§ 1 Zeta function of arithmetic schemes

Def. An <u>arithmetic scheme</u> is a scheme X of finite type over $\mathbb{Z}$, i.e.

$$X = \bigcup_{i=1}^{r} \mathcal{U}_i \quad \text{fin open cover}$$

with $\mathcal{U}_i = \operatorname{Spec}\left(\mathbb{Z}[x_1, \dots, x_{n_i}] / I_i \right)$

Ex:

- $X = \operatorname{Spec} \mathbb{Z}$
- $X = \operatorname{Spec} \mathcal{O}_K$, $K/\mathbb{Q}$ fin , $\mathcal{O}_K = \overline{\mathbb{Z}}^K$
- $\mathbb{F}_q$ = field with $q = p^n$ elts and X fin type / $\mathbb{F}_q$

Non-Examples

- $\operatorname{Spec} \mathbb{C}[x]$
- $\operatorname{Spec} \mathbb{F}_p(T)$

Property: X arith scheme , $x \in X$ closed pt

$\Rightarrow$ $\mathcal{K}(x) := \mathcal{O}_{X,x} / m_x$ fin field

set $N(x) := |\mathcal{K}(x)|$

Def (Zeta function)

X arith scheme

$$\zeta(X,s) := \prod_{\substack{x \in X \\ \text{closed}}} \frac{1}{1 - N(x)^{-s}}$$

Ex. $X = \operatorname{Spec} \mathbb{Z}$

$$\longrightarrow \quad \zeta(s) = \prod_p \frac{1}{1-p^{-s}} \quad \text{Riemann Zeta function}$$

$$= \sum_{n \geq 1} \frac{1}{n^s}$$

- $X = \operatorname{Spec} \mathbb{F}_q$

$$\longrightarrow \quad \zeta(X,s) = \frac{1}{1-q^{-s}}$$

- $X_p := X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{F}_p$

$$\Longrightarrow \quad \zeta(X,s) = \prod_p \zeta(X_p, s)$$

Thm $\zeta(X,s)$ converges abs on $\{ s \in \mathbb{C} \mid \operatorname{Re}(z) > \dim X \}$

i.e. $\displaystyle\prod_{\substack{x \in X \\ \text{close}}} \left| \frac{1}{1-N(x)^{-z}} \right|$ conv, $\neq 0$ $\quad \forall z \in \mathbb{C}$, $\operatorname{Re}(z) > \dim X$

(By Noether Normaliz reduced to $X = \mathbb{A}^n_{\mathbb{F}_p}$)

<u>Prop</u> $X / \mathbb{F}_q$

$$\Rightarrow \quad \zeta(X, s) = \exp\left(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \, \frac{q^{-ns}}{n} \right)$$

where
$$|X(\mathbb{F}_{q^n})| = \mathrm{Hom}_{\mathbb{F}_q}(\operatorname{spec} \mathbb{F}_{q^n}, X)$$

<u>pf</u> uses:
$$|X(\mathbb{F}_{q^n})| = \sum_{\substack{x \in X \text{ closed} \\ \deg x \mid n}} \deg x$$

<u>Ex</u>:

(1) $$\zeta(\mathbb{A}^1_{\mathbb{F}_q}, s) = \exp\left(\sum_{n=1}^{\infty} \frac{(q^{(1-s)})^n}{n} \right) \underset{\left[\log(1-x) = -\sum_{j=1}^{\infty} \frac{x^j}{j}\right]}{=} \frac{1}{1 - q^{1-s}}$$

(3) $$\zeta(\mathbb{P}^1_{\mathbb{F}_q}, s) = \zeta(\{\infty\}, s) \cdot \zeta(\mathbb{A}^1, s) = \frac{1}{(1-q^{-s})(1-q^{1-s})}$$

(2) $$\zeta(X \times \mathbb{A}^1, s) = \zeta(X, 1-s)$$

[wlog $X = X_p / \mathbb{F}_p$, then use Prop]

§ 2 The Weil conjectures

$X/\mathbb{F}_q$ sep fin typ

Def.

$$Z(X/\mathbb{F}_q, t) := \exp\left(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \frac{t^n}{n} \right)$$

$$= \prod_{\substack{x \in X \\ \text{closed}}} \frac{1}{1 - t^{\deg(x)}}, \quad \deg x = [k(x) : \mathbb{F}_q]$$

Rmk:

- $Z(X/\mathbb{F}_q, q^{-s}) = \zeta(X, s)$

- $Z(\mathbb{P}^1/\mathbb{F}_q, t) = \dfrac{1}{(1-t)(1-qt)}$

- $z_0(X) := \bigoplus_{\substack{x \in X \\ \text{closed}}} \mathbb{Z} \cdot [x]$ "zero-cycles on X"

$$\Rightarrow Z(X/\mathbb{F}_q, t) = 1 + \sum_{d \geq 1} |\{ \gamma \in z_0(X)^{\geq 0} \mid \deg \gamma = d \}| \, t^d$$

$$\in 1 + t\,\mathbb{Z}[\![t]\!]$$

Weil conjectures (= Thm's of Weil, Dwork, Grothendieck, Deligne)

X smooth proj. $/\mathbb{F}_q$ $\quad n = \dim X$

geom. conn.

