Verbal subgroups of hyperbolic groups have infinite width

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Verbal set and verbal subgroup

Let $F(X)$ be a free group on a countable generating set $X = \{x_1, x_2, \ldots\}$. Let $w \in F(X)$.

Let G be a group. $g \in G$ is called a w-element if g is an image of w under a homomorphism $F(X) \to G$.

One can think of w as a monomial $w = w(x_1, x_2, \ldots, x_k)$. Then $w(g_1, g_2, \ldots, g_k) \in G$ is the image of w under homomorphism extending map $x_i \to g_i$.

The set of w-elements in G, also called the set of values of w in G, is denoted $w[G]$:

$$\{g \in G \mid g = w(g_1, \ldots, g_k)\} = w[G].$$

The subgroup generated by $w[G]$ is denoted by $w(G)$:

$$\langle w[G] \rangle = w(G).$$

$w(G)$ is called w-verbal subgroup of G.

Examples:

- $w = x^{-1}y^{-1}xy$. $w(G) = [G, G]$.
- $w = x^2$, $G = \mathbb{Z}$. $w(G) = 2\mathbb{Z}$.
- $w = x^5y^{-2}$. $w(G) = w[G] = G$ since $g = g^5(g^2)^{-2} = w(g, g^2)$.

Represent w as

$$w = x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k} w',$$

where $w' \in [F, F]$. Denote $e(w) = \gcd(m_1, m_2, \ldots, m_k)$, or $e(w) = 0$ if all $m_i = 0$.

If $e(w) = d > 0$, then every d-th power $g^d \in w[G]$. Indeed, there are $d_1, \ldots, d_k$ such that

$$d_1 m_1 + \ldots + d_k m_k = d.$$

Then $w(g^{d_1}, g^{d_2}, \ldots, g^{d_k}) = g^d$.

In particular, if $e(w) = 1$, then $w[G] = G$.

Words $w \in F$ s.t. $w \neq 1$ in F and $e(w) \neq 1$ are called proper.

Question: can elements of $w(G)$ be represented as a product of bounded number of values of $w^{\pm 1}$?

For a $g \in w(G)$, define its w-width:

$$l_w(g) = \min\{n \mid g = g_1 g_2 \cdots g_n,\ g_i^{\pm 1} \in w[G]\}.$$

w-width of G is defined to be

$$l_w(G) = \sup\{l_w(g) \mid g \in w(G)\},$$

which is a non-negative integer or infinity. If $l_w(G) < \infty$ for any w, we say that G is verbally elliptic:

$$\exists l \ w(G) \subseteq w^{\pm 1}[G]^l$$

If $l_w(G) = \infty$ for any proper w, we say that G is verbally parabolic:

$$\forall l \ w(G) \not\subseteq w^{\pm 1}[G]^l$$

History

- Ore's Conjecture (1951): Commutator width of non-abelian finite simple groups is 1. Established by Liebeck, O'Brian, Shalev and Tiep (2010).
- Serre's Conjecture: If G is a finitely generated profinite group then every subgroup of finite index is open. Proved by Nikolov and Segal (2007). Proof based on establishing uniform bounds on verbal width in finite groups.

In infinite groups, study was initiated by P. Hall.

- Stroud (1960's): All finitely generated abelian-by-nilpotent groups G are verbally elliptic.
- Rhemtulla (1968): All free products (except for infinite dihedral group) are verbally parabolic.

- Merzlyakov (1967): All linear algebraic groups are VE.
- Romankov (1982): All f.g. virtually nilpotent and virtually polycyclic groups are VE.
- Grigorchuk (1996): Groups in a wide class of amalgamated free products and HNN-extensions are VP, commutator words.
- Bardakov: Braid groups are VP (1992), HNN-extensions with proper associated subgroups and one relator groups with at least three generators are VP (1997).
- Dobrynina (2000): Groups in a wide class of amalgamated free products are VP.

Theorem. Every non-elementary hyperbolic group G is VP, i.e., every proper verbal subgroup of G has infinite width.

Rhemtulla's proof

In 1968, Rhemtulla showed that w-verbal subgroups free products (with exception to infinite dihedral group) have infinite width for every proper w.

Essentially, given $G = A * B$, Rhemtulla fixes an element $b \in B$ and counts

$$\text{number of subwords of the form } bu_1u_2\ldots u_kb$$
$$\text{minus}$$
$$\text{number of subwords of the form } b^{-1}u_1u_2\ldots u_kb^{-1}$$

for every k.

Turns out, for a given w and bounded l, for any $g \in w^{\pm 1}[G]^l$ this value must be a multiple of $e(w)$, except for finitely many values of k.

Provided that, to disprove finite width, one can easily construct an element $g \in w(G)$ where arbitrarily many values of k give subwords that occur exactly 1 time.

In other words, Rhemtulla builds a function $\gamma(g)$ that counts number of k's such that subwords $bu_1 \dots u_k b$ are "irregular".

