SIMPLICIAL SETS & SPECTRA

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Abstract. These are notes for my talk at the Wuppertal Summer School on Derived and Triangulated Categories.

1. Introduction

1.1. Notation.

- $\mathcal{S}\mathrm{et}, \mathcal{T}\mathrm{op}$ is the category of sets resp. topological spaces.
- $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the category of functors $\mathcal{C} \to \mathcal{D}$, for given categories $\mathcal{C}, \mathcal{D}$.
- $\mathcal{C}(x, y)$ is the hom-set $\hom_{\mathcal{C}}(x, y)$ for objects x, y in a category $\mathcal{C}$.

2. Simplicial sets

In mainly follow [GJ09].

2.1. The category of simplicial sets.

Definition. The *simplex category* $\mathbf{\Delta}$ is the category with finite, linearly ordered sets as objects, and order preserving functions as morphisms.

For $n \in \mathbb{N}$, we write $[n] = \{0, 1, 2, \ldots, n\}$ (with obvious order) for the unique (up to isomorphism) object in $\mathbf{\Delta}$ with $n + 1$ elements.

Definition. A *simplicial set* is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$. The *category of simplicial sets* is $\mathrm{s}\mathcal{S}\mathrm{et} := \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$.

In order to describe simplicial sets, it is convenient to know that the morphisms in $\mathbf{\Delta}$ are generated by the following special ones. Consider the

coface maps:	$d^i : [n-1] \to [n]$	for $0 \le i \le n$
codegeneracy maps:	$s^j : [n+1] \to [n]$	for $0 \le j \le n$

defined as follows.

- d^i is the unique injective map in $\mathbf{\Delta}$ which misses $i \in [n]$.
- s^j is the unique surjective map in $\mathbf{\Delta}$ which hits $j \in [n]$ twice.

Now every morphism in $\mathbf{\Delta}$ can be written as a composition of coface and codegenarcy maps. In fact, we can say more. For this, we first observe that these maps together satisfy certain relationships in $\mathbf{\Delta}$, called the *cosimplicial identities*:

$$\begin{aligned} d^j d^i &= d^i d^{j-1} && \text{if } i < j \\ s^j d^i &= d^i s^{j-1} && \text{if } i < j \\ s^j d^j = \mathrm{id} &= s^j d^{j+1} && \\ s^j d^i &= d^{i-1} s^j && \text{if } i > j+1 \\ s^j s^i &= s^i s^{j+1} && \text{if } i \le j \end{aligned}$$

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Now let $\mathcal{C}$ be the category where an object is a series of sets $\{X_n\}_{n\geq 0}$ together with

face maps:	$d_i : X_n \to X_{n-1}$	for $0 \leq i \leq n$
degeneracy maps:	$s_j : X_n \to X_{n+1}$	for $0 \leq j \leq n$

satisfying the *simplicial identities*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{if } i < j \\ d_i s_j &= s_{j-1} d_i && \text{if } i < j \\ d_j s_j = \mathrm{id} &= d_{j+1} s_j && \\ d_i s_j &= s_j d_{i-1} && \text{if } i > j+1 \\ s_i s_j &= s_{j+1} s_i && \text{if } i \leq j \end{aligned}$$

A morphism $\{X_n\}_{n\geq 0} \to \{Y_n\}_{n\geq 0}$ is a sequence of functions $\{f_n : X_n \to Y_n\}_{n\geq 0}$, commuting with the d_i's and s_j's. Since the simplicial identities are exactly dual to the cosimplicial identities, we have a functor

$$H : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{C}$$

which sends a simplicial set $X : \mathbf{\Delta} \to \mathcal{S}\mathrm{et}$ to the sequence of sets $X_n := X([n])$, together with the maps $d^i := X(d_i)$ and $s^j := X(s_j)$. Now the coface and codegenarcy maps plus to cosimplicial identities generate $\mathbf{\Delta}$ in the following precise sense:

Proposition. *The functor H is an equivalence.*

Fundamentally, simplicial sets are a combinatorial way to think of *spaces*. To give the idea, let $X \in \mathrm{s}\mathcal{S}\mathrm{et}$ be given. The elements of X_n are called *n-simplices*. We think of an n-simplex $\sigma \in X_n$ as an 'n-cell' in the 'space' X, with faces $d_0\sigma, \ldots, d_n\sigma$. For example, we think of a 1-simplex γ as a path from the point $d_1\gamma$ to the point $d_0\gamma$, and a two-simplex τ as a triangle with faces $d_0\tau, d_1\tau, d_2\tau$, and vertices

$$d_0 d_0 \tau = d_0 d_1 \tau \qquad d_1 d_2 \tau = d_1 d_1 \tau \qquad d_0 d_2 \tau = d_1 d_0 \tau$$

We can see that these are the vetices by drawing a picture, and see that this agrees with the simplicial identities.

