Workshop: The Mordell–Weil Theorem

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Summer semester 2024

A particularly interesting type of variety in algebraic geometry is the abelian variety. By definition, it is a tuple consisting of a complete variety and (multiplication, identity and inverse) morphisms satisfying the usual group laws. However, a more accessible way of looking at these varieties is through rational points. It turns out that the sets of rational points of abelian varieties obtain a (functorial) group structure and as expected, one can show that these groups are abelian. Nevertheless, in contrast to what the name might suggest, abelian varieties are a lot more than 'just' varieties with a commutative group structure. An example of an abelian variety is the elliptic curve. One of the benefits of elliptic curves is that they are given by *one* explicit polynomial equation, which makes direct computations (albeit tedious) doable. In the case of higher dimensional abelian varieties, it is no longer possible to find such nice polynomials in general.

Now let A be an abelian variety defined over a field K. The group of K-rational points of A is defined as $A(K) := \mathrm{Hom}_{\mathrm{Spec}\, K}(\mathrm{Spec}\, K, A)$, more intuitively thought of as the points of A with coordinates in K. Already in 1901 Poincaré speculated that the group $E(\mathbb{Q})$ is finitely generated for elliptic curves E defined over $\mathbb{Q}$. A quick check shows however that $A(K)$ cannot be finitely generated for any field K: elliptic curves over $\mathbb{C}$ are complex tori, and these have uncountably many points. Nevertheless, Mordell proved the fact that $E(\mathbb{Q})$ is finitely generated in 1922. Soon after, in 1928, Weil generalised this result to what we now know as one of the most fundamental theorems in arithmetic geometry.

Theorem 1 (Mordell–Weil)**.** *Let $K/\mathbb{Q}$ be a finite field extension. Let A be an abelian variety over K. Then $A(K)$ is finitely generated.*

Typically, one of the first steps in proving the Mordell–Weil theorem is to prove the following theorem:

Theorem 2 (Weak Mordell–Weil)**.** *Let $K/\mathbb{Q}$ be a finite field extension. Let A be an abelian variety over K. Let $m \geq 2$ be an integer. Then $A(K)/mA(K)$ is finite.*

One sees easily that the Mordell–Weil theorem implies the weak Mordell–Weil theorem. However, proving the reverse implication requires some extra steps. Roughly spoken, we will introduce a notion of 'complexity' of the coordinates of rational points, which we will call the *height* of a point. This height function satisfies a certain set of properties, and we will show that these properties allow us to prove Mordell–Weil from weak Mordell–Weil.

The main goal of this workshop is to understand the meaning of the Mordell–Weil theorem and to work through a modern proof provided by Silverman in [Sil86].

Talk 1: Schemes & varieties

Description: Classically, one considers varieties to be algebraic structures which (locally) look like zeroes of polynomial equations. However, a more modern and more general approach in algebraic geometry is to use the notion of a scheme. In this talk, we compare the classical and modern approach and discuss the relevant algebro-geometric preliminaries of this workshop.

Tasks: We give a rough outline of the topics that should be discussed. All of the material can be found in more detail in [Har77] or [GW20]. Feel free to add clarifying results and examples where you see fit.

Briefly define the following: scheme, morphism of schemes and schemes over some other scheme [GW20, Definition 3.1]. Define rational points of a scheme [EvdGM, 0.3] and discuss (but do not prove in detail) the equivalence [GW20, Theorem 3.37]. Motivated by this equivalence and [GW20, Example 9.17], define a variety [EvdGM, 0.4]. Throughout this process, provide some clarifying examples of the translation between classical and scheme theoretic varieties.

From this point on you may choose to work with a smooth, projective variety X over some field k. Define what an invertible sheaf is [Har77, §II.5], recall that it can be canonically identified with a line bundle, and describe the group structure on the Picard group [Har77, Proposition II.6.12]. Define Weil divisors, Cartier divisors [Har77, §II.6] and, in case k is algebraically closed, linear systems [Har77, §II.7]. Show the isomorphisms between the class group, Cartier class group and the Picard group of a scheme (you do not need to prove that these are isomorphisms). [Har77, Proposition II.6.13]. Then, show the equivalence between the following concepts: morphisms from X into the projective space $\mathbb{P}_k^N$, globally generated invertible sheaves on X and base point free linear systems on X (again, you do not need to prove this) [Har77, Theorem II.7.1, Remark II.7.8.1].

Talk 2: Elliptic curves

Description: For the second introductory talk we zoom in on a particular type of variety: the elliptic curve. We show that it is in general given by a Weierstrass equation and we describe the addition of its points in multiple ways.
Tasks: An excellent source for the theory of elliptic curves is [Sil09]. It is however written in the language of classical varieties; for a scheme theoretic translation, one may look at [EvdGM, Example 1.8]. Define the genus of a smooth projective curve and define an elliptic curve. Using the Riemann–Roch theorem and the material discussed in the first part of this talk, show that any elliptic curve can be embedded into the projective plane, and that the image is given by a Weierstrass equation [Sil09, Proposition III.3.1]. Also briefly discuss smoothness of elliptic curves in terms of its discriminant [Sil09, Proposition III.1.4(a)(i)]. Now describe how the rational points of an elliptic curve admit a group structure in two different ways:

- Give the 'algebraic group law', following for example [Sil09, Proposition III.3.4]
- Give the 'geometric group law', following for example [Sil09, Section III.2]. Also describe this group law explicitly on projective coordinates.

Remark that these group structures are the same.

