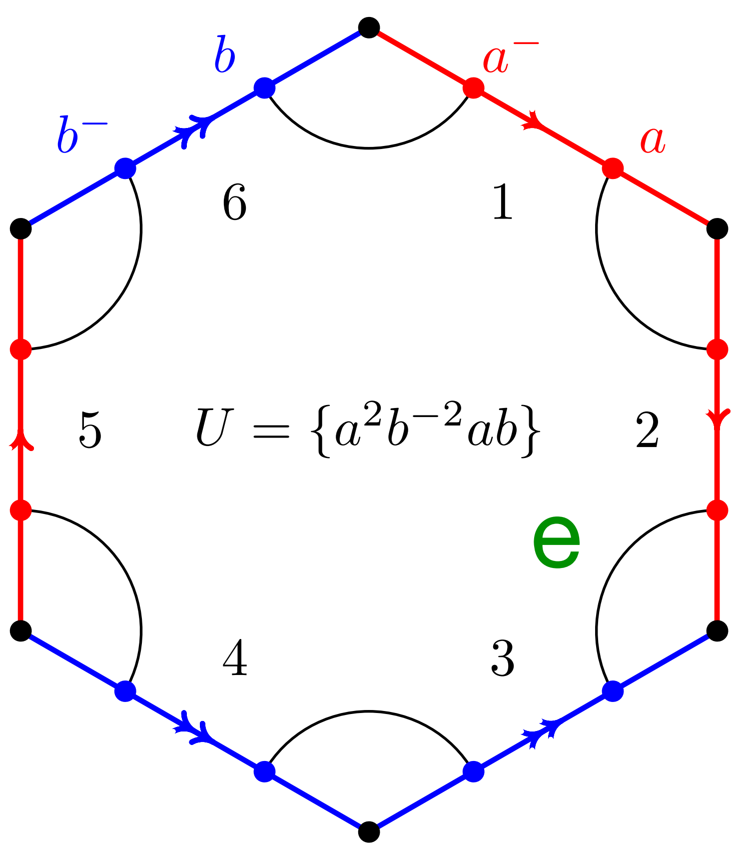
Tiling Conjecture (K.–Wilton ’09)
A minimal diskbusting list of words is polygonal.

Detecting Polygonality

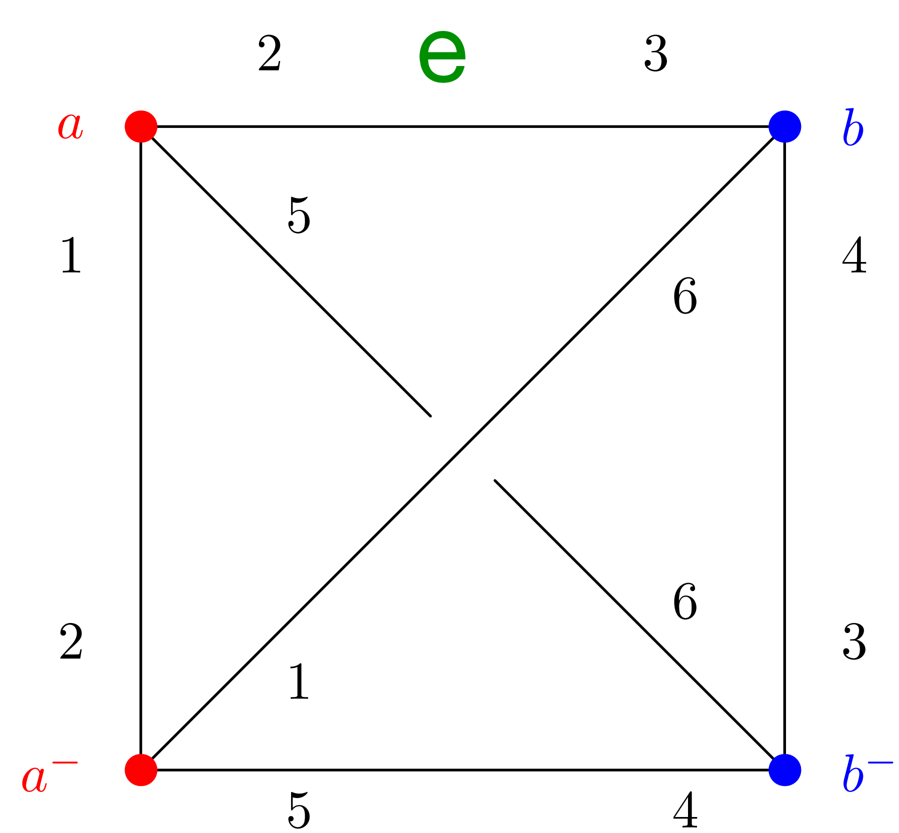
Recall $\pi_1(Z(U)) = F/\langle\langle U \rangle\rangle$.