An n-simplex σ in X is called *degenerate* if there is some $(n-1)$-simplex τ such that $\sigma = s_j\tau$ for some τ. For example, for a 0-simplex $x \in X$, we think of $s_0 x$ as the trivial path at x. A simplex which is not degenerate is called *non-degenerate*.

Example. Let $n \geq 0$. The most fundamental example of a simplicial set is the *standard n-simplex*, written $\Delta[n]$. As a functor, this is nothing but the image of $[n]$ under Yoneda $\mathbf{\Delta} \to \mathrm{s}\mathcal{S}\mathrm{et}$, so $\Delta[n] = \mathbf{\Delta}(-, [n])$. As a space, we think of $\Delta[n]$ as the topological n-simplex, which is the convex hull of its $n+1$ vertices. As a series of sets, we have that $\Delta[n]_0$ is the set of vertices, labelled $0, 1, \ldots, n$. The 1-simplices are the edges and the degenerate vertices, and so on for higher dimensions. There is a unique non-degenerate simplex in highest level, namely $\iota_n := \mathrm{id}_{[n]} \in \Delta[n]_n = \mathbf{\Delta}([n], [n])$. It corresponds to the n-cell which is the whole topological n-simplex.

The *boundary* $\partial\Delta[n]$ of $\Delta[n]$ is the simplicial set generated by simplices of level $< n$. It is the largest subcomplex of $\Delta[n]$ that contains all the faces $d_i\iota_n$.

For $0 \leq k \leq n$, the *k-th horn* $\Lambda^k[n]$ is the largest subcomplex of $\Delta[n]$ which contains all faces $d_i\iota_n$ except the k-th face $d_k\iota_n$. We have inclusions $\Lambda^k[n] \subset \partial\Delta[n] \subset \Delta[n]$. It is instructive to draw pictures for low n.

By Yoneda, we have

$$X \cong \operatorname*{colim}_{\Delta[n] \to X} \Delta[n]$$

for any $X \in \mathrm{s}\mathcal{S}\mathrm{et}$. In particular, X_n is naturally isomorphic to $\mathrm{s}\mathcal{S}\mathrm{et}(\Delta[n], X)$.

Example. Let $\mathcal{C}$ be a category. The *nerve* of $\mathcal{C}$ is the simplicial set $N(\mathcal{C})$, where $N(\mathcal{C})_n$ is the set of diagrams in $\mathcal{C}$ of the form

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} x_n$$

The maps d_i act by composing f_{i-1} with f_i (or removing f_0 resp. f_{n-1} when $i = 0$ resp. $i = n$), while s_j acts by inserting an identity at x_j.

More abstractly, we consider $[n] \in \mathbf{\Delta}$ as a category in the obvious way. Then the simplicial set $N(\mathcal{C})$ is the functor $\mathrm{Ob}(\mathrm{Fun}(-, \mathcal{C}))$. This also shows that the nerve construction is natural in $\mathcal{C}$: the functor

$$N(-) : \mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{S}\mathrm{et} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$$

is the 'transpose' of the hom-set functor $\mathcal{C}\mathrm{at} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$.

2.2. **Geometric realization.** Write $|\Delta[n]|$ for the topological n-simplex.

Example. The *singular complex* $\mathrm{Sing}(Y)$ of a topological space Y is the simplicial set

$$[n] \mapsto \mathcal{T}\mathrm{op}(|\Delta[n]|, Y)$$

This gives a functor $\mathrm{Sing}(-) : \mathcal{T}\mathrm{op} \to \mathrm{s}\mathcal{S}\mathrm{et}$.

Later, we will see that the singular complex can be used to define singular homology. First, we will see how it gives an adjunction between simplicial sets and topological spaces, which is one step towards formalizing the idea of thinking of simplicial sets as spaces.

