Rotating Navier-Stokes Equations in $\mathbb{R}^3_+$ with Initial Data Nondecreasing at Infinity: The Ekman Boundary Layer Problem

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Abstract

We prove time-local existence and uniqueness of solutions to a boundary layer problem in a rotating frame around the stationary solution called Ekman spiral. We choose initial data in the vector-valued homogeneous Besov space $\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ for $2 < p < \infty$. Here the L^p-integrability is imposed in the normal direction, while we may have no decay in tangential components, since the Besov space $\dot{\mathcal{B}}^0_{\infty,1}$ contains nondecaying functions such as almost periodic functions. A crucial ingredient is theory for vector-valued homogeneous Besov spaces. For instance we provide and apply an operator-valued bounded H^∞-calculus for the Laplacian in $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^n; \mathsf{E})$ for a general Banach space E.

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1. Introduction

We study the initial-boundary value problem for the three-dimensional rotating Navier-Stokes equations in a half-space $\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3); x_3 > 0\}$ with initial data nondecreasing at infinity:

$$\partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} + \Omega e_3 \times \mathbf{U} + \nu curl^2 \mathbf{U} = -\nabla p,\ \nabla\cdot\mathbf{U} = 0, \qquad (1.1)$$

$$\mathbf{U}(t,x)|_{x_3=0} = (U_1(t,x), U_2(t,x), U_3(t,x))|_{x_3=0} = (0,0,0), \qquad (1.2)$$

$$\mathbf{U}(t,x)|_{t=0} = \mathbf{U}_0(x) \tag{1.3}$$

where $x = (x_1, x_2, x_3)$, $\mathbf{U}(t,x) = (U_1, U_2, U_3)$ is the velocity field and p is the pressure. In Eqs. (1.1) e_3 denotes the vertical unit vector, $\nu > 0$ is a constant viscosity parameter. The constant $\Omega \in \mathbb{R}$ is called Coriolis parameter and equals to twice the frequency of rotation around x_3 axis. Eqs. (1.1)-(1.3) are the 3D Navier-Stokes equations written in a rotating frame. The initial velocity field $\mathbf{U}_0(x)$ depends on three variables x_1, x_2 and x_3. We require the velocity field $\mathbf{U}(t,x)$ to satisfy Dirichlet (no slip) boundary conditions on the plane $\{x_3 = 0\}$.

Ekman spiral is the famous exact solution (time-independent) of the full nonlinear problem (1.1)-(1.2). It describes rotating boundary layers in geophysical fluid dynamics (atmospheric and oceanic boundary layers cf. [11], [19], [14], [5], [17]). The boundary layer in the theory of rotating fluids known as the Ekman layer is between a geostrophic flow and a solid boundary at which the no slip condition applies. In the geostrophic flow region corresponding to large x_3 (far away from the solid boundary at $x_3 = 0$), there is a uniform flow with velocity U_∞ in the x_1 direction. Associated with U_∞, there is a pressure gradient in the x_2 direction. The Ekman spiral solution in $\mathbb{R}^3_+$ matches this uniform velocity for large x_3 with the no slip boundary condition at $x_3 = 0$. The corresponding velocity field $\mathbf{U}^E(x_3)$: $\mathbf{U}^E(x_3) = (U_1^E(x_3), U_2^E(x_3), 0)$ depends only on the vertical variable x_3:

$$U_1^E(x_3) = U_\infty \left(1 - e^{-\frac{x_3}{\delta}} \cos(\frac{x_3}{\delta})\right),\ U_2^E(x_3) = U_\infty e^{-\frac{x_3}{\delta}} \sin(\frac{x_3}{\delta}), \tag{1.4}$$

where δ is the rotating boundary layer (Ekman layer) thickness:

$$\delta = \left(\frac{2\nu}{|\Omega|}\right)^{1/2}. \tag{1.5}$$

The corresponding pressure field $p^E(x_2)$ depends only on x_2 and it is given by

$$p^E(x_2) = -\Omega U_\infty x_2. \tag{1.6}$$

Clearly, the nonlinear term in (1.1) is zero for $\mathbf{U} = \mathbf{U}^E(x_3)$ and, therefore, the pair of $(\mathbf{U}^E(x_3), p^E(x_2))$ which is called 'Ekman spiral', is an exact solution of the full nonlinear problem. Remarkable persistent (stability) of the Ekman spiral in atmospheric and oceanic rotating boundary layers has been noticed in geophysical literature. Observe that the velocity field satisfies

$$\lim_{x_3 \to +\infty} \mathbf{U}^E(x_3) = (U_\infty, 0, 0). \tag{1.7}$$

We make a trivial remark that will be important for future estimates of norms. We note that the velocity field corresponding to the Ekman spiral solution is bounded

$$|\mathbf{U}^E(x_3)| \le 2U_\infty. \tag{1.8}$$

In spite of the importance of the Ekman layer in geophysics it seems that a mathematical treatment of the phenomenon has not been given so far. Our aim is to give a mathematical theory for time-local solvability of the nonstationary problem around the stationary Ekman spiral solution. Since the Ekman spiral has velocity field nondecreasing at infinity, it is essential in the theory of geophysical rotating boundary layers to study solvability of (1.1)-(1.3) for initial data in spaces of functions nondecreasing at infinity.

We write

$$\mathbf{U}(t,x_1,x_2,x_3) = \mathbf{U}^E(x_3) + \mathbf{V}(t,x_1,x_2,x_3), \tag{1.9}$$

$$p(t,x_1,x_2,x_3) = p^E(x_2) + q(t,x_1,x_2,x_3). \tag{1.10}$$

Since the Ekman spiral is an exact solution of the full nonlinear problem, the vector field $\mathbf{V}(t,x_1,x_2,x_3)$ satisfies the following equations

$$\partial_t \mathbf{V} + (\mathbf{V}\cdot\nabla)\mathbf{V} + (\mathbf{U}^E(x_3)\cdot\nabla)\mathbf{V} + V_3\frac{\partial \mathbf{U}^E}{\partial x_3} + \Omega e_3\times\mathbf{V} + \nu curl^2\mathbf{V} = -\nabla q, \tag{1.11}$$

$$\nabla\cdot\mathbf{V} = 0, \tag{1.12}$$

$$\mathbf{V}(t,x)|_{x_3=0} = (V_1(t,x),V_2(t,x),V_3(t,x))|_{x_3=0} = (0,0,0), \tag{1.13}$$

$$\mathbf{V}(t,x)|_{t=0} = \mathbf{V}_0(x). \tag{1.14}$$

Let $\mathbf{J}$ be the matrix such that $\mathbf{J}\mathbf{a} = e_3\times\mathbf{a}$ for any vector field $\mathbf{a}$, i.e.

$$\mathbf{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.15}$$

Let $\mathbf{P}_+$ be the Helmholtz projection operator on solenoidal fields in $\mathbb{R}^3_+$. We define the Stokes operator $\mathbf{A}(\nu)$:

$$\mathbf{A}(\nu)\mathbf{V} = \nu\mathbf{P}_+ curl^2\mathbf{V} = -\nu\mathbf{P}_+\Delta\mathbf{V} \tag{1.16}$$

on solenoidal vector fields $\mathbf{V}$. The operator $\mathbf{P}_+$ can be represented by

$$\mathbf{P}_+ f = r\mathbf{P}Ef. \tag{1.17}$$

Here, r is the restriction operator to the half-space and $\mathbf{P}$ is the Helmholtz projection operator in the whole space, defined by

$$\mathbf{P} = \{P_{ij}\}_{i,j=1,2,3},\ P_{ij} = \delta_{ij} + R_iR_j; \tag{1.18}$$

$R_j (j=1,2,3)$ are the Riesz operators $\frac{\partial}{\partial x_j}(-\Delta)^{-1/2}$ with the symbols $\frac{i\xi_j}{|\xi|}$ (see e.g. [26]). Besides, the operator E is defined as follows:

Definition 1.1. *(1) For a function $h(x)$ on $\mathbb{R}^3_+$ we define an extended function $e^{\pm}h$ by*

$$\left(e^{\pm}h\right)(x) = \begin{cases} h(x) & \text{if } x_3 > 0, \\ \pm h(x^*) & \text{if } x_3 < 0, \end{cases}$$

where $x^* = (x_1, x_2, -x_3)$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+$.
(2) For a vector field $f(x) = (f^1, f^2, f^3)$ on $\mathbb{R}^3_+$ we define an extended vector field Ef by

$$i\text{–th component of } (Ef)(x) = \begin{cases} \left(e^{+}f^i\right)(x) & \text{for } 1 \le i \le 2, \\ \left(e^{-}f^3\right)(x) & \text{for } i = 3. \end{cases}$$

That is, $Ef = diag[e^+, e^+, e^-]\left({}^T f\right)$, where diag represents a diagonal matrix, ${}^T f$ is a transposed vector field of f.