Def W(U) = Link(*, Z(U)) is the *Whitehead graph* of U ⊆ F.

<u>Link</u> of a vertex on S → **<u>Simple cycle</u>** on W(U)



$Z(U)$



$W(U) = Link(v, Z(U))$

Rank Two Case

Theorem (K.–Oum)

Tiling Conjecture is true for $n = 2$.

Corollary (K.–Oum) For U : set of words in F_2, TFAE:

(1) D(U) is one-ended;

(2) D(U) contains a hyperbolic surface group.

Regular/geometric Lists of Words

U⊊F is *geometric*, if U is a multicurve on ∂(handlebody).

(Gordon–Wilton) U: geometric ⇒ D(U) contains a surface gp.

Theorem (K '09)

A minimal, diskbusting, geometric list is polygonal.

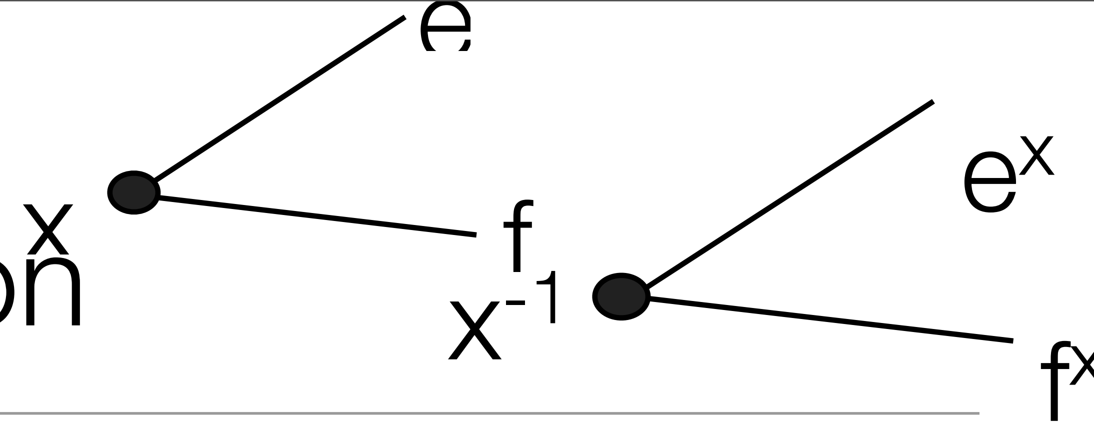
We say U⊊F is *k-regular*, if each generator appears k times.

(Manning) A k-regular, non-planar, minimal (+α) list is **not** virtually geometric.

Theorem (K.–Oum)

A k-regular, minimal, diskbusting list is polygonal.

Graph Theoretic Formulation



k(u,v)=maxflow(u,v)=mincut(u,v) in a graph.

(Whitehead, Berge..) U⊆F is minimal and diskbusting ⇔

W(U) is connected and $k(x,x^{-1})=\deg(x)$ for each vertex x.

Tiling Conjecture (Graph Theoretic Form)
Let $\Gamma=(V,E)$ be a graph with ≥ 4 vertices, equipped with an involution $x \leftrightarrow x^{-1}$ on V, and pairing $e\in\delta(x) \leftrightarrow e^x\in\delta(x^{-1})$.

Assume Γ is connected, and $k(x,x^{-1})=\deg(x)$ for each x.
Then there exist cycles C_1, C_2,..., C_r such that
for each pair of incident edges (e,f) at x:

#of C_i's containing e and f = #of C_i's with e^x and f^x

Graph of Free Groups with **Z**–Edge groups

Proposition (K.–Wilton)

Assume Tiling Conjecture for any rank.