Then:

(1) (Rationality)

$$Z(X/\mathbb{F}_q, t) = \frac{P_1(t)\, P_3(t) \cdots P_{2n-1}(t)}{P_0(t)\, P_2(t) \cdots P_{2n}(t)}$$

where $P_0(t) = 1 - t$, $\quad P_{2n}(t) = 1 - q^n t$

$$P_i(t) \in 1 + t\,\mathbb{Z}[t]$$

(2) (Functional eqn) $\quad E := \sum_{i=0}^{2n} (-1)^i \deg P_i(t)$

$$\Rightarrow Z\left(X/\mathbb{F}_q, \frac{1}{q^n t}\right) = \pm\, q^{nE/2}\, t^E\, Z(X/\mathbb{F}_q, t)$$

(3) (Analog of Riemann Hyp)
Write $P_i(t) = \prod_{j=1}^{b_i} (1 - \alpha_{ij} t)$ in $\overline{\mathbb{Q}}[t]$

$\Rightarrow$ $\alpha_{ij} \in \overline{\mathbb{Z}}$ and $\forall\ \iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$

we have $|\iota(\alpha_{ij})| = q^{i/2}$

Rmk: · $\dim X = 1$: Weil ~ 1949

(1), (2) : Dwork 1960

rationality Grothendieck et al ~ 1970

(3) Deligne 1974

The goal of the course is to sketch Weil's proof in the curve case i.e.

Thm (Weil)

$C/\mathbb{F}_q$ sm proj, geom. con. curve of genus g

Then

(1) $Z(C/\mathbb{F}_q, t) = \dfrac{P(t)}{(1-t)(1-qt)}$, $P(t) \in 1 + t\,\mathbb{Z}[t]$, $\deg P(t) = 2g$

(2) $Z(C/\mathbb{F}_q, 1/qt) = q^{1-g} t^{2-2g} Z(C/\mathbb{F}_q, t)$

(3) $P(t) = \prod_{i=1}^{2g} (1-\alpha_i t)$, $\alpha_i \in \overline{\mathbb{Z}}$

$\Rightarrow$ $|\iota(\alpha_i)| = \sqrt{q}$ $\forall i$

$\forall \iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$

Ex: $\cdot$ $C = \mathbb{P}^1$

$$Z(\mathbb{P}^1, t) = \frac{1}{(1-t)(1-qt)} \quad , \quad g(\mathbb{P}^1) = 0$$

(3) empty, (2) : check directly.

Kor: $\zeta(C,s) = Z(C/\mathbb{F}_q, q^{-s})$

holomorphic on $\mathbb{C} \setminus \left\{ \frac{2\pi i}{\log q}\mathbb{Z}, 1 + \frac{2\pi i}{\log q}\mathbb{Z} \right\}$

and if $\zeta(C,s) = 0 \Rightarrow \mathrm{Re}(s) = \frac{1}{2}$

Pf: $\zeta = \dfrac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$

$\zeta = 0 \Leftrightarrow q^{s} = \alpha_{i_0}$ (α_i as above)

$$\Rightarrow \underset{\underset{q^{\mathrm{Re}(s)}}{\parallel}}{|z(q^{s})|} = |\alpha_i| = \sqrt{q} \Rightarrow \mathrm{Re}(s) = \frac{1}{2}$$

$\square$

Compare to

Riemann Conjecture $\zeta(\mathbb{Z},s) = 0$ and $0 \leq \mathrm{Re}(s) \Rightarrow \mathrm{Re}(s) = \frac{1}{2}$

under Rationality (1) + Fct eqn (2)

the analog of RH (3) is equivalent to

Thm (*): $C/\mathbb{F}_q$ as above

$$1+q^n - 2g\sqrt{q^n} \le |C(\mathbb{F}_{q^n})| \le 1+q^n + 2g\sqrt{q^n}$$

(2) => Thm:

$$\exp\left(\sum_{n\geq 1} |C(\mathbb{F}_{q^n})| \frac{t^n}{n}\right) = \frac{\prod_{i=1}^{2g}(1-\alpha_i t)}{(1-t)(1-qt)}$$

apply log and compare coeff

Ex: $F(X,Y,Z) \in \mathbb{F}_p[X,Y,Z]$ hom of degree d

Assume $(F, \partial_X F, \partial_Y F, \partial_Z F)$ have no common solution in $\overline{\mathbb{F}}_p$

Then $F(X,Y,Z) = 0$ has a non-trivial solution in $\mathbb{F}_{p^n}$

if $(d-1)(d-2) \le \sqrt{p^n}$

(use $g = \frac{(d-1)(d-2)}{2}$)

- Proof (to be explained)

(weak version) Rationality + Fct eq $\Leftarrow$ Riemann-Roch

- Thm (*) $\Leftarrow$ Intersection thy on surfaces + RR on surfaces + Hodge Index

$\Rightarrow$ Thm.

- There is also a more elementary proof by Bombieri (Stepanov) from 1974 using only RR for curves.