By employing the Helmholtz projection operator $\mathbf{P}_+$, we transform system (1.11)-(1.14) into an abstract operator differential equation for $\mathbf{V}$,

$$\mathbf{V}_t + \mathbf{A}(\nu)\mathbf{V} + \Omega\mathbf{S}\mathbf{V} + \mathbf{C}_E\mathbf{V} + \mathbf{P}_+(\mathbf{V}\cdot\nabla)\mathbf{V} = 0, \qquad \mathbf{V}|_{t=0} = \mathbf{V}_0, \tag{1.19}$$

where

$$\mathbf{S} = \mathbf{P}_+\mathbf{J}\mathbf{P}_+,\ \mathbf{C}_E\mathbf{V} = \mathbf{P}_+\left((\mathbf{U}^E(x_3)\cdot\nabla)\mathbf{V} + V_3\frac{\partial \mathbf{U}^E}{\partial x_3}\right) \tag{1.20}$$

and we have used $\mathbf{P}_+\mathbf{J}\mathbf{V} = \mathbf{P}_+\mathbf{J}\mathbf{P}_+\mathbf{V}$ on solenoidal vector fields. Let us compare the situation here with the one in the whole space as treated in [10]. The main difference between the problem in a half-space $\mathbb{R}^3_+$ with the problem in $\mathbb{R}^3$ is that the Stokes operator $\mathbf{A} = \mathbf{A}(\nu)$ and the operator $\mathbf{S} = \mathbf{P}_+\mathbf{J}\mathbf{P}_+$ do not commute in $\mathbb{R}^3_+$ and there is an additional 'Ekman operator' $\mathbf{C}_E$ in Eqs. (1.19). Motivated by real physical applications, where physical fields are a superposition of non-monochromatic modes having different horizontal wavenumbers (periodicity and almost periodicity in the variables x_1 and x_2), we consider initial data $\mathbf{V}_0(x)$ for Eqs. (1.19) in spaces of solenoidal vector fields nondecreasing at infinity in x_1, x_2. The consideration of solutions not decaying at infinity in x_1, x_2 is essential in the development of rigorous mathematical theory of the Ekman rotating boundary layer problem. In view of (1.7) it is natural to consider vector fields $\mathbf{V}$, which belong to L^p, $1 < p < +\infty$ in x_3.

The first step in the analysis of the nonlinear problem (1.19) is to show that the corresponding linear operator generates a semigroup in appropriate spaces (L^p, $1 < p < +\infty$ or L^∞). We note that the L^p, $1 < p < +\infty$ case is usually simpler than the L^∞ case due to the fact that Riesz operators are bounded operators in L^p but not in L^∞. We recall that for $\Omega = 0$ (non-rotating case) Green's function of the Stokes operator in $\mathbb{R}^3$ and $\mathbb{R}^3_+$ (half-space) belong to $L^1(\mathbb{R}^3)$ implying that the corresponding operator generates a semigroup in $L^\infty(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3_+)$. On the other hand, for $\Omega \neq 0$ Green's function of the (Stokes+Coriolis) problem in $\mathbb{R}^3$ does not belong to $L^1(\mathbb{R}^3)$ (see [10]). Moreover, it behaves as $|x|^{-3}$ for large $|x|$ and the corresponding integral operator is not a bounded operator in $L^\infty(\mathbb{R}^3)$. One needs to restrict initial data on a subspace of $L^\infty(\mathbb{R}^3)$ ([10, Appendix A]). Similar situation of unboundedness in $L^\infty(\mathbb{R}^2)$ (for horizontal x_1, x_2 planes) holds for the linear (Stokes+Coriolis) problem in a half-space. One needs to restrict initial data on a subspace of $L^\infty(\mathbb{R}^2)$ where Riesz operators and, consequently, the operator $\mathbf{P}_+\mathbf{J}\mathbf{P}_+$ are bounded. The natural space for this purpose for initial data $\mathbf{V}_0$ is the space $X = \dot{B}^0_{\infty,1}(\mathbb{R}^2; L^p(\mathbb{R}_+))$, the space of all $L^p(\mathbb{R}_+)$-valued $\dot{B}^0_{\infty,1}$ functions in $\mathbb{R}^2$. Here, $\mathbb{R}_+ := (0,\infty)$, and $\dot{B}^0_{\infty,1}$ is the homogeneous Besov space which is strictly smaller than L^∞.

Related to the Navier-Stokes equations, the space $\dot{B}^0_{\infty,1}$ was first used in Sawada-Taniuchi [23] to solve the Boussinesq equations for nondecaying initial data. It is known [27] that $\nabla f \in \dot{B}^0_{\infty,1}$ if f and $\nabla^2 f$ are in L^∞, hence, the space $\dot{B}^0_{\infty,1}$ contains nondecreasing functions such as almost periodic functions of the form $\sum_{j=1}^\infty \alpha_j \exp(\sqrt{-1}\lambda_j \cdot x)$ with $\{\alpha_j\}_{j=1}^\infty \in l^1$, $\{\lambda_j\} \subset \mathbb{R}^3 \setminus \{0\}$. Since our space X is an L^p-valued Besov space, it includes functions nondecreasing in tangential direction x_1, x_2, and decreasing in the normal direction x_3. Moreover, we can prove, as shown in Corollary 2.12, that the Riesz operators are bounded in vector-valued Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^N; L^p(\mathbb{R}_+))$ for all indices $1 \le p, q \le \infty$, $s \in \mathbb{R}$, and every space dimension $N = 1, 2, 3, \ldots$. The boundedness is essentially a consequence of Theorem 2.5 in Section 2 that is an extension of the Mikhlin type multiplier theorem, obtained by Amann [1] in the inhomogeneous Besov spaces $B^s_{p,q}(\mathbb{R}^N; \mathsf{E})$ for a general Banach space E, to the homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^N; \mathsf{E})$.

In order to estimate the nonlinear term we also employ the spaces $\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ and $\mathrm{BUC}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$, where the latter one denotes the space of all $L^p(\mathbb{R}_+)$-valued bounded uniformly continuous functions on $\mathbb{R}^{n-1}$. Note that we always work in general space dimension $n \ge 2$, as long as the Coriolis and Ekman operators are not involved. The key for the nonlinear estimate is the embedding between the above spaces (see Lemma 2.3). However, homogeneous Besov spaces are usually defined as a quotient spaces (modulo polynomials), which are not suitable for the study of partial differential equations (PDE). This is the reason why we use rather "script" Besov spaces, $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ instead of $\dot{B}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ (see Definition 2.1). The key embedding result (for E-valued functions, where E is a Banach space) now reads as

$$\dot{\mathcal{B}}^0_{\infty,1} \hookrightarrow \mathcal{BUC} \hookrightarrow \dot{\mathcal{B}}^0_{\infty,\infty}.$$

Here, $\mathcal{BUC}$ is the subspace of BUC such that $\mathrm{BUC} = \mathcal{BUC} \oplus \{1\}$, where $\{1\}$ denotes the space of constant functions. By $\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ we denote the solenoidal part of $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ (see (2.13) and what follows for the definition).

In this paper we construct a local-in-time solution of the rotating Navier-Stokes equations (1.1)-(1.3) in the space $\mathrm{BC}([0,T_0);\mathcal{BUC}_\sigma(\mathbb{R}^2;L^p(\mathbb{R}_+)))$ under the condition that the initial velocity $\mathbf{V}_0 \in \dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2;L^p(\mathbb{R}_+))$, $2<p<+\infty$. Here, $\mathcal{BUC}_\sigma$ denotes a solenoidal part of $\mathcal{BUC}$ (see definition (2.4)). Also, we denote by $\mathrm{BC}(I;\mathsf{E})$ the space of all bounded continuous E-valued functions on the interval $I\in\mathbb{R}$. In particular, the current work is concerned with the so-called mild solutions, the solutions of the corresponding integral equation to (1.19).