Definition. The *geometric realization functor* $|-| : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$ is the left Kan extension of the functor $|-| : \mathbf{\Delta} \to \mathcal{T}\mathrm{op}$ along the Yoneda embedding $\Delta[-] : \mathbf{\Delta} \to \mathrm{s}\mathcal{S}\mathrm{et}$.

In concrete terms, for a simplicial set X we have that

$$|X| = \operatorname*{colim}_{\Delta[n] \to X} |\Delta[n]|$$

This reflects the idea of simplicial sets as spaces explained earlier: to get the geometric realization of X, we take a topological n-simplex $|\Delta[n]|$ for each n-simplex in X, and then glue them together according to the face- and degeneracy maps of X.

Example. The simplicial n-sphere S^n is the simplicial set $\Delta[n]/\partial\Delta[n]$. Its geometric realization is an n-sphere.

Example. Let G be a group, considered as a category with 1 element and with G as hom-set. Then the classifying space of G is $|NG|$.

Proposition. *The singular complex is right adjoint to the geometric realization:*

$$|-| : \mathrm{s}\mathcal{S}\mathrm{et} \rightleftarrows \mathcal{T}\mathrm{op} : \mathrm{Sing}(-)$$

The proof is a straightforward application of Yoneda. We will later see that this adjunction can be turned into a Quillen equivalence.

One can show that the geometric realization of a simplicial set is a CW complex. In particular, it is a compactly generated Hausdorff space.[1] Write $\mathcal{T}\mathrm{op}^{\mathrm{cgh}}$ for the category of compactly generated Hausdorff space. Then the functor

$$|-| : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}^{\mathrm{cgh}}$$

preserves all finite limits.

[1]Recall that a topological space Y is *compactly generated* if $F \subset Y$ is closed if and only if $F \cap K$ is closed in K for all compacta $K \subset Y$.

2.3. **Dold-Kan correspondence.** We generalize the idea of simplicial sets to arbitrary categories:

Definition. Let $\mathcal{C}$ be a category. Then a *simplicial object* in $\mathcal{C}$ is a functor $\boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{C}$. The category of such is $\mathrm{s}\mathcal{C} \coloneqq \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{C})$.

Of particular interest is when $\mathcal{C}$ is the category $\mathcal{A}\mathrm{b}$ of abelian groups. The *Dold-Kan* correspondence tells us that we can think of $\mathrm{s}\mathcal{A}\mathrm{b}$ as chain complexes $\cdots X_2 \to X_1 \to X_0$ in positive degrees. Writing the category of such as $\mathrm{Ch}_{\geq 0}$, then we have

Proposition. *There is an equivalence of categories*

$$N : \mathrm{s}\mathcal{A}\mathrm{b} \rightleftarrows \mathrm{Ch}_{\geq 0} : \Gamma$$

To explain the functor N, we need a few definitions. Let $A \in \mathrm{s}\mathcal{A}\mathrm{b}$ be given.

- The *Moore complex* CA is the chain complex where $CA_n \coloneqq A_n$, and with boundary maps the alternating face maps

$$\partial \coloneqq \sum\nolimits_{0 \leq i \leq n} (-1)^i d_i : CA_n \to CA_{n-1}$$

- The *degeneracies* in the Moore complex is the subcomplex DA where DA_n is the subgroup of CA_n generated by degenerate simplices.
- The *normalized chain complex* is the subcomplex NA of CA with $\bigcap_{1 \leq i \leq n} \ker d_i$ in degree n, and with d_0 as boundary maps.

Lemma. *For the natural maps $NA \to CA \to CA/DA$, it holds*

- *$NA \to CA/DA$ is an isomorphism;*
- *$NA \to CA$ is a quasi-isomorphism (in fact a chain homotopy equivalence).*

For a chain complex C, one has that $\Gamma(C)_n$ is the direct sum $\bigoplus_{[n] \twoheadrightarrow [k]} C_k$ indexed over surjections $[n] \twoheadrightarrow [k]$. We omit the description of the face and degeneracy maps.