Talk 3: Abelian varieties

Description: In this talk we generalise the notion of an elliptic curve to one of an abelian variety. We discuss the basics and the most fundamental results on the structure of abelian varieties.
Tasks: Throughout this talk you only have to work with schemes defined over some field. All of the material can be found in texts like [EvdGM] and [Mil86]. Define the terms group scheme, group scheme homomorphism and group variety [EvdGM, Definition 3.1, Definition 1.2]. Using the Yoneda Lemma, discuss [EvdGM, Proposition 3.6]. Provide some examples from [EvdGM, Example 3.8]. Define what an abelian variety is [EvdGM, Definition 1.3]. Define translations on a group variety [EvdGM, Definition 1.4] and prove [EvdGM, Proposition 1.14] using the Rigidity Lemma. Also discuss [EvdGM, Corollary 1.15]. Define the kernel of a homomorphism [EvdGM, Definition 3.13] . Then define the 'multiplication by n' map and mention that it is surjective and has finite kernel [Mil86, Theorem 8.2] (you do not need to prove the statements here). Finally, discuss smoothness [EvdGM, Proposition 1.5] and projectiveness [EvdGM, 2.26].

Talk 4: Number fields

Description: To understand abelian varieties over number fields we need to get a good number theoretical understanding. More specifically, we are interested in prime ideals of rings of integers and their associated valuations. In this talk all of the relevant number theoretic concepts and results will be discussed.
Tasks: We refer to [Keu23]. However, feel free to use any book on algebraic number theory you like. Define what a number field is [Keu23, Definition 1.1]. Also give the definition of the ring of integers $\mathcal{O}_K$ associated to a number field K [Keu23, Definition 1.14]. Define the norm of an element in K [Keu23, Definition 1.15]. Define what a Dedekind domain is [Keu23, Definition 2.3] and show that ideals admit a unique (up to order) factorisation into prime ideals [Keu23, Theorem 2.11]. You may freely use the equivalent characterisations in Definition 2.3 and Theorem 2.43 in [Keu23]. Define fractional ideals [Keu23, Definition 2.29]. Also define the $\mathfrak{p}$-adic valuation on ideals and elements of $K^\times$ for prime ideals $\mathfrak{p}$ [Keu23, Definition 2.37]. Show that $\mathcal{O}_K$ is a Dedekind domain [Keu23, Theorem 2.46]. Define the ramification index and residue class degrees of prime ideals in $\mathcal{O}_K$ [Keu23, Definition 3.3]. Show [Keu23, Theorem 3.4] and prove it if time allows. Define the norm of an ideal in $\mathcal{O}_K$ and show its relation to the norm of elements in $\mathcal{O}_K$ [Keu23, 3.18-3.20]. Also give the norm of a prime ideal.

Talk 5: Heights on projective space

Description: In this talk we introduce the concept of *heights*, and we show how particular height functions can aid us in proving the Mordell–Weil theorem.
Tasks: Show (but do not prove) [Sil86, Lemma 5.1] and [Sil86, Theorem 5.2]. In the first part of the talk we discuss absolute values. Give the definition of absolute values, equivalence of absolute values and places [Keu23, Definition 10.1, 10.4]. Highlight the difference between Archimedean and non-Archimedean absolute values [Keu23, Proposition 10.6, Definition 10.7]. Show how (discrete) valuations and embeddings in $\mathbb{R}$ and $\mathbb{C}$ induce absolute values [Keu23, Example 10.2, 10.3]. Classify the places on a number field (Theorem 10.12 and Corollary 10.22 in [Keu23], at least sketch the proofs). Show the product formula of places [Keu23, Theorem 10.24]).

Now define the height of a point relative to a number field and the absolute (logarithmic) height of a point (see [Sil86, §2], be careful of the difference in notation between [Sil86] and [Keu23]). Finally, show that this height is well-defined.

Talk 6: Heights on varieties

Description: We continue our search for a suitable height function. In the previous talk we constructed heights on projective space, which we now generalise to heights on smooth projective varieties.
Tasks: Define the (logarithmic) height on a variety relative to a morphism into projective space [Sil86, §3]. Recall [Har77, Theorem II.7.1, Remark II.7.8.1] and define the height on a variety relative to an invertible sheaf [Sil86, §3]. Show that, up to bounded difference, this height is only dependent on the chosen invertible sheaf [Sil86, Theorem 3.1]. Construct the height machine [Sil86, Theorem 3.3]. Finally, introduce (and prove if time allows) the Theorem of the Cube and discuss its consequences on the structure of height functions on abelian varieties [Sil86, Theorem 4.1, Corollary 4.2]. You may also want to consult [Mil86, §6].

Talk 7: The weak Mordell–Weil theorem

Description: An important part of proving the Mordell–Weil theorem is proving the weak Mordell–Weil theorem. We will look at the proof of said theorem in this talk.
Tasks: Follow section C.1 from [HS00]. More specifically, discuss and prove Lemma C.1.1, Proposition C.1.2, Lemma C.1.3, and Proposition C.1.5. Theorem C.1.4 and Proposition C.1.6 can be taken as black boxes. Provide additional details from [HS00, Section A.9.1, A.9.4] wherever necessary. Finally, prove the weak Mordell–Weil theorem.

Talk 8: The Mordell–Weil theorem

Description: The goal of this talk is to deduce the Mordell–Weil theorem from the weak Mordell–Weil theorem, using the material that has been discussed in all the preceding talks.

Tasks: Prove [Sil86, Lemma 5.1]. Next, prove the finiteness theorems [Sil86, Theorem 2.1, Corollary 3.4]. Finally, prove the Mordell–Weil theorem [Sil86, Theorem 5.2].

References

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