Deformation theory III

Set up:

- k field of characteristic exponent $p \geq 1$,
- X_0 scheme of finite type,
- G abstract group acting on X_0,
- A Artin local ring with $A/m_A = k$,
- (X, φ) deformation of X_0 over A.

Suppose G-action X_0 extends to X.
Let $0 \to I \to A' \to A \to 0$ extension with $I \simeq k$

<u>Question</u>: Can we extend (X, φ) over A to some (X', φ') over A', together with the G-action?

Has to do with group cohomology $H^r(G, H^s(X_0, \Theta_{X_0}))$
Here $\Theta_{X_0/k} = \underline{Hom}(\Omega^1_{X_0/k}, \mathcal{O}_{X_0})$.

Application :

Rössler - S (2020) Constructed a families of abelian varieties $X_0 \longrightarrow \mathbb{P}^n$ with $c_1 = 0$. for $p \geq 2$. Using Beauville - Bogomolov decomposition to show that X_0 has *no projective deformations* over rings like $R = \mathbb{Z}_p = \varprojlim \boxed{\mathbb{Z}/p^n\mathbb{Z}}$.

Using the action of $G = \{\pm 1\}$ on X_0 and Takayama - S (2018) we showed that there is no *formal deformation* $\mathcal{X} = (X_n)_{n \geq 0}$.

Introduce categories:

$$\mathcal{E} = \{ \text{deformations } (X,\varphi) \text{ of } X_0 \text{ over some } A \}$$

$$\mathcal{F} = (\mathrm{Art}_\Lambda)^{opp} = \left\{ \begin{array}{l} \text{Spectra } S \text{ of Artin local} \\ \Lambda\text{-algebras } A \text{ with } A/m_A = k \end{array} \right\}$$

Here Λ is a local noetherian ring with $\Lambda/m_A = k$

e.g. $\Lambda = k$ or $\Lambda = \mathbb{Z}_p$, $k = \mathbb{F}_p$.

Have functor

$$\mathcal{E} \longrightarrow \mathcal{F} \qquad (X,\varphi) \longmapsto S$$

Recall: deformations are cartesian squares

$$\begin{tikzcd}
X_0 \arrow[r, "\varphi"] \arrow[d] & X \arrow[d] \\
\mathrm{Spec}(k) \arrow[r, hook] & \mathrm{Spec}(A)
\end{tikzcd}$$

Write $\xi \in \mathcal{E}$ and $S \in \mathcal{F}$

Grothendieck: cartesian maps maps sense for abstract categories $\mathcal{E}, \mathcal{F}$:

$$\begin{tikzcd}
\eta \arrow[dr] \arrow[drr, dashed] & & \\
& \xi \arrow[r, "\varphi"] \arrow[d, mapsto] & \xi' \arrow[d, mapsto] \\
& S \arrow[r] & S'
\end{tikzcd}$$

induces $\mathrm{Hom}_S(\eta, \xi) = \mathrm{Hom}_{S\to S'}(\eta, \xi')$

for all $\eta \in \mathcal{E}_S$

Define

$$\mathrm{Lif}(\xi, S') = \left\{ \begin{matrix} \xi \xrightarrow{f} \xi' \text{ cartesian} \\ \text{over } S \to S' \end{matrix} \right\} \Big/ \simeq$$

Has G-action via transport of structure

$$\sigma . (\xi \xrightarrow{f} \xi') = (\xi \xrightarrow{\sigma^{-1}} \xi \xrightarrow{f} \xi')$$

Consider diagram

$$\begin{tikzcd}
 & & & G \arrow[dl, dashed] \arrow[d] \\
1 \arrow[r] & \mathrm{Aut}_{\xi}(\xi') \arrow[r] & \mathrm{Aut}_{S'}(\xi') \arrow[r] & \mathrm{Aut}_{S}(\xi)
\end{tikzcd}$$

Lemma A The image of G is contained in the image of $\mathrm{Aut}_{S'}(\xi')$ if and only if $[f] \in \mathrm{Lif}(\xi, S')$ is G-fixed.

Suppose this is the case. Pullback gives

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Aut}_{\xi}(\xi') \arrow[r] & \tilde{G} \arrow[r] & G \arrow[r] \arrow[l, dashed, bend right] & 1
\end{tikzcd}$$

Lemma B The G-action on ξ extends to ξ' if and only if the extension splits.

Suppose $\mathrm{Aut}_{\mathfrak{z}}(\mathfrak{z}')$ is abelian. Becomes G-module via

$$\sigma \cdot h = \tilde{\sigma} \cdot h \, \tilde{\sigma}^{-1}$$

The equation $c_{\sigma\eta} \cdot \widetilde{\sigma\eta} = \tilde{\sigma} \cdot \tilde{\eta}$ defines 2-cocycle $c : G^2 \longrightarrow \mathrm{Aut}_{\mathfrak{z}}(\mathfrak{z}')$, yields class

$$[\tilde{G}] = [c] \in H^2(G, \mathrm{Aut}_{\mathfrak{z}}(\mathfrak{z}'))$$

Fact: Extension $\tilde{G}$ splits $\Longleftrightarrow$ $[\tilde{G}] = 0$.

Back to deformation theory:

$$\mathcal{E} = \{ \text{deformations } (X, u) \text{ of } X_0 \text{ over } A \}$$

$$\mathcal{F} = (\mathrm{Art}_\Lambda)^{opp}$$

The category $\mathcal{E}$ has a tangent space

$$T = T_{\mathcal{E}/\mathcal{F}} = \mathcal{E}_{k[\varepsilon]} / \simeq$$

is k-vector space, with G-action. Here $T = H^1(X_0, \Theta_{X_0/k})$ if X_0 is smooth.

Moreover, saw that $L = \mathrm{Lif}(S, S')$ is T-torsor, and has a G-action.

<u>Serre</u>: $H^1(G, T) = \{ T\text{-torsors } L \text{ with compatible } G\text{-action} \} / \simeq$

and $L^G \neq \emptyset \iff [L] = 0$

Likewise: category of automorphisms of deformations has tangent space

$$\Lambda = \mathrm{Aut}_{S_0}(S_0 \otimes k[\varepsilon])$$

Thm (Rim 1980, Takayama-S 2018)

(I) Suppose $L = \mathrm{Lif}(\xi, \xi')$ is non-empty.
then there is a G-fixed point if and only if
$[L] = 0$ in $H^1(G, T)$

(II) Suppose there is some $\xi' \in L^G$. Then the
G-action extends from ξ to ξ' if and only if
$[\widetilde{G}] = 0$ in $H^2(G, \Lambda)$

Suppose G is finite, with order $n = |G|$.

Let (X, ψ) be some deformation of X_0 over A.

<u>Corollary</u> 1 Suppose $(p, n) = 1$. Then the G-action on X extends to some deformation (X', ψ') over A'.

<u>pf</u>: $H^r(G, M) = 0$, $r \geq 1$ for k-vector spaces M.

Now apply Thm $\square$

Let $H \subset G$ be a p-Sylow-group.

<u>Corollary</u> 2 the G-action on X extends to (X', ψ') if and only if the H-action extends.

<u>pf</u>: The restriction maps $H^r(G, M) \to H^r(H, M)$, $r \geq 1$ is bijective for all k-vector spaces M. $\square$