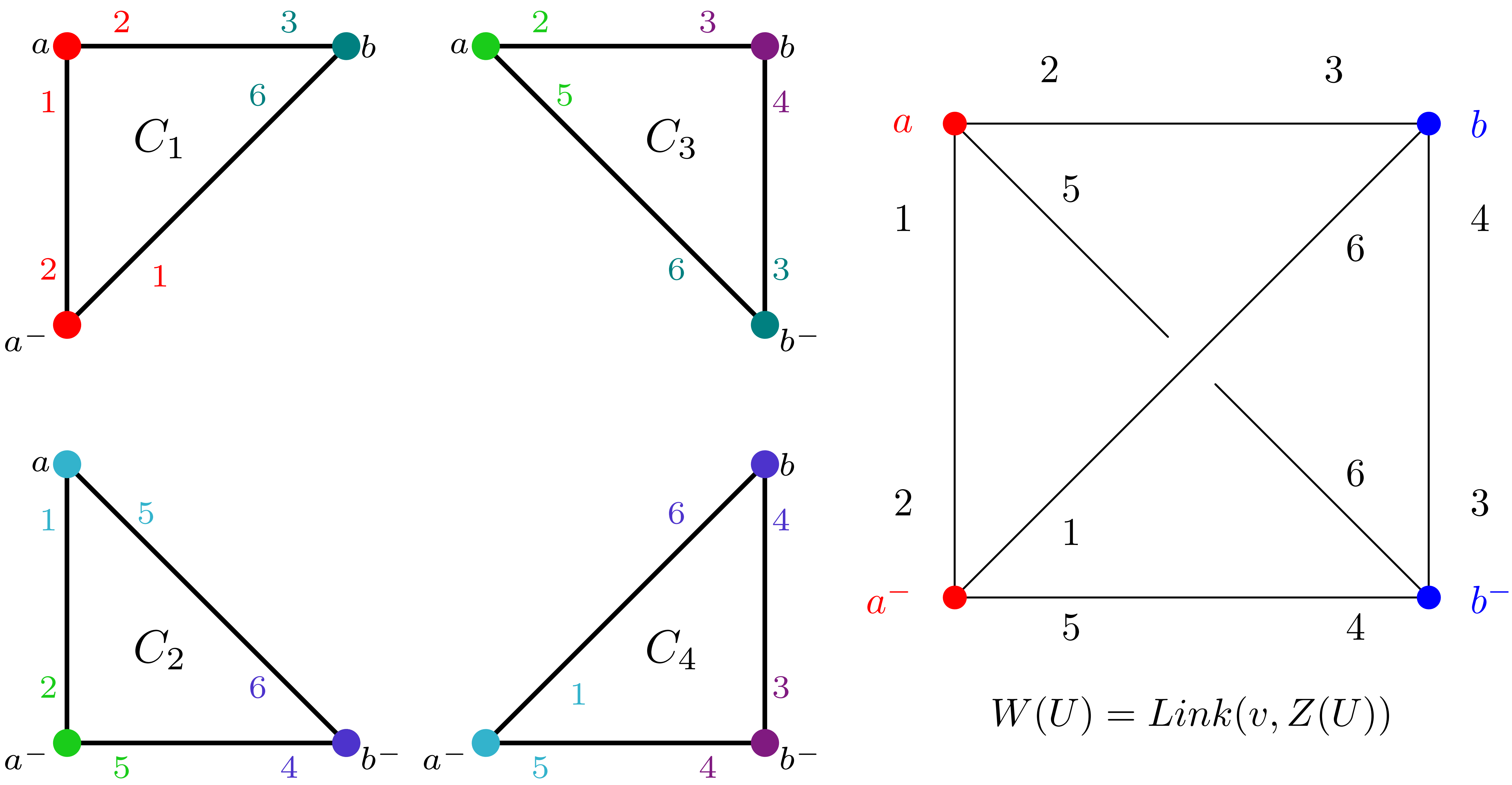
Suppose G is a one-ended graph of (v.) free groups with (v.) cyclic edge groups, such that $g^n \not\sim g^m$ for $|n| \neq |m|$ and $g \neq 1$.

Then either:

(1) G contains a hyperbolic surface group; or,

(2) G is virtually $Z \times F_n$.

Example : U = {aba²b⁻²}



Hence, U = {aba²b⁻²} is polygonal.

Thank you.