Example. Let $\mathbb{Z} : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathrm{s}\mathcal{A}\mathrm{b}$ be the functor that takes free abelian groups level-wise. Then we recover the singular homology functor as the composition

$$\mathcal{T}\mathrm{op} \xrightarrow{\mathrm{Sing}(-)} \mathrm{s}\mathcal{S}\mathrm{et} \xrightarrow{\mathbb{Z}} \mathrm{s}\mathcal{A}\mathrm{b} \xrightarrow{C(-)} \mathrm{Ch}_{\geq 0} \xrightarrow{H_n} \mathcal{A}\mathrm{b}$$

2.4. **Model structure on** $\mathrm{s}\mathcal{S}\mathrm{et}$**.** Recall that a map of topological spaces $Y \to Y'$ is a weak equivalence (or simply: *equivalence*) if it induces isomorphisms on all homotopy groups.

Definition. We call a map $f : X \to X'$ in $\mathrm{s}\mathcal{S}\mathrm{et}$ a

- *cofibrations* if it is a the monomorphisms, i.e., a levelwise injection,
- *equivalence* if $|f|$ is an equivalence,
- *Kan fibration* if it has the right lifting property with respect to all horn inclusions $\Lambda^k[n] \to \Delta[n]$, meaning that for each solid commutative diagram

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & X' \end{array}$$

  the dotted arrow exists, making the diagram commutative.

Proposition. *These fibration, cofibrations and equivalences define a model structure on* $\mathrm{s}\mathcal{S}\mathrm{et}$.

This model structure is called the *standard* or *Quillen* model structure, and is designed to model spaces by simplicial sets. There are other model structures available, with other purposes. For example, one has the Joyal model structure, which is designed to model ∞-categories by simplicial sets, but we won't go into this.

Neither will we go into the (non-trivial) proof of the above statement, but refer to e.g. [Hov99, §3.2]. From here on, we endow $\mathrm{s}\mathcal{S}\mathrm{et}$ with the Quillen model structure.

Remark. In fact, $\mathrm{s}\mathcal{S}\mathrm{et}$ has more structure: it is cartesian closed, where the internal mapping space $\mathrm{Map}(X, X')$ is the simplicial set $[n] \mapsto \mathrm{s}\mathcal{S}\mathrm{et}(X \times \Delta[n], X')$. In fact, this makes $\mathrm{s}\mathcal{S}\mathrm{et}$ into a simplicial model category, which means that the model structure is compatible with the enrichment over $\mathrm{s}\mathcal{S}\mathrm{et}$ in a specified way that we won't go in to.

Definition. A *Kan complex* is a fibrant simplicial set, that is, a simplicial set K for which each horn $\Lambda^k[n] \to K$ has an extension $\Delta[n] \to K$.

Let K be a Kan complex, and $x \in K$ a 0-simplex. We can define the *homotopy groups* $\pi_n(K, x)$ of K at x as follows. Consider $S^n = \Delta[n]/\partial\Delta[n]$ as pointed, with point $\partial\Delta[n]$. Then an element of $\pi_n(K, x)$ is an equivalence class of a pointed map $(S^n, \partial\Delta[n]) \to (K, x)$, where the equivalence relation is homotopy of maps relative to $\partial\Delta[n]$. The latter is not hard to define, by drawing the same diagrams as one would in the topological case, using $\Delta[1]$ as the unit interval $[0, 1]$, etc.

One can show that the $\pi_n(K, x)$ are actually groups, using the horn-filling condition. For example, for $g, h \in \pi_n(K, x)$ with representatives $\gamma, \eta : \Delta[1] \to K$, we take the horn $\tau : \Lambda^1[2] \to K$ with edges γ, η. Then a filler $\sigma : \Delta[2] \to K$ of τ gives a concatenation of paths: $\gamma \cdot \eta = d^1\sigma$.

For a general simplicial set X, one puts $\pi_n(X, x) := \pi_n(K, x)$, where K is a fibrant replacement of X.

Theorem. *The adjunction* $|-| \dashv \mathrm{Sing}(-)$ *is a Quillen equivalence.*

In particular, up to fibrant/cofibrant replacements, it does not matter whether we compute homotopy groups in $\mathrm{s}\mathcal{S}\mathrm{et}$ or in $\mathcal{T}\mathrm{op}$. Likewise, we have

Proposition. *For* $A \in \mathrm{s}\mathcal{A}\mathrm{b}$*, it holds* $\pi_n(A, 0) \cong H_n(NA) \cong H_n(CA)$.