For the linear Stokes problem we employ the solution formula derived in Desch-Hieber-Prüss [7] for the Stokes resolvent in terms of the resolvent of the Dirichlet Laplacian and certain remainder terms. Detailed information on the full linear problem (Stokes + Coriolis + Ekman) is then used to construct a (local-in-time) mild solution to the nonlinear rotating Navier-Stokes equations in $\mathbb{R}^3_+$. To derive the estimates for the linear part we will employ theory for E-valued Besov spaces. The main ingredient will be an operator-valued version of Mikhlin's multiplier result. It will be the basis for an operator-valued bounded H^∞-calculus for the Laplacian on E-valued homogeneous Besov spaces, which serves as a useful tool in estimating the formulas for the Helmholtz projection and the resolvent of the Stokes operator. The generation result for the Stokes operator and a standard perturbation argument will then lead to the generation result for the full linear operator (Stokes+Coriolis+Ekman).

Our main result reads as

Theorem 1.2. *Let $2<p<\infty$. For each $\mathbf{V}_0\in\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^2;L^p(\mathbb{R}_+))$ there exist $T_0>0$ and a unique mild solution $\mathbf{V}$ of (1.19) such that*

$$(t\mapsto \mathbf{V}(t))\in BC([0,T_0);\mathcal{BUC}_\sigma(\mathbb{R}^2;L^p(\mathbb{R}_+))), \tag{1.21}$$

$$(t\mapsto t^{1/2}\nabla\mathbf{V}(t))\in BC([0,T_0);\mathcal{BUC}(\mathbb{R}^2;L^p(\mathbb{R}_+))). \tag{1.22}$$

Furthermore, the solution $\mathbf{V}$ satisfies

$$t^{1/2}||\nabla'\mathbf{V}(t)||_{\mathcal{BUC}(\mathbb{R}^2;L^p(\mathbb{R}_+))}\to 0 \qquad \text{as} \quad t\to 0. \tag{1.23}$$

Here, $\nabla'=(\partial_{x_1},\partial_{x_2})$.

Remark 1.3. (i) As lower estimate for existence time T_0 we get for every $\varphi_0\in(0,2\pi)$ and every $\delta\in(0,1/2]$ that

$$T_0\geq \min\left\{1,\left(\frac{1}{C_{\varphi_0,\delta,p}e^{2\omega_1}||\mathbf{V}_0||_{\dot{B}^0_{\infty,1}(\mathbb{R}^2;L^p(\mathbb{R}_+))}}\right)^{\frac{1}{-(1/2p)-\delta+(1/2)}}\right\}. \tag{1.24}$$

Here, the constants $C_{\varphi_0,\delta,p}>0$ and $\omega_1>0$ are determined in Proposition 4.5 -Lemma 5.1 and Proposition 4.5, respectively, and depend on the Coriolis parameter Ω and $||\mathbf{U}^E||_{W^{1,\infty}}$, where $W^{1,\infty}=W^{1,\infty}(\mathbb{R}^3_+):=\{u\in L^\infty(\mathbb{R}^3_+): \nabla u\in L^\infty(\mathbb{R}^3_+)\}$.

(ii) If we assume that $\nabla^j \mathbf{V}_0 \in \mathcal{BUC}$ for some positive integer j, then the solution $\mathbf{V}$ satisfies

$$\nabla^j \mathbf{V}, t^{1/2}\nabla^{j+1}\mathbf{V} \in BC([0,T_0); \mathcal{BUC}(\mathbb{R}^2; L^p(\mathbb{R}_+))).$$

For $j=1$ this fact can be shown by applying ∇ to the corresponding integral equation to (1.19) and using (1.22). The inductive procedure $j \to j+1$ is similarly shown by applying ∇^j to the integral equation.
(iii) We also get higher regularity on the interval $[\epsilon, T_0)$ with arbitrary small $\epsilon > 0$, namely that

$$\nabla^k \mathbf{V} \in BC([\epsilon, T_0); \mathcal{BUC}(\mathbb{R}^2; L^p(\mathbb{R}_+))) \quad \text{for any positive integer } k. \tag{1.25}$$

This follows by an iterative use of remark (ii) and (1.21)-(1.22).

Note that in general we do not expect the solutions of the nonlinear equations to be an element of the space of initial data $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2; L^p(\mathbb{R}_+))$. This is essentially due to the fact that normal derivatives act merely on the L^p part of the space $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ (see Remark 4.2). To overcome this problem we apply the contraction mapping principle in the larger space $\mathcal{BUC}(\mathbb{R}^2; L^p(\mathbb{R}_+))$. The unboundedness of the Helmholtz projection in that space is handled by using a splitting of $\mathbf{P}_+\partial_3$ in a term with pure normal derivative and terms containing only tangential derivatives and Riesz operators. This leads to the slightly technical Section 4.

In what follows we write vector fields in small letters as u, v instead of $\mathbf{U}$, $\mathbf{V}$ except the Ekman spiral solution $\mathbf{U}^E$.

The plan of the paper is as follows. In Section 2 we prepare the definition of the vector-valued homogeneous Besov spaces $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$, and ensure boundedness of the Helmholtz projection in this space by modifying Mikhlin's theorem for the inhomogeneous spaces obtained by Amann [1]. We also set up the notion of an operator-valued bounded H^∞-calculus in the vector-valued Besov spaces $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; \mathsf{E})$ for a general Banach space E, and prove that the Laplacian admits this property. The operator-valued bounded H^∞-calculus will play a key role in Section 4. In Section 3 we will give resolvent estimates for the full linear part (Stokes+Coriolis+Ekman) based on estimates for the resolvent of the Stokes operator and a standard perturbation argument. The resolvent estimates for the Stokes operator are obtained by applying the results of Section 2 to an explicit representation for the Stokes resolvent. As a consequence, the full operator is proved to generate a holomorphic semigroup in our vector-valued Besov space. In the technical Section 4 we first collect some basic facts about the Laplacian in the considered spaces. Using this, we then proceed by providing estimates for the semigroup generated by the full linear operator, which will represent the basis for the application of the contraction mapping principle. In Section 5 we prove Theorem 1.2 utilizing the estimates obtained in the sections before. Finally, in the appendix we give a characterization of the vector-valued solenoidal Besov space defined in Section 2 and an estimate for fractional powers of the Laplacian applied to the heat kernel.

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2. Basic ingredients

In this section we define E-valued homogeneous Besov spaces and provide the required basics for the treatment of the linear and nonlinear problems in the subsequent sections.

In standard monographs, see e.g. [28], the homogeneous Besov space $\dot{B}^s_{r,q}(\mathbb{R}^N)$ for $N \in \mathbb{N}$ is defined as

$$\dot{B}^s_{r,q}(\mathbb{R}^N) := \{f \in \mathcal{Z}'(\mathbb{R}^N) : \|f\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N)} < \infty\},$$

where $\|f\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N)} = \left(\sum_{j\in\mathbb{Z}}(2^{sj}\|\phi_j * f\|_{L^r(\mathbb{R}^N)})^q\right)^{1/q}$ for $1 \le r, q \le \infty$, $s \in \mathbb{R}$. (see also [3]). Here $\mathcal{Z}'(\mathbb{R}^N)$ denotes the topological dual of

$$\mathcal{Z}(\mathbb{R}^N) := \{f \in \mathcal{S}(\mathbb{R}^N) : D^\alpha \hat{f}(0) = 0,\ \alpha \in \mathbb{N}_0^N := \mathbb{N} \cup \{0\}\},$$

where $D^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_N}^{\alpha_N}$ for $\alpha = (\alpha_1, \dots, \alpha_N)$. By $\mathbb{Z}$ we denote the set of all integers and by $\mathbb{N}$ the set of all natural numbers. The space $\mathcal{Z}'(\mathbb{R}^N)$ can be identified with $\mathcal{S}'(\mathbb{R}^N)$ modulo all polynomials in $\mathbb{R}^N$, where $\mathcal{S}'(\mathbb{R}^N)$ denotes the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^N)$. Hence $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N)$ is a space of equivalence classes whose elements in general possess different derivatives. This leads to the fact that it is not appropriate to construct solutions of a concrete PDE in such a space. In such a situation it is desirable to have a space of functions, which motivates the alternative definition given below.