γ is bounded on $w^{\pm 1}[G]^l$ and unbounded on $w(G)$.

Plan

How do we adopt this approach to the case of hyperbolic groups?

1. Decide occurrences of what to count.

2. Figure out things repeat $e(w)$ times.

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Alternative approach: adopt Fujiwara's treatment of second bounded cohomologies in hyperbolic groups.

Idea of Proof

1. Restrict consideration to a set $R \subseteq G$ that consist of products of "big" powers of some fixed elements.

By the Big Powers Condition in hyperbolic groups, this gives a well defined notion of a subword.

Function $\gamma : R \to \mathbb{Z}$ counts number of “irregular” subwords in g, that is the number of subwords of specific form that occur a number of times not divisible by $e(w)$.

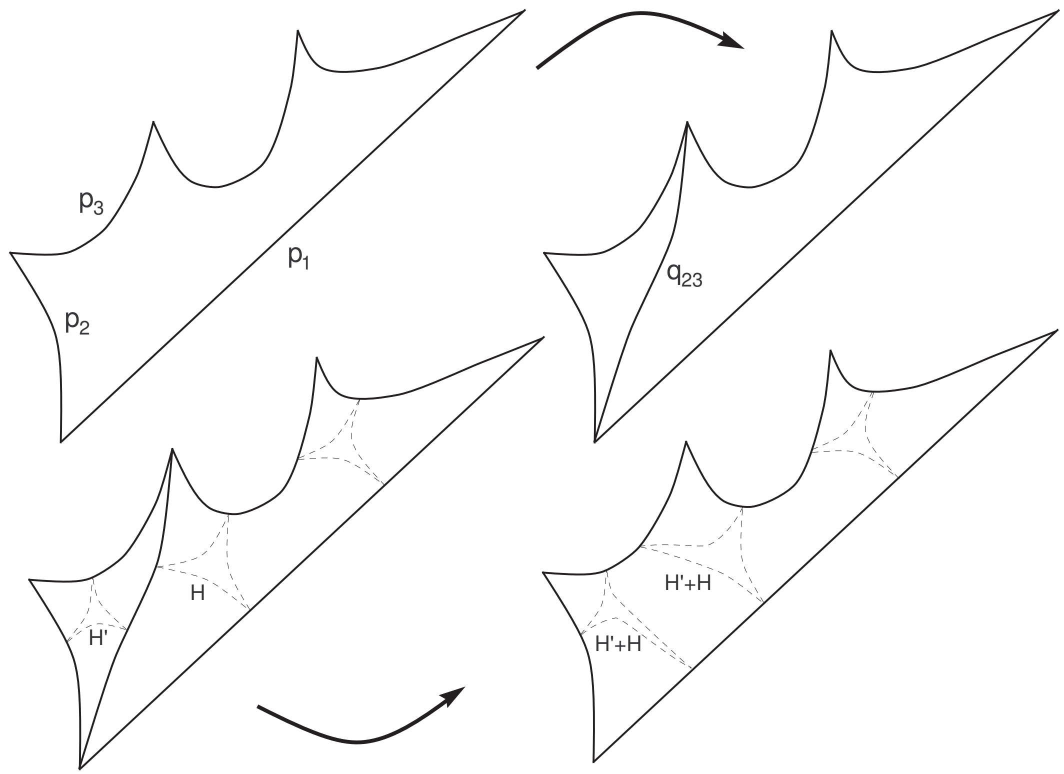
We show that γ is bounded on $R \cap w^{\pm 1}[G]^l$ and unbounded on $R \cap w(G)$.

Idea of Proof

2. Draw equality $g = w(g_1, \ldots, g_k)$ in the Cayley graph of G, get a "thin" polygon.

Thin hyperbolic n-gons

Since triangles in a hyperbolic space are δ-thin, all geodesic n-gons are also δ'-thin (where δ' depends on n):



This allows to organize $g = w(g_1, \ldots, g_k)$ into a "cancelation" picture. (See blackboard)

Once we've shown that γ is bounded on $R \cap w^{\pm 1}[G]^l$, it is not hard to show that γ is unbounded on $R \cap w(G)$, similarly to the case $G = A * B$.

Consequences

Observation: if a group G has a verbally parabolic homomorphic image, then G is verbally parabolic. Therefore, the following groups are VP (by original Rhemtulla's result):

- non-abelian residually free groups;
- pure braid groups (also follows from Bardakov's results);
- non-abelian right angled Artin gorups.

Consequence of the main result: non-elementary groups hyperbolic relative to proper residually finite subgroups (Osin) are VP. Thus, the following non-elementary groups are VP:

- the fundamental groups of complete finite volume manifolds of pinched negative curvature;
- $CAT(0)$ groups with isolated flats;
- groups acting freely on $\mathbb{R}^n$–trees.