GRK Ring Lecture:
Brauer groups and obstructions, Part I

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Eminent mathematicians

Who are these eminent mathematicians?

Émile Picard
(1856–1941)

Richard Brauer
(1901–1977)

Picard group and Brauer group

Picard group and the **Brauer group** are named after them, two extremely important constructions. I give a somewhat unusual definition, for rings R:

The group $\mathrm{Pic}(R)$ comprises isomorphism classes of R-modules L such that $L \otimes R' \simeq R'$ for some faithfully flat $R \subset R'$. Such L are called **invertible modules**.

The group $\mathrm{Br}(R)$ comprises equivalence classes of R-algebras A such that $A \otimes R' \simeq \mathrm{Mat}_n(R')$ with $n \geq 1$ and some faithfully flat $R \subset R'$. Such A are called **Azumaya algebras**.

In both cases, group structure comes from $\otimes$.

Examples for invertible modules

Examples for invertible modules over rings R:

Fields, local rings, principal ideal domains?

Factorial rings?

Dedekind domains?

More examples

Coordinate rings $R = \Gamma(C \smallsetminus \{z\}, \mathscr{O}_C)$ for projective curve C.

The point $z \in C$ yields invertible sheaf $\mathscr{L} = \mathscr{O}_C(z)$. Then

$$\mathrm{Pic}(R) = \mathrm{Pic}(C)/\mathbb{Z}\mathscr{L}.$$

Projective line $C = \mathbb{P}^1$ gives $\mathrm{Pic}(R) = \mathbb{Z}/d\mathbb{Z}$, with $d = [\kappa(z) : k]$.

For elliptic curve $C\colon y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ one gets $\mathrm{Pic}(R) = C(k)$, when z is the origin.

Examples of Azumaya algebras

Examples of Azumaya algebras: The quaternion algebra

$$\mathbb{H} = \{aE + bI + cJ + dK \mid a, b, c, d \in \mathbb{R}\}$$

generated by the complex matrices

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Clearly $\mathbb{H} \otimes \mathbb{C} = \mathrm{Mat}_2(\mathbb{C})$. Surprising formula:

$$(aE + \ldots + dK) \cdot (aE - \ldots - dK) = a^2 + \ldots + d^2 \in \mathbb{R}_{\geq 0}$$

So non-zero quaternions are invertible, hence $\mathbb{H} \neq \mathrm{Mat}_2(\mathbb{R})$.

Cohomology

Usually it is almost impossible to compute $\mathrm{Pic}(R)$ and $\mathrm{Br}(R)$ from definitions. But can be expressed in terms of **cohomology**!

Invertible modules L are **twisted forms** of R. Such correspond to first cohomology with coefficients in $\underline{\mathrm{Aut}}(R) = \mathbb{G}_m$, so

$$\mathrm{Pic}(R) = H^1(R, \mathbb{G}_m).$$

Interpretation via cocycles: Choose basis $e_1 \in L \otimes R'$. Then write $(e_1 \otimes 1) = \lambda \cdot (1 \otimes e_1)$ for some $\lambda \in (R' \otimes R')^\times$. Satisfies cocycle condition, yields cohomology class

$$[\lambda] = [L] \in H^1(R, \mathbb{G}_m).$$

Exponential Sequence

This works not only for rings, but for ringed spaces, or ringed topoi. Cohomological interpretation relates $\mathrm{Pic}(X)$ to other groups:

Let X be a complex space. Have **exponential sequence**

$$0 \longrightarrow \mathbb{Z}_X \xrightarrow{2\pi i} \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \longrightarrow 1.$$

Gives long exact sequence

$$H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow \mathrm{Pic}(X) \longrightarrow H^2(X, \mathbb{Z}).$$

For $\alpha \in H^1(X, \mathscr{O}_X)$, the resulting sheaf $\mathscr{L}$ is **obstruction** to make α integral.

The coboundary defines **first Chern class** $c_1(\mathscr{L}) \in H^2(X, \mathbb{Z})$.

Divisor sequence

Let X be an integral scheme. Have **divisor sequence**

$$1 \longrightarrow \mathscr{O}_X^\times \longrightarrow \mathscr{R}_X^\times \longrightarrow \underline{\mathrm{Div}}_X \longrightarrow 0.$$

Yields long exact sequence

$$\Gamma(X, \mathscr{R}_X^\times) \longrightarrow \mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow H^1(X, \mathscr{R}_X^\times)$$

Term on the right vanishes, because coefficients are constant and X is irreducible: **No obstructions!**

So each invertible sheaf $\mathscr{L}$ is of the form $\mathscr{O}_X(D)$ for some Cartier divisor $D \in \mathrm{Div}(X)$.

Azumaya algebras

Now back to Brauer group $\mathrm{Br}(R)$ and Azumaya algebras A. These are twisted forms of matrix algebras $\mathrm{Mat}_n(R)$.

By Skolem–Noether, each automorphism of $\mathrm{Mat}_n(R)$ is locally given by conjugacy, so we get class

$$[A] \in H^1(R, \mathrm{PGL}_n)$$

However, elements in $\mathrm{Br}(R)$ are **equivalence classes**, modulo

$$A \sim A' \quad \Longleftrightarrow \quad A \otimes \mathrm{Mat}_r(R) \simeq A' \otimes \mathrm{Mat}_s(R).$$

Gives inverses, via identification $A \otimes A^{\mathrm{op}} = \mathrm{End}_R(A) \simeq \mathrm{Mat}_n(R)$.

Cocycle construction

Cocycle construction: Choose $\varphi : A \otimes R' \to \mathrm{Mat}_n(R')$.