Recall that a Littlewood-Paley decomposition is given by a family of functions $\phi_j \in \mathcal{S}(\mathbb{R}^N)$ satisfying $\sum_{j\in\mathbb{Z}} \widehat{\phi_j}(\xi) = 1$ for $\xi \in \mathbb{R}^N \setminus \{0\}$, where $\widehat{\phi_j}(\xi) := \widehat{\phi_0}(2^{-j}\xi)$ and $0 \ne \phi_0 \in \mathcal{S}(\mathbb{R}^N)$ such that $\mathrm{supp}\widehat{\phi_0} \subseteq \{1/2 \le |\xi| \le 2\}$. Moreover, for a Banach space E, we denote by $\mathcal{S}'(\mathbb{R}^N; \mathsf{E})$ the space of all E-valued linear continuous functionals on $\mathcal{S}(\mathbb{R}^N)$, i.e. $\mathcal{S}'(\mathbb{R}^N;\mathsf{E}) := \mathcal{L}(\mathcal{S}(\mathbb{R}^N);\mathsf{E})$. Note that then

$$\mathcal{S}(\mathbb{R}^N;\mathsf{E}) \hookrightarrow L^q(\mathbb{R}^N;\mathsf{E}) \hookrightarrow \mathcal{S}'(\mathbb{R}^N;\mathsf{E}), \quad q \in [1,\infty].$$

Being aware of these facts we can proceed with a rigorous definition of E-valued homogeneous Besov spaces. The authors are not sure about the fact of who introduced these spaces first. Our definition follows the approach for inhomogeneous E-valued Besov spaces of Amann in [1]. The inhomogeneous versions were introduced first by Grisvard [12] and Muramatu [18].

Definition 2.1. *Let E be a Banach space, $1 \le r,q \le \infty$, $s \in \mathbb{R}$, and $\{\phi_j\}_{j\in\mathbb{Z}}$ a Littlewood-Paley decomposition. If*

$$\text{either} \quad s < N/r \qquad \text{or} \qquad s = N/r \quad \text{and} \quad q = 1, \tag{2.1}$$

then the E-valued homogeneous Besov space $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})$ is defined by

$$\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E}) :=$$
$$\{f \in \mathcal{S}'(\mathbb{R}^N;\mathsf{E}) : \|f\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})} < \infty,\ f = \sum_{j\in\mathbb{Z}} \phi_j * f \ \text{in } \mathcal{S}'(\mathbb{R}^N;\mathsf{E})\},$$

where

$$\|f\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})} := \left(\sum_{j\in\mathbb{Z}} (2^{sj}\|\phi_j * f\|_{L^r(\mathbb{R}^n;\mathsf{E})})^q \right)^{1/q}.$$

On the other hand, if E is additionally the dual space of a Banach space F, $s \in \mathbb{R}$, $1 < r,q \le \infty$, and

$$\text{either} \quad s > N/r \qquad \text{or} \qquad s = N/r \quad \text{and} \quad q \ne 1, \tag{2.2}$$

we set

$$\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E}) := (\dot{\mathcal{B}}^{-s}_{r',q'}(\mathbb{R}^N;\mathsf{F}))'. \tag{2.3}$$

Remark 2.2. (1) Definition 2.1 relies on the fact that under condition (2.1) the series $\sum_{j\in\mathbb{Z}} \phi_j * f$ converges in $\mathcal{S}'(\mathbb{R}^N;\mathsf{E})$ for $f \in \mathcal{S}'(\mathbb{R}^N;\mathsf{E})$ with $\|f\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})} < \infty$. For $\mathsf{E} = \mathbb{C}$, the set of all complex numbers, a proof of this fact can be found in [3], [16]. We omit the proof here, since the one given in [16] directly transfers to the E-valued case. Note that $\|f\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})} < \infty$ is not sufficient for the convergence of $\sum_{j\in\mathbb{Z}} \phi_j * f$ in $\mathcal{S}'(\mathbb{R}^N;\mathsf{E})$, if the parameters s,r,q satisfy the inverse condition (2.2). Therefore we used definition (2.3) in that case. Also note that the first ones who made use of definition (2.2) in the case $\mathsf{E} = \mathbb{C}$ for the space $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^N)$ related to the Navier-Stokes equations were O. Sawada and Y. Taniuchi in [23] and O. Sawada in [22].

(2) By standard arguments it can be easily shown that $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})$ is a Banach space.

(3) Requiring f to have the representation $f = \sum_{j\in\mathbb{Z}} \phi_j * f$ ensures that

(E-valued) constants are not element of $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^N;\mathsf{E})$. This yields the continuous embedding

$$\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^N;\mathsf{E}) \hookrightarrow \mathrm{BUC}(\mathbb{R}^N;\mathsf{E}).$$

(Observe that $\|c\|_{\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^N;\mathsf{E})} = 0$ for $c \in \mathsf{E}$!)

(4) In this work we do not make use of $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})$ for r,q,s satisfying (2.2) with $r=1$ or $q=1$. Therefore we skipped a proper definition of those spaces.

(5) In the scalar-valued case $\mathsf{E}=\mathbb{C}$ for all values of the parameters s,r,q as in Definition 2.1 the space $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N)$ is isomorphic to $\dot{B}^s_{r,q}(\mathbb{R}^N)$, see [3], [16].

The embedding in Remark 2.2 (3) is of crucial importance for estimating the nonlinear term in Section 4. But, since the Helmholtz projection $\mathbf{P}_+$ is expected to be unbounded in $\mathrm{BUC}(\mathbb{R}^2;L^p(\mathbb{R}_+))$, it is necessary to employ the larger space $\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^2;L^p(\mathbb{R}_+))$, which admits the boundedness of $\mathbf{P}_+$. For this purpose we define

$$\mathcal{BUC}(\mathbb{R}^N;\mathsf{E}) := \{f \in \mathrm{BUC}(\mathbb{R}^N;\mathsf{E}); f = \sum_{j\in\mathbb{Z}} \phi_j * f \text{ in } \mathcal{S}'(\mathbb{R}^N;\mathsf{E})\}.$$

Since the series $\sum_{j\in\mathbb{Z}} \phi_j * f$ converges in $\mathcal{S}'(\mathbb{R}^N;\mathsf{E})$ for $f \in \mathrm{BUC}(\mathbb{R}^N;\mathsf{E})$ this space is well-defined and it is isomorphic to $\mathrm{BUC}(\mathbb{R}^N;\mathsf{E})$ modulo constants. We also define the solenoidal part of $\mathcal{BUC}$ by

$$\mathcal{BUC}_\sigma(\mathbb{R}^N;\mathsf{E}) := \{f \in \mathcal{BUC}(\mathbb{R}^N;\mathsf{E}); \mathrm{div} f = 0,\ f|_{\partial\mathbb{R}^N_+} = 0\}. \tag{2.4}$$

An essential ingredient for the calculations in Section 4 will be

Lemma 2.3. *Let $N \in \mathbb{N}$ and E be the dual of a Banach space F. Then*

$$\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^N;\mathsf{E}) \hookrightarrow \mathcal{BUC}(\mathbb{R}^N;\mathsf{E}) \hookrightarrow \dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^N;\mathsf{E}).$$

Proof. The first embedding can be proved along the line of Remark 2.2(3). It remains to prove the second embedding. To this end let $f \in \mathcal{BUC}(\mathbb{R}^N;\mathsf{E})$ and $\varphi \in \dot{\mathcal{B}}^0_{1,1}(\mathbb{R}^N;\mathsf{F})$. Since $\dot{\mathcal{B}}^0_{1,1}(\mathbb{R}^N;\mathsf{F}) \subseteq L^1(\mathbb{R}^N;\mathsf{F})$, we can form the dual pairing of f and φ and compute

$$\begin{aligned} |\langle f,\varphi\rangle| &= \Big|\int_{\mathbb{R}^N} \langle f(x), \sum_{j\in\mathbb{Z}} \phi_j * \varphi(x)\rangle_{\mathsf{F},\mathsf{E}} \mathrm{d}x\Big| \\ &\le \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^N} \|f(x)\|_{\mathsf{E}} \|\phi_j * \varphi(x)\|_{\mathsf{F}} \mathrm{d}x \\ &\le \|f\|_{L^\infty(\mathbb{R}^N;\mathsf{E})} \|\varphi\|_{\dot{\mathcal{B}}^0_{1,1}(\mathbb{R}^N;\mathsf{F})}. \end{aligned}$$

Thus, by definition, $f \in \dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^N;\mathsf{E})$ and we have

$$\|f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^N;\mathsf{E})} = \sup_{\varphi\in\dot{\mathcal{B}}^0_{1,1}(\mathbb{R}^N;\mathsf{F}), \|\varphi\|_{\dot{\mathcal{B}}^0_{1,1}(\mathbb{R}^N;\mathsf{F})}=1} |\langle f,\varphi\rangle| \le \|f\|_{\mathcal{BUC}(\mathbb{R}^N;\mathsf{E})}.$$

□

Next we state some density results of certain subspaces of smooth functions. This will turn out to be helpful in proving the strong continuity of the semigroups derived in Section 3.