2.5. **Pointed simplicial sets.** The category of *pointed simpicial sets* is the under-category $\mathrm{s}\mathcal{S}\mathrm{et}_* := \mathrm{s}\mathcal{S}\mathrm{et}_{\Delta[0]/}$.

Definition. For $X, X' \in \mathrm{s}\mathcal{S}\mathrm{et}_*$,

- the *smash product* is $X \wedge X' := (X \times X')/(X \vee X')$, where $X \vee X'$ is the coproduct in $\mathrm{s}\mathcal{S}\mathrm{et}_*$,
- the *mapping space* is the simplicial set $[n] \mapsto \mathrm{s}\mathcal{S}\mathrm{et}_*(X \wedge (\Delta[n] \sqcup \{*\}), X')$, pointed by the trivial map to the point of X'.

This makes $\mathrm{s}\mathcal{S}\mathrm{et}_*$ into a closed monoidal category, in the sense that we have an adjuntion

$$(-) \wedge X \dashv \mathrm{Map}_*(X, -)$$

Let $X \in \mathrm{s}\mathcal{S}\mathrm{et}_*$ be given. The *suspension* of X is $\Sigma X := X \wedge S^1$, while the *loop space* of X is $\mathrm{Map}_*(S^1, X)$. By the closed monoidal structure on $\mathrm{s}\mathcal{S}\mathrm{et}_*$, these form an adjoint pair $\Sigma \dashv \Omega$. In fact, this is a Quillen adjunction.

Observe, for $X \in \mathrm{s}\mathcal{S}\mathrm{et}_*$ it holds that $\pi_n(\Omega X) \cong \pi_{n+1}(X)$.

3. Spectra

I mainly follow [BF78] and [Lev16].

One idea of spectra is to stabilize the adjoint pair $\Sigma \dashv \Omega$ on $\mathrm{s}\mathcal{S}\mathrm{et}_*$ to an adjoint equivalence by inverting Σ. It turns out that the resulting homotopy category is a triangulated category. An analogy is with chain complexes. The homotopy category of $\mathrm{Ch}_{\geq 0}$ is not triangulated, since the suspension $\Sigma = (-)[1]$ is not an equivalence. The solution is by going to unbounded chain complexes, and thus admitting negative homology groups. Similarly, it will turn out that by inverting Σ on $\mathrm{s}\mathcal{S}\mathrm{et}$ we will introduce negative homotopy groups, and by passing to the homotopy category, we get a triangulated category.

Another motivation for spectra comes from the fact that they represent homology theories, but we will not go into this. There are also other models for spectra available, part of the reason being that people wanted the category of spectra to be symmetric monoidal. We will not touch this at all.

3.1. **The category of spectra.**

Definition. A *spectrum* E is a sequence $E = (E_0, E_1, \dots)$ of pointed simplicial sets, together with bonding maps $\sigma_n^E : S^1 \wedge E_n \to E_{n+1}$. A *map of spectra* $f : E \to E'$ is a series of maps $f_n : E_n \to E'_n$ commuting with the bonding maps. The category of spectra is written $\mathcal{S}\mathrm{p}$.

Example. For $X \in \mathrm{s}\mathcal{S}\mathrm{et}_*$, the *suspension spectrum* is $\Sigma^\infty X = (X, S^1 \wedge X, \dots, S^n \wedge X, \dots)$, with obvious bonding maps. This defines a functor $\Sigma^\infty : \mathrm{s}\mathcal{S}\mathrm{et}_* \to \mathcal{S}\mathrm{p}$, with right adjoint

$$\Omega^\infty : \mathcal{S}\mathrm{p} \to \mathrm{s}\mathcal{S}\mathrm{et}_* : E \mapsto E_0$$

Example. The *sphere spectrum* is $\mathbb{S} \coloneqq \Sigma^\infty S^0$, that is, it has $\mathbb{S}_n \coloneqq S^n$, and obvious bonding maps, using that $S^1 \wedge S^n \cong S^{n+1}$.

Example. Let $X \in \mathrm{s}\mathcal{S}\mathrm{et}_*$ and $E \in \mathcal{S}\mathrm{p}$. Then $E \wedge X$ is the spectrum where $(E \wedge X)_n \coloneqq E_n \wedge X$, with obvious bonding map. We write $\Sigma(E) \coloneqq E \wedge S^1$.

