Toric Varieties
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Jakob Bergqvist and Ivo Kroon

Toric varities form a large class of objects in algebraic geometry. Though they can be defined over more general bases, in this workshop we consider them only over $\mathbb{C}$. They are characterised by containing an open, dense torus, i.e. a copy of $(\mathbb{C}^*)^n$, such that the action of this torus on itself extends to the whole space in an algebraic manner. Some examples of toric varieties are $\mathbb{C}^n$, the rational cuspidal curve, the projective space $\mathbb{P}^n$, as well as the less trivial Hirzebruch surfaces $\mathcal{H}_r$. More generally, they arise 'in nature' in the compactification of certain moduli problems.

As we will see in this workshop, there is a correspondence between affine toric varieties and *semigroups* inside a free abelian group of rank n. Furthermore, if the toric variety is also *normal*, this correspondence extends to certain *cones* inside a n-dimensional $\mathbb{R}$-vector space. In the non-affine case, one instead needs a collection of cones, which is called a *fan*. Using these correspondences, it becomes possible to investigate many of the geometric properties of the variety using the tools of semigroups and linear algebra. This allows for a certain computational machinery not usually available in algebraic geometry.

Many results are obtained much more easily for toric varieties. For this reason, they form an excellent testing ground for new results and conjectures. As an example, the resolution of singularities can be constructed very directly on a toric variety using the computational machinery of fans. Also, as we will see, the topological fundamental group of a toric variety can be computed directly in terms of lattices. For a more advanced case, the chow ring and algebraic cobordism are also more easily computable for toric varieties (see [5, Chapter 5] and [4]).

Toric varieties are well-suited for the students of this workshop, as they can be studied without using specialised machinery in algebraic geometry. Our main source will be the tome [1]. We also recommend speakers and anyone interested take a look at the survey paper [3] which summarizes the initial development of the theory in the 70's. For a more concise source, look at [2]. Note that in some older litterature, such as [2], toric varieties are called torus/toroidal embeddings.

Talk 1: Definitions and affine toric varieties

The first talk should cover both definitions of toric varieties: both in terms of the open dense torus [1, Def. 1.1.3 and 3.1.1] and through gluing spectra of semigroup rings [1, §1.1] or [3, 2.2]. Make sure to note that a semigroup in our case includes a neutral element. After some examples exemplifying both definitions (e.g. $\mathbb{C}^n$, rational cuspidal curve), give the argument that the definitions are indeed equivalent (Cf. [2, page 2-3] or [1, Thm. 1.1.17]). Give the criterion of normality for toric varieties and restrict to this case. Define cones and their associated semigroups in the *dual* vector space. Outline the correspondence with normal affine toric varieties and certain types of cones. Give examples of toric varieties associated to certain cones (E.g. [1, Example 1.2.12]).

Talk 2: General toric varieties and their properties

The second talk should discuss how to extend the correspondence from the previous talk to *non-affine* toric varieties. For this, introduce the notion of a *fan* and its associated toric variety [1, p. 106-107]. Give examples: $\mathbb{P}^2$, the blowup of $\mathbb{A}^2$ in the origin and $\mathbb{P}^1 \times \mathbb{P}^1$. Outline the central points of the orbit-cone correspondence (cf. [1, Thm. 3.2.6]). Also mention how to construct toric morphisms from the side of cones and fans. Give, and (if time permits) prove, the criterion for smoothness (which will be used in the fourth talk) [1, Thm. 3.1.19 (a) and Def. 1.2.16 (a)].

Talk 3: Divisors on toric varieties

Introduce the definition of a (Weil) divisor, principal divisors and the class group. Also introduce the divisor of a ray, and of a character through [1, Prop. 4.1.2]. Show that the Class group is generated by divisors coming from rays and argue that divisors of characters lie are zero in the class group (see 4.1.3 loc. cit.). Introduce Cartier divisors, and argue similarly for the Picard group. State and prove Prop. 4.2.6 to give a criterion of smoothness in terms of divisors.

Talk 4: Singularities on toric surfaces

The main goal of this talk is to cover the algorithm to resolve singularities on a toric surface. First mention that each affine toric chart has at most a single singularity. Show that, in terms of lattices and cones, this singularity can be put into a standard form (cf. [1, Prop. 10.1.1]). Explain the algorithm of refining the fan to resolve the singularity, cf. Thm. 10.1.10 loc. cit. It is sufficient to treat the case of an affine toric surface.

Show that the exceptional fibre is a chain of $\mathbb{P}^1$'s by recalling that each subdivision corresponds to a blow-up. Apply this algorithm to an example. Remark that this algorithm calculates a continued fraction, whose length equals the number of components in the exceptional fibre (Cf. §10.2 loc. cit). Mention how, more generally, the continued fraction encodes the self-intersection numbers of these components. If time permits, outline the connection with cyclic quotient singularities (Cf. Prop. 10.1.2 loc. cit).

Talk 5: Classification of smooth toric surfaces

The main goal of this talk should be to prove the classification of smooth projective toric surfaces ([1, Thm. 10.4.3]). Recall the fans of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and introduce Hirzebruch surfaces $\mathcal{H}_r$ through their fans. Mention that a smooth projective toric variety has a primitive collection summing to zero (Prop. 7.3.6 loc. cit). State and prove Theorem

10.4.3 loc. cit. If time permits, state Theorem 10.4.4 loc. cit and calculate the intersection matrix of the Hirzebruch surface (see Example 10.4.6 loc. cit.).

Talk 6: Computing the topological fundamental group

Outline [1, §12.1] with a particular focus on proving Thm. 12.1.9. Outline, and if time permits, prove Theorem 12.1.10. Apply these theorems to an explicit example, for example 12.1.11.

References

[1] Cox, David A. and Little, John B. and Schenck, Henry K., *Toric Varieties*, Graduate Studies in Mathematics, vol. 124. American Mathematical Society, Providence, RI, 2011

[2] Kempf, G. and Knudsen, Finn Faye and Mumford, D. and Saint-Donat, B., *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York, 1973.

[3] Danilov, V. I., *The Geometry of Toric Varieties*, Volume 33 of *Russ. Math. Surv.*. The British Library and The London Mathematical Society, 1978.

[4] Krishna, Amalendu and Uma, Vikraman, *The algebraic cobordism ring of toric varieties*, *Int. Math. Res. Not. IMRN.* Oxford University Press, 2013.

[5] Fulton, William, *Introduction to toric varieties*, Annals of Mathematics Studies vol. 131. Princeton University Press, Princeton, NJ, 1993.