Lemma 2.4. *Let $n \in \mathbb{N}, 1 \le p < \infty$ and $D \subseteq \mathbb{R}$ be an open set. Let E be a Banach space and s, r, q be as in condition (2.1). Then*

(1) $\Big\{u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n;\mathsf{E}) : D^\alpha u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n;\mathsf{E}),\ \alpha \in \mathbb{N}_0^n\Big\} \overset{d}{\hookrightarrow} \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n;\mathsf{E})$.

(2) $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};\bigcap_{k\in\mathbb{N}_0} W^{k,p}(D)) \overset{d}{\hookrightarrow} \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(D))$.

(3) $\Big\{u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(D)) : D^\alpha u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(D)),\ \alpha \in \mathbb{N}_0^n\Big\}$
$\overset{d}{\hookrightarrow} \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(D))$.

Proof. (1) Choose a mollifier φ_ε, i.e. $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi_0(x/\varepsilon)$ with $0 \ne \varphi_0 \in C_c^\infty(\mathbb{R}^n)$, $\varphi_0 \ge 0$, and $\int_{\mathbb{R}^n}\varphi_0(x)\mathrm{d}x = 1$. We claim that $\varphi_\varepsilon * u \to u$ in $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n;\mathsf{E})$ if $\varepsilon \to 0$ for each $u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n;\mathsf{E})$. Indeed,

$$\begin{aligned}
&\|\varphi_\varepsilon * u - u\|^r_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n;\mathsf{E})}\\
&\le \sum_{k\in\mathbb{Z}}\left(\sum_{j=k-2}^{k+2} 2^{ks}\|(\varphi_\varepsilon * \phi_j - \phi_j) * \phi_k * u\|_{L^q(\mathbb{R}^n;\mathsf{E})}\right)^r\\
&\le \sum_{k\in\mathbb{Z}}\left(2^{ks}\|\phi_k * u\|_{L^q(\mathbb{R}^n;\mathsf{E})}\sum_{j=k-2}^{k+2}\|\varphi_\varepsilon * \phi_j - \phi_j\|_{L^1(\mathbb{R}^n)}\right)^r, \qquad (2.5)
\end{aligned}$$

where we applied the vector-valued version of Young's inequality, see [1, page 13]. Since $\sum_{j=k-2}^{k+2}\|\varphi_\varepsilon * \phi_j - \phi_j\|_{L^1(\mathbb{R}^n)} \le \sum_{j=k-2}^{k+2}(\|\varphi_\varepsilon * \phi_j\|_1 + \|\phi_j\|_1) \le 10\|\phi_0\|_1$, $\varepsilon \in (0,1)$, it is easy to see, that the series in (2.5) converges uniformly in $\varepsilon \in (0,1)$. Thus, we may interchange passing to the limit and summation, which yields $\phi_\varepsilon * u \to u$, due to $\|\varphi_\varepsilon * \phi_j - \phi_j\|_{L^1(\mathbb{R}^n)} \to 0$ if $\varepsilon \to 0$. This implies (1) in view of

$$\begin{aligned}
\|D^\alpha\varphi_\varepsilon * u\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n;\mathsf{E})} &\le \sum_{k\in\mathbb{Z}}\left(\sum_{j=k-2}^{k+2} 2^{ks}\|\phi_j * D^\alpha\varphi_\varepsilon\|_1\|\phi_k * u\|_{L^q(\mathbb{R}^n;\mathsf{E})}\right)^r\\
&\le 5\|\phi_0\|_1\|D^\alpha\varphi_\varepsilon\|_1\|u\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^n;\mathsf{E})}, \quad \alpha \in \mathbb{N}_0^n.
\end{aligned}$$

(2) First suppose $D = \mathbb{R}$. Here we choose a mollifier in the last component, i.e. a function $\psi_\delta := \frac{1}{\delta}\psi_0(x_n/\delta)$ with $0 \ne \psi_0 \in C_c^\infty(\mathbb{R})$, $\psi_0 \ge 0$, and $\int_{\mathbb{R}}\psi_0(x_n)\mathrm{d}x_n = 1$. By similar arguments as in the proof of (1) we obtain $\psi_\delta *_{x_n} u \to u$ in $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R}))$ if $\delta \to 0$ for each $u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R}))$, (here $*_{x_n}$ denotes the convolution w.r.t. the last component x_n). For $u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(D))$ we set

$$u_\delta := r\psi_\delta *_{x_n} E_0 u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; \textstyle\bigcap_{k\in\mathbb{N}_0} W^{k,p}(D)), \ \delta > 0,$$

where $r_D : \mathbb{R}^n \to D$ is the restriction and $E_0 : D \to \mathbb{R}^n$ the trivial extension. We compute

$$\begin{aligned}\|u_\delta - u\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(D))} &= \|r_D\psi_\delta *_{x_n} E_0u - r_D E_0 u\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(D))} \\ &\le \|\psi_\delta *_{x_n} E_0 u - E_0 u\|_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R}))} \longrightarrow 0\end{aligned}$$

if $\delta \to 0$, which yields (2).

(3) This follows by combining the mollifier arguments in the proof of (1) and (2), i.e. here it can be shown that $\varphi_\varepsilon *_{x'} r_D\psi_\delta *_{x_n} E_0 u \to u$ if $(\varepsilon,\delta)\to 0$ for each $u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(D))$. □

The following operator-valued Mikhlin type multiplier result is fundamental for the treatment of the linearized equations in Section 3. Although it is not explicitly included there, essentially it is a consequence of results obtained by Amann in [1]. (Observe that Amann does not deal with homogeneous Besov spaces. The dotted spaces appearing in [1] have a different meaning.)

Theorem 2.5. *Let $N \in \mathbb{N}$, E be a Banach space. Let $s \in \mathbb{R}$, $1 \le r, q \le \infty$ be satisfying condition (2.1). Furthermore, let $m \in C^{N+1}(\mathbb{R}^N \setminus \{0\}, \mathcal{L}(\mathsf{E}))$ such that*

$$\|m\|_{M(\mathsf{E})} := \max_{|\alpha|\le N+1} \sup_{\xi\in\mathbb{R}^N\setminus\{0\}} |\xi|^{|\alpha|} \|D^\alpha m(\xi)\|_{\mathcal{L}(\mathsf{E})} < \infty. \tag{2.6}$$

Then $\mathcal{F}^{-1}m\mathcal{F}$ is a bounded operator on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})$ and we have

$$\|\mathcal{F}^{-1}m\mathcal{F}\|_{\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E}))} \le C\|m\|_{M(\mathsf{E})}, \tag{2.7}$$

where $C = C(n) > 0$ is independent of r,q,s and m.

Remark 2.6. If E is the dual of a Banach space F, by definition the assertion is also valid for $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})$, if s, r, q satisfy condition (2.2).

A function $m : \mathbb{R}^n \setminus \{0\} \to \mathcal{L}(\mathsf{E})$, satisfying the assumptions of Theorem 2.5, is called an operator-valued multiplier on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})$. Easy examples of operator-valued multipliers are given by scalar-valued multipliers, i.e. functions $m : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ that satisfy the assumptions of Theorem 2.5 with $\mathsf{E} = \mathbb{C}$. Indeed, by the identification $m = m \cdot I$, where I is the identity on E, it is easy to verify that m is also an operator-valued multiplier. The key for the proof of Theorem 2.5 is the following lemma.