Likewise, $\mathrm{Map}(X, E)$ is the spectrum where $\mathrm{Map}(X, E)_n \coloneqq \mathrm{Map}_*(X, E_n)$, and with bonding map $S^1 \wedge \mathrm{Map}_*(X, E_n) \to \mathrm{Map}_*(X, E_{n+1})$ the adjoint of the map

$$S^1 \wedge \mathrm{Map}_*(X, E_n) \wedge X \xrightarrow{S^1 \wedge \mathrm{ev}} S^1 \wedge E_n \xrightarrow{\sigma_n^E} E_{n+1}$$

We write $\Omega(E) \coloneqq \mathrm{Map}(S^1, E)$.

These constructions are adjoint to one another: we have $(-) \wedge X \dashv \mathrm{Map}(X, -)$. We can extend these definition to unpointed simplicial sets, by adding a disjoint base point. This makes $\mathcal{S}\mathrm{p}$ tensored and cotensored over $\mathrm{s}\mathcal{S}\mathrm{et}$.

For $X \in \mathrm{s}\mathcal{S}\mathrm{et}_*$, the unit of the adjunction $\Sigma \dashv \Omega$ gives maps

$$\Sigma : \pi_m(X) \to \pi_m(\Omega\Sigma X) \cong \pi_{m+1}(\Sigma X) \cong \pi_{m+1}(S^1 \wedge X)$$

Definition. For $n \in \mathbb{Z}$, the *n-th homotopy group* of $E \in \mathcal{S}\mathrm{p}$ is

$$\begin{aligned}\pi_n(E) &\coloneqq \mathrm{colim}_k\, \pi_{n+k}(E_k) \\ &= \mathrm{colim}_k \left(\cdots \to \pi_{n+k}(E_k) \xrightarrow{\Sigma} \pi_{n+k+1}(S^1 \wedge E_k) \xrightarrow{\sigma_k^E} \pi_{n+k+1}(E_{k+1}) \to \dots \right)\end{aligned}$$

A map of spectra $f : E \to E'$ is a *stable weak equivalence* if it induces isomorphisms on all homotopy groups.

3.2. **Ω-spectra.**

Definition. An *Ω-spectrum* is a spectrum E for which the bonding maps $S^1 \wedge E_n \to E_{n+1}$ induce equivalence $E_n \to \Omega E_{n+1}$ via the adjunction $S^1 \wedge (-) \cong \Sigma \dashv \Omega$.

Lemma. *There is a functor $Q : \mathcal{S}\mathrm{p} \to \mathcal{S}\mathrm{p}$, together with a natural transformation $\eta : \mathrm{id} \to Q$, such that for each $E \in \mathcal{S}\mathrm{p}$ the map $\eta_E : E \to QE$ is a stable weak equivalence, and with QE an Ω spectrum.*

From here on, we choose such Q, η.

3.3. **The stable model category structure.**

Definition. Let $f : E \to F$ be a map of spectra. Then f is called a

- *stable cofibration* if $E_0 \to F_0$ is a cofibration, and all the maps

$$E_{n+1} \cup_{S^1 \wedge E_n} (S^1 \wedge F_n) \xrightarrow{(f_{n+1}, \sigma_n^F)} F_{n+1}$$

  are cofibrations as well,
- *stable fibration* if each f_n is a fibration, and each square

$$\begin{array}{ccc} E_n & \xrightarrow{\eta} & (QE)_n \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle Qf_n} \\ F_n & \xrightarrow{\eta} & (QF)_n \end{array}$$

  is a homotopy pullback (= right derived functor of ordinary pullback) in $\mathrm{s}\mathcal{S}\mathrm{et}_*$.

Proposition. *The stable equivalence, cofibrations and fibrations together with the tensoring and cotensoring $(-) \wedge X \dashv \mathrm{Map}(X, -)$ form a simplicial model structure on $\mathcal{S}\mathrm{p}$.*

From here on, we endow $\mathcal{S}\mathrm{p}$ with this model structure. The *stable homotopy category* $\mathcal{SH} := \mathrm{Ho}(\mathcal{S}\mathrm{p})$ is the homotopy category of $\mathcal{S}\mathrm{p}$.

4. The triangulated structure on the stable homotopy category

It turns out that $\mathcal{SH}$ is triangulated: we want to get a taste of the proof of this. Note that $\mathcal{S}\mathrm{p}$ is pointed by the zero-object $0 := \Sigma^\infty(*)$.