Write $(\varphi \otimes 1) = \psi \circ (1 \otimes \varphi)$ for some $\psi \in \mathrm{PGL}_n(R' \otimes R')$.

Choose lift $\tilde{\psi} \in \mathrm{GL}_n(R' \otimes R')$, after refining R'.

Cocycle condition usually fails for lift; obstruction is 2-cochain

$$\alpha = \tilde{\psi}_{12} \cdot \tilde{\psi}_{02}^{-1} \cdot \tilde{\psi}_{01} \in \mathbb{G}_m(R' \otimes R' \otimes R')$$

Now cocycle conditions holds, gives $[\alpha] = [A] \in H^2(R, \mathbb{G}_m)$.

Grothendieck's interpretation

Works for any ring, or ringed space, or ringed topos. Gives **Grothendieck's cohomological interpretation**

$$\mathrm{Br}(X) \subset H^2(X, \mathbb{G}_m)$$

Using liftings to SL_n instead of GL_n, one sees that Brauer group is torsion.

But cohomology is not torsion, in general: Exponential sequence for complex spaces gives

$$H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathscr{O}_X) \longrightarrow H^2(X, \mathscr{O}_X^\times) \longrightarrow H^3(X, \mathbb{Z}).$$

Projective representations

Application in group theory: Let G be an finite group, and $\rho : G \to \mathrm{PGL}_n(\mathbb{C})$ be a projective representation. Can one lift it to a linear representation?

For each $g \in G$ choose lift $A_g \in \mathrm{GL}_n(\mathbb{C})$ of the class $\rho(g)$. Write

$$A_g \cdot A_h = \alpha_{g,h} A_{gh}$$

for some $\alpha_{g,h} \in \mathbb{C}^\times$. Satisfies cocycle relation, gives group cohomology class $[\alpha] \in H^2(G, \mathbb{C}^\times)$. Classically called **Schur multiplier**. Can be viewed as Brauer group of **classifying space** $X = BG$.

Cup products

Set $\mu_n = \mathbb{G}_m[n]$. Canonical pairing $\mathbb{Z}/n\mathbb{Z} \times \mu_n \to \mathbb{G}_m$, induces **cup product**

$$\cup : H^1(k, \mathbb{Z}/n\mathbb{Z}) \times H^1(k, \mu_n) \longrightarrow H^2(k, \mathbb{G}_m) = \mathrm{Br}(k).$$

Let $k \subset K$ be a Galois extension with group $\mathbb{Z}/n\mathbb{Z} =< \sigma >$, and $\beta \in k^\times$. Then

$$A = K[T]/(t^n - \beta, \lambda T - T\sigma(\lambda))$$

is Azumaya algebra, satisfy $[A] = [K] \cup [\beta]$.

These are called **cyclic algebras**, generalize quaternion algebras $\mathbb{H}$.

Brauer–Severi varieties

Let k be a ground field. A scheme X is called **Brauer–Severi variety** if

$$X \otimes k' \simeq \mathbb{P}^n \otimes k'$$

for some field extension $k \subset k'$. In other words, X is a twisted form of $\mathbb{P}^n$.

Example: **quadric curves** $X \subset \mathbb{P}^2$. Indeed:

$$X : T_0^2 + T_1^2 + T_2^2 = 0$$

has no rational point over $k = \mathbb{R}$, but becomes $\mathbb{P}^1$ over $k' = \mathbb{C}$.

Cohomology classes

Using the universal property of $\mathbb{P}^n$, one sees that

$$\underline{\mathrm{Aut}}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}\,.$$

So Brauer–Severi varieties give rise to classes $[X] \in \mathrm{Br}(k)$, as do Azumaya algebras.

In fact, the categories of twisted forms of $\mathrm{Mat}_{n+1}(k)$ and twisted forms of $\mathbb{P}^n$ are equivalent.

Obstruction against rational points

Theorem: *Let X be a Brauer–Severi variety. The cohomology class*

$$[X] \in H^2(k, \mathbb{G}_m)$$

is the obstruction against the existence of a rational point $a \in X$.

The proof relies on an interpretation of $\mathbb{P}^n$ as a **moduli space**, serves as baby example for moduli problems of second lecture.

Preliminary considerations

Preliminary considerations:

Write $P = \mathbb{P}^n = \mathbb{P}(E)$, and consider the **dual projective space** $P^* = \mathbb{P}(E^*)$.

The rational points $a \in P$ correspond to hyperplanes $H \subset P^*$.

The resulting invertible sheaf $\mathscr{L} = \mathscr{O}_{P^*}(H)$ is very ample, with $h^0(\mathscr{L}) = n + 1$, and defines an isomorphism $P^* \to \mathbb{P}(V)$, for the linear system $V = H^0(P, \mathscr{L})$.

Likewise, Brauer–Severi variety X comes with a dual variety X^*.

Proof of Theorem

Proof for the Theorem:

Suppose there is a rational point $a \in X$. Corresponds to hyperplane $H \subset X^*$, yields invertible sheaf $\mathscr{L} = \mathscr{O}_{X^*}(H)$ and isomorphism $X^* \to \mathbb{P}(V)$ as above. **Biduality** $X = X^{**}$ gives $X \simeq \mathbb{P}^n$, hence $[X] = 0$.

Conversely, suppose $[X] \in H^2(k, \mathbb{G}_m)$ is trivial. Then Brauer–Severi variety comes from some $H^1(k, \mathrm{GL}_{n+1})$. But this group is trivial by **Hilbert 90**. Thus $X \simeq \mathbb{P}^n$, which contains rational points.

Thank you very much for the attention!