Lemma 2.7. [1, Lemma 4.2(i)] *Let $N \in \mathbb{N}$, $s \in \mathbb{R}$, $1 \le r, q \le \infty$, and E be a Banach space. Given $j \in \mathbb{Z}$, suppose that $m \in C^{N+1}(\mathbb{R}^N \setminus \{0\}, \mathcal{L}(\mathsf{E}))$ such that*

$$\mu_j := \max_{|\alpha|\le N+1} \sup_{2^{j-1}\le|\xi|\le 2^{j+1}} |\xi|^{|\alpha|} ||D^\alpha m(\xi)||_{\mathcal{L}(\mathsf{E})} < \infty. \tag{2.8}$$

Then $\mathcal{F}^{-1}(m\widehat{\phi}_j) \in L^1(\mathbb{R}^N; \mathcal{L}(\mathsf{E}))$ *and*

$$||\mathcal{F}^{-1}(m\widehat{\phi}_j)||_{L^1(\mathbb{R}^N;\mathcal{L}(\mathsf{E}))} \leq C\mu_j,$$

where $C = C(n) > 0$ *is independent of* m *and* j.

Proof of Theorem 2.5. Since $\sum_{j\in\mathbb{Z}} \hat{\phi}_j = 1$ (except at 0) and $\mathrm{supp}\widehat{\phi_j} \cap \mathrm{supp}\hat{\phi}_k = \emptyset$ for $|j-k| \geq 3$, we calculate

$$\begin{aligned} ||\mathcal{F}^{-1}m\mathcal{F}f||^q_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})} &= \sum_{k=-\infty}^{\infty} (2^{ks}||\phi_k * (\mathcal{F}^{-1}m\mathcal{F}f)||_r)^q \\ &\leq \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-2}^{k+2} 2^{ks}||\phi_j * (\mathcal{F}^{-1}m\mathcal{F}f) * \phi_k||_r \right)^q. \end{aligned}$$

It follows from $\phi_j * (\mathcal{F}^{-1}m\mathcal{F}f) = \phi_j * (\mathcal{F}^{-1}m) * f = (\mathcal{F}^{-1}(m\widehat{\phi}_j)) * f$ that

$$||\mathcal{F}^{-1}m\mathcal{F}f||^q_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})} \leq \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-2}^{k+2} 2^{ks}||\phi_j * (\mathcal{F}^{-1}m\widehat{\phi}_j) * f * \phi_k||_r \right)^q.$$

Lemma 2.7 and again Young's inequality in the general form as given in [1, page 13] yield

$$\begin{aligned} &||\mathcal{F}^{-1}m\mathcal{F}f||^q_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})} \\ &\leq \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-2}^{k+2} 2^{ks}||\mathcal{F}^{-1}(m\widehat{\phi}_j)||_{L^1(\mathbb{R}^N;\mathcal{L}(\mathsf{E}))}||f * \phi_k||_r \right)^q \\ &\leq \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-2}^{k+2} 2^{ks}C(n)\mu_j||f * \phi_k||_r \right)^q, \end{aligned}$$

where μ_j is defined by (2.8). Since assumption (2.6) implies that $\sup_{j\in\mathbb{Z}} \mu_j \leq ||m||_{M(\mathsf{E})}$, we have

$$\begin{aligned} ||\mathcal{F}^{-1}m\mathcal{F}f||_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})} &\leq 5C(n)||m||_{M(\mathsf{E})}\big(\sum_{k=-\infty}^{\infty} (2^{ks}||f * \phi_k||_r)^q \big)^{1/q} \\ &\leq 5C(n)||m||_{M(\mathsf{E})}||f||_{\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})}. \end{aligned}$$

We have proved Theorem 2.5. □

In the sequel we will also make use of the following type of an operator-valued bounded H^∞-calculus.

Definition 2.8. *Let $N \in \mathbb{N}$, $\phi \in (0, \pi)$, and E be a Banach space. Let $s \in \mathbb{R}$, $1 \le r, q \le \infty$ be as in condition (2.1). Let A be a sectorial operator in $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E})$, i.e., A is a closed operator in $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E})$ with a dense domain $D(A)$ and dense range $R(A)$ satisfying*

$$||(\lambda + A)^{-1}||_{\mathcal{L}(\mathsf{E})} \le C/|\lambda|, \quad \lambda \in \Sigma_\theta$$

for some $\theta \in (0, \pi)$ with $C > 0$ independent of λ, where Σ_θ is the sector $\{z \in \mathbb{C} \setminus \{0\}; |\arg z| < \theta\}$. We say that A admits an ($\mathcal{L}(\mathsf{E})$-) operator-valued bounded H^∞-calculus on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E})$ if there exists a $C_\phi > 0$ such that

$$\|h(A)\|_{\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E}))} \le C_\phi \|h\|_{L^\infty(\Sigma_\phi;\mathcal{L}(\mathsf{E}))} \tag{2.9}$$

for all $h \in H^\infty(\Sigma_\phi; \mathcal{K}_A(\mathsf{E})) := \{h : \Sigma_\phi \to \mathcal{K}_A(\mathsf{E}) : h$ bounded and holomorphic$\}$, where

$$\mathcal{K}_A(\mathsf{E}) := \{T \in \mathcal{L}(\mathsf{E}) : T(\lambda + A)^{-1} = (\lambda + A)^{-1}T, \ \lambda \in \rho(-A)\}. \tag{2.10}$$

We denote the class of all operators admitting an operator-valued bounded H^∞-calculus on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E})$ by $\mathcal{H}^\infty_{Op}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E}))$. The angle

$$\phi^\infty_{Op}(A) := \inf\{\phi \in (0, \pi) : \text{ there is a } C_\phi > 0 \text{ such that } (2.9) \text{ holds}\}$$

is called the (operator-valued) H^∞-angle of A in $\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N; \mathsf{E})$.

We would like to note the following facts about this definition.

Remark 2.9. (a) For a comprehensive introduction to an operator-valued bounded H^∞-calculus we refer to [15] and [13], for the scalar-valued case see [4] and [6]. But we also emphasize that Definition 2.8 is different from the definition in [15] and [13]. There operator-valued means on the whole space X (here $X = \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E})$), whereas in our definition operator-valued is just with respect to the E part of the space. Of course the definition in [15] and [13] is more general but therefore also stronger in our specific situation (estimate (2.9) has to be satisfied for a larger class of bounded holomorphic functions).

(b) It is clear that the definition above extends to arbitrary E-valued Banach spaces.

(c) Setting $\mathsf{E} = \mathbb{C}$, we see that $A \in \mathcal{H}^\infty_{Op}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E}))$ in particular implies a scalar bounded H^∞-calculus, i.e. $A \in \mathcal{H}^\infty(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N))$.

(d) Set $g(z) := z^s$, $s \in [0, 1]$. Then $g : \Sigma_\phi \to \Sigma_\phi$ for $\phi \in (0, \pi)$. This shows that $h \circ g \in H^\infty(\Sigma_\phi; \mathcal{K}_A(\mathsf{E}))$ for $h \in H^\infty(\Sigma_\phi; \mathcal{K}_A(\mathsf{E}))$. Assuming that $A \in \mathcal{H}^\infty_{Op}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E}))$, it is not too difficult to see that $A^s = g(A)$ is sectorial on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E})$ (see e.g. [6]) and that $h(g(A)) = h \circ g(A)$. Thus

$$\begin{aligned}\|h(g(A))\|_{\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E}))} &= \|h \circ g(A)\|_{\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E}))} \\ &\le C_\phi \|h \circ g\|_{L^\infty(\Sigma_\phi;\mathcal{L}(\mathsf{E}))} \\ &\le C_\phi \|h\|_{L^\infty(\Sigma_\phi;\mathcal{L}(\mathsf{E}))}\end{aligned}$$

for $h \in H^\infty(\Sigma_\phi; \mathcal{K}_A(\mathsf{E}))$. Consequently we see that the property of having an operator-valued bounded H^∞-calculus transfers to fractional powers, i.e., $A \in \mathcal{H}^\infty_{Op}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E}))$ implies $A^s \in \mathcal{H}^\infty_{Op}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E}))$ with $\phi^\infty_{Op}(A^s) \leq \phi^\infty_{Op}(A)$ for $s \in [0, 1]$.