4.1. **$\mathcal{SH}$ is additive.** The steps as carried out in [Sch12] are:

(1) $\mathcal{SH}$ has finite products. In fact, the localization functor $\gamma : \mathcal{S}\mathrm{p} \to \mathcal{SH}$ preserves these, so finite products in $\mathcal{SH}$ are given by finite products of spectra.
(2) For $E, F \in \mathcal{S}\mathrm{p}$, the maps

$$E \cong E \times 0 \to E \times F \qquad \& \qquad F \cong 0 \times F \to E \times F$$

exhibit $E \times F$ as coproduct of E and F.
(3) Let $E \in \mathcal{S}\mathrm{p}$. The maps $\Delta : E \to E \times E$ and the first inclusion $E \to E \cup E \cong E \times E$ induce a map $E \times E \to E \times E$ by the previous point. We ask that this is an equivalence.

The additive structure on $\mathcal{SH}(E, F)$ is then given by

$$(f + g) = E \xrightarrow{\Delta} E \times E = E \cup E \xrightarrow{(f,g)} E$$

for $f, g \in \mathcal{SH}(E, F)$.

4.2. **The equivalence Σ.** In this part, I mainly follow [Jar07]. The idea is to relate $\Sigma \dashv \Omega$ to a Quillen equivalence via natural stable equivalences.

Definition. For $n \in \mathbb{Z}$ and $E \in \mathcal{S}\mathrm{p}$, we define the spectrum $E[n]$ via

$$E[n]_m := \begin{cases} 0 & \text{if } n+m < 0 \\ E_{n+m} & \text{if } n+m \geq 0 \end{cases}$$

Clearly, we have an adjunction $(-)[1] \dashv (-)[-1]$. Also, for $n, m \in \mathbb{Z}$, it holds

$$\pi_n(E[m]) = \operatorname{colim}_k \pi_{n+k}(E_{k+m}) \cong \operatorname{colim}_k \pi_{n-m+k}(E_k) \cong \pi_{n-m}(E)$$

Lemma. *If $E \to F$ is a stable cofibration, then so is $E[1] \to F[1]$.*

Proof. This is straightforward, by using that $E \to F$ being a stable cofibration implies that all $E_n \to F_n$ are cofibrations. □

Proposition. *The adjunction $(-)[1] \dashv (-)[-1]$ is a Quillen equivalence $\mathcal{S}\mathrm{p} \rightleftarrows \mathcal{S}\mathrm{p}$.*

Proof. Since $(-)[1]$ preserves cofibrations and trivial cofibrations, the adjunction is Quillen. Since $(-)[m]$ shifts homotopy groups, it holds that $E \to F[-1]$ is a stable weak equivalence if and only if $E[1] \to F$ is. □

Proposition. *There are natural stable equivalences $\Sigma \simeq (-)[1]$ and $\Omega \simeq (-)[-1]$. Consequently, $\Sigma \dashv \Omega$ is a Quillen adjunction, and Σ is an equivalence on $\mathcal{SH}$.*

4.3. **Distinguished triangles.** One can define distinguished triangles explicitly using mapping cones, as in [Wei94, §10.9]. We will instead use homotopy theory in a slightly informal way.

Definition. Let $f : E \to F$ be a map of spectra (or a map in any pointed model category). Then the *homotopy cofiber* $\operatorname{hocofib}(f)$ is the homotopy pushout $F \cup^h_E 0$ (i.e., the left derived functor of the ordinary cofiber functor).

One can show that there is a natural stable equivalence

Also mention in terms of cone

$$\operatorname{hocofib}(E \to 0) \simeq \Sigma E$$

Consequently, for any $f : E \to F$ in $\mathcal{S}\mathrm{p}$, we have a sequence of maps

$$E \xrightarrow{f} F \to \operatorname{hocofib}(f) = F \cup^h_E 0 \to 0 \cup^h_E 0 \simeq \Sigma E$$

Definition. A *distinguished triangle* in $\mathcal{SH}$ is any sequence $E' \to F' \to G \to \Sigma E'$ isomorphic to the image of a sequence as above.

Theorem. *The stable homotopy category $\mathcal{SH}$ with translation functor Σ, the above distinguished triangles, and additive structure from §4.1, is a triangulated category.*

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