Note that for

$$h \in H_0^\infty(\Sigma_\phi; \mathcal{K}_A(\mathsf{E})) := \Big\{ h \in H^\infty(\Sigma_\phi; \mathcal{K}_A(\mathsf{E})) : \\ \|h(z)\|_{\mathcal{L}(\mathsf{E})} \leq C \frac{|z|^s}{(1+|z|)^{2s}}, z \in \Sigma_\phi, \text{for some } C, s > 0 \Big\}$$

the operator $h(A)$ is defined by

$$h(A) := \frac{1}{2\pi i} \int_\Gamma h(\lambda)(\lambda - A)^{-1} \mathrm{d}\lambda,$$

where Γ is the path $\Gamma := \{r\mathrm{e}^{i\theta}; \infty > r \geq 0\} \cup \{r\mathrm{e}^{-i\theta}; 0 \leq r < \infty\}$ for $\theta \in (0, \phi)$, passing from $\infty\mathrm{e}^{i\theta}$ to $\infty\mathrm{e}^{-i\theta}$. This representation explains the restriction of the values of the functions h to the subalgebra $\mathcal{K}_A(\mathsf{E})$. Otherwise there would be a second, possibly different, way to define $h(A)$, namely by the integral

$$h(A) := \frac{1}{2\pi i} \int_\Gamma (\lambda - A)^{-1} h(\lambda) \mathrm{d}\lambda.$$

This differs from the scalar-valued case, where these two definitions always coincide. Thus, in order to obtain a compatible definition for the operator-valued case it is reasonable to use this restriction.

By the additional decay in 0 and ∞ it is obvious that $h(A)$ belongs to the class $\mathcal{L}(\dot{\mathcal{B}}^s_{p,q}(\mathbb{R}^N; \mathsf{E}))$ for $h \in H_0^\infty(\Sigma_\phi; \mathcal{K}_A(\mathsf{E}))$. To define $h(A)$ for arbitrary $h \in H^\infty(\Sigma_\phi; \mathcal{K}_A(\mathsf{E}))$ we take $z \mapsto g(z) := z/(1+z)^2 \in H_0^\infty(\Sigma_\phi; \mathcal{K}_A(\mathsf{E}))$ and set

$$h(A) := (hg)(A) g(A)^{-1}$$

initially defined on $D(A) \cap R(A)$. Since the convergence lemma (see [4]) is still true for operator-valued holomorphic functions (see [13]), as in the scalar-valued case it suffices to prove (2.9) for all $h \in H_0^\infty(\Sigma_\phi; \mathcal{K}_A(\mathsf{E}))$ in order to obtain the validity of (2.9) for all $h \in H^\infty(\Sigma_\phi; \mathcal{K}_A(\mathsf{E}))$.

Examples of operators that admit an operator-valued bounded H^∞-calculus on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E})$ are in order. The first one is the Laplacian $\Delta = \sum_{j=1}^N \partial_j^2$, $\partial_j = \partial/\partial x_j$.

Proposition 2.10. *Let $N \in \mathbb{N}$ and E be a Banach space. Let $s \in \mathbb{R}$, $1 \leq r, q \leq \infty$ satisfy condition (2.1). The Laplacian $-\Delta$ in $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E})$ with domain $D(-\Delta) = \{u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E}) : D^\alpha u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E}), \alpha \in \mathbb{N}_0^N, |\alpha| \leq 2\}$ admits an operator-valued bounded H^∞-calculus on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N; \mathsf{E})$ with H^∞-angle $\phi^\infty_{Op}(-\Delta) = 0$.*
By duality, estimate (2.9) still holds for $A = -\Delta$, if E is a dual space and s, r, q satisfy (2.2).

Remark 2.11. Observe that this result can not be obtained as a trivial consequence of results in the previous literature, as for instance the results proved in [6]. This is due to the fact that related to Mikhlin type estimates in [6] the authors, forced by their theoretical approach, always have to assume that E is a space of class $\mathcal{HT}$ (or equivalently that E is a UMD space), whereas in our situation E is an arbitrary Banach space with no further restriction. Moreover, in [6] there are no results on Besov spaces.

Proof. Note that the sectoriality of $-\Delta$ in $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})$ with spectral angle $\phi_{-\Delta}=0$ is an immediate consequence of Theorem 2.5 and Lemma 2.4 (1). Indeed, it is well-known that $\mathcal{F}\lambda(\lambda-\Delta)^{-1}\mathcal{F}^{-1}=\lambda(\lambda+|\xi|^2)^{-1}$ satisfies the scalar Mikhlin conditions also for $|\alpha|\le N+1$ (instead of $|\alpha|\le[N/2]+1$) and for all $\lambda\in\Sigma_{\pi-\varphi_0}$, and arbitrary $\varphi_0\in(0,\pi)$.

Now let $\phi\in(0,\pi)$ and $h\in H_0^\infty(\Sigma_\phi;\mathcal{K}_A(\mathsf{E}))$. Taking Fourier transform yields

$$\mathcal{F}h(-\Delta)=\frac{1}{2\pi i}\int_\Gamma h(\lambda)\mathcal{F}(\lambda-(-\Delta))^{-1}\mathrm{d}\lambda=h(|\cdot|^2),$$

and by assumption obviously

$$\sup_{\xi\in\mathbb{R}^N\setminus\{0\}}\|h(|\xi|^2)\|_{\mathcal{L}(\mathsf{E})}\le C.$$

We will prove now that $\xi\mapsto h(|\xi|^2)$ satisfies the Mikhlin condition of Theorem 2.5. To this end one can copy the proof for scalar valued h (i.e. $\mathsf{E}=\mathbb{C}$, see e.g. [20]) verbatim, simply replacing absolute value $|\cdot|$ by the operator norm $\|\cdot\|_{\mathcal{L}(\mathsf{E})}$. But for the readers convenience we give the proof here.

We have to calculate $D^\alpha h(|\cdot|^2)$ for any multi index α satisfying $|\alpha|\le N+1$. By induction we deduce for arbitrary $m\in\mathbb{N}$

$$D_j^m h(|\xi|^2)=\sum_{k=0}^{\left[\frac{m}{2}\right]}a_k h^{(m-k)}(|\xi|^2)\xi_j^{m-2k},\quad \xi\in\mathbb{R}^N\setminus\{0\},\ j=1,\dots,N,$$

with certain coefficients $a_k\in\mathbb{N}_0$ for $k\in\{0,\dots,\left[\frac{m}{2}\right]\}$, where $[r]:=\max\{\ell\in\mathbb{N}_0:\ell\le r\}$ for $r\ge0$. For an arbitrary multi index $\alpha\in\mathbb{N}_0^N$ iterative application of D^α then leads to

$$\begin{aligned}D^\alpha h(|\xi|^2)&=D_N^{\alpha_N}\cdots D_2^{\alpha_2}D_1^{\alpha_1}h(|\xi|^2)\\&=\sum_{\beta\le\left[\frac{\alpha}{2}\right]}a_\beta h^{(|\alpha|-|\beta|)}(|\xi|^2)\xi^{\alpha-2\beta},\quad \xi\in\mathbb{R}^N\setminus\{0\},\end{aligned}\tag{2.11}$$

where $\beta\le\alpha$ and $[\alpha]$ for multi indices $\alpha,\beta\in\mathbb{N}_0^N$ has to be understood componentwise. In order to estimate the derivatives of the holomorphic function h we define

$$r(t):=\frac{t}{2}\sin(\phi),\quad t\in(0,\infty).$$

Then the ball $B_{r(t)}(t)$ is contained in the sector Σ_ϕ for each $t \in (0,\infty)$. Thus, by Cauchy's formula we may conclude

$$\begin{aligned}\|h^{(k)}(t)\|_{\mathcal{L}(\mathsf{E})} &\le C\frac{k!}{r(t)^k}\max_{|z|=r(t)}\|h(z)\|_{\mathcal{L}(\mathsf{E})}\\ &\le \frac{C(k,\phi)}{t^k}\|h\|_\infty, \quad t\in(0,\infty),\ k\in\mathbb{N}_0,\end{aligned}$$

where we put $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\Sigma_\phi;\mathcal{L}(\mathsf{E}))}$ for simplicity. This fact applied to (2.11) for $D^\alpha h(|\cdot|^2)$ yields

$$\begin{aligned}|\xi|^{|\alpha|}\|D^\alpha h(|\xi|^2)\|_{\mathcal{L}(\mathsf{E})} &\le C\|h\|_\infty \sum_{\beta\le[\frac{\alpha}{2}]} a_\beta |\xi|^{|\alpha|}|\xi|^{-2(|\alpha|-|\beta|)}|\xi^{\alpha-2\beta}|\\ &\le C\|h\|_\infty \sum_{\beta\le[\frac{\alpha}{2}]} a_\beta \frac{|\xi|^{|\alpha-2\beta|}}{|\xi|^{|\alpha|-2|\beta|}}\\ &= C\|h\|_\infty \sum_{\beta\le[\frac{\alpha}{2}]} a_\beta \le C\|h\|_\infty, \quad \xi\in\mathbb{R}^N\setminus\{0\},\end{aligned} \tag{2.12}$$

since $|\alpha-2\beta| = |\alpha|-2|\beta|$ for $\beta \le \left[\frac{\alpha}{2}\right]$. Hence, the conditions of Theorem 2.5 are satisfied and in view of (2.7) and (2.12) we obtain

$$\|h(-\Delta)\|_{\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E}))} = \|\mathcal{F}^{-1}h(|\cdot|^2)\mathcal{F}\|_{\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E}))} \le C\|h\|_\infty$$

for all $h \in H_0^\infty(\Sigma_\phi;\mathcal{L}(\mathsf{E}))$ which proves the claim. □

By the preparations above we are in the situation to give an elegant proof of the boundedness of the Helmholtz projection on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))$.

Corollary 2.12. *Let $n\in\mathbb{N}$, $n\ge 2$, $1<p<\infty$. Let $s\in\mathbb{R}$, $1\le r,q\le\infty$ be as in Definition 2.1. The Helmholtz projection $\mathbf{P}_+$ is bounded on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))$.*

Proof. We use the representation

$$\mathbf{P}_+ = r(I+RR^T)E$$

as given in (1.17) and (1.18). Obviously $r\in\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R})),\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+)))$ and $E\in\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+)),\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R})))$. It remains to prove the boundedness of $R=(R_1,\ldots,R_n)$ on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R}))$. For $j=1,\ldots,n-1$ we write formally

$$R_j = \partial_j(-\Delta)^{-1/2} = R_j' h(-\Delta'),$$

where $R_j' := \partial_j(-\Delta')^{-1/2}$ is the tangential Riesz operator and

$$h:\Sigma_\phi\to\mathcal{K}_{-\Delta'}(L^p(\mathbb{R})), \quad h(z) := [z(z-\Delta_n)^{-1}]^{1/2}$$

for some $\phi \in (0,\pi)$ and $\Delta_n := \partial_n^2$. Theorem 2.5 easily yields $R'_j = \mathcal{F}^{-1}\left[\frac{i\xi_j}{|\xi'|}\cdot I\right]\mathcal{F} \in \mathcal{L}(\dot{\mathcal{B}}^0_{\infty,q}(\mathbb{R}^{n-1};L^p(\mathbb{R})))$, since $\frac{i\xi_j}{|\xi'|}$ satisfies the scalar Mikhlin conditions. Furthermore, from well-known resolvent estimates for the Laplacian $-\Delta_n$ on $L^p(\mathbb{R})$ we obtain

$$\|z(z-\Delta_n)^{-1}\|_{\mathcal{L}(L^p(\mathbb{R}))} \le C_\phi, \quad z\in\Sigma_\phi.$$

(This is easily proved by applying the standard Mikhlin's theorem.) This implies $h\in H^\infty(\Sigma_\phi,\mathcal{K}_{-\Delta'}(L^p(\mathbb{R})))$ and therefore $h(-\Delta')\in\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R})))$ by Proposition 2.10, which proves the boundedness of R_j for $j=1,\dots,n-1$.

In the case $j=n$ we directly write $R_n = g(-\Delta')$ with

$$g:\Sigma_\phi\to\mathcal{K}_{-\Delta'}(L^p(\mathbb{R})), \quad g(z)=\partial_n(z-\Delta_n)^{-1/2}.$$

Again by well-known estimates for $-\Delta_n$ we deduce $g\in H^\infty(\Sigma_\phi;\mathcal{K}_{-\Delta'}(L^p(\mathbb{R})))$ implying $R_n\in\mathcal{L}(\dot{\mathcal{B}}^0_{\infty,q}(\mathbb{R}^{n-1};L^p(\mathbb{R})))$ and the proof is complete. □

Corollary 2.12 allows us to define the solenoidal part of $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))$ as

$$\dot{\mathcal{B}}^s_{r,q,\sigma}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+)) := \mathbf{P}_+(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))). \tag{2.13}$$

Since $\mathbf{P}_+$ is a bounded projection, this is a closed subspace of $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))$. At least for the most important case in this note we will prove in the Appendix (Lemma A.2) the validity of the usual characterization

$$\begin{aligned}&\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))\\ &\quad=\{u\in\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+));\mathrm{div}u=0, u\cdot\nu|_{\partial\mathbb{R}^n_+}=0\}\end{aligned}$$

for $1<p<\infty$. The crucial step will be to give a meaning to the trace $u\cdot\nu|_{\partial\mathbb{R}^n_+}=0$.

As another consequence of Proposition 2.10 and Remark 2.9 (c) we obtain the following operator-valued bounded H^∞-calculus for the Poisson operator. It will turn out to be the key-ingredient in the proof of the resolvent estimates of the Stokes operator in Theorem 3.1.

Corollary 2.13. *Let $N\in\mathbb{N}$, E be a Banach space, and $s\in\mathbb{R}$, $1\le r,q\le\infty$ be as in (2.1). The Poisson operator $|\nabla| := (-\Delta)^{1/2}$ admits an operator-valued bounded H^∞-calculus on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^N;\mathsf{E})$ with H^∞-angle $\phi^\infty_{\mathcal{O}p}(|\nabla|)=0$. By duality, estimate (2.9) still holds for $A=|\nabla|$, if E is a dual space and s,r,q satisfy (2.2).*

3. The linear problem

In this section we consider the linear problem (Stokes + Coriolis + Ekman):

$$\begin{cases} \partial_t \mathbf{\Phi} - \nu \Delta \mathbf{\Phi} + \Omega \mathbf{e}_3 \times \mathbf{\Phi} + (\mathbf{U}^E(x_3)\cdot\nabla)\mathbf{\Phi} + \Phi_3 \frac{\partial \mathbf{U}^E}{\partial x_3} &= -\nabla \pi, \\ \nabla \cdot \mathbf{\Phi} &= 0, \\ \mathbf{\Phi}(t,x)|_{x_3=0} &= 0, \\ \mathbf{\Phi}(t,x)|_{t=0} &= \mathbf{\Phi}_0(x), \end{cases} \tag{3.1}$$

for $x \in \mathbb{R}^n_+$ and $t \in (0,\infty)$. After applying the Helmholtz projection $\mathbf{P}_+$, the above equation (3.1) can be written in operator form as follows

$$\mathbf{\Phi}_t + \mathbf{A}\mathbf{\Phi} + \Omega \mathbf{S}\mathbf{\Phi} + \mathbf{C}_E \mathbf{\Phi} = 0, \ \mathbf{\Phi}(t)|_{t=0} = \mathbf{\Phi}_0, \tag{3.2}$$

where $\mathbf{A}$ is the Stokes operator in a half-space, $\mathbf{S} = \mathbf{P}_+\mathbf{J}\mathbf{P}_+$ is the Coriolis operator in $\mathbb{R}^3_+$, and $\mathbf{C}_E$ is the Ekman operator. Most of the results below are stated in arbitrary dimension $n \geq 2$. Only if the Coriolis and the Ekman operators come into play we restrict dimension to the case $n = 3$. Since the results here are based on the results in Section 2 the proofs work simultaneously in all homogeneous Besov spaces $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ as defined in Definition 2.1. Therefore, throughout this section we assume $1 < p < \infty$ and s, r, q to be given as in condition (2.1) or condition (2.2) and set for simplicity $X := \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ and $X_\sigma := \dot{\mathcal{B}}^s_{r,q,\sigma}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+)) = \mathbf{P}_+(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+)))$. We start by stating the generation result for the Stokes operator. Without the loss of generality we may assume $\nu = 1$ for the viscosity parameter. By the equality $(\lambda + \nu\mathbf{A})^{-1} = \frac{1}{\nu}(\frac{\lambda}{\nu} + \mathbf{A})^{-1}$ all the proved results for $\mathbf{A}$ easily follow also for the operator $\nu\mathbf{A}$ and any fixed $\nu > 0$. Hence the Stokes operator is given as

$$\begin{aligned} \mathbf{A} &= \mathbf{A}_{\mathbb{R}^n_+} = -\mathbf{P}_+\Delta, \\ D(\mathbf{A}) &= D(\Delta_D) \cap X_\sigma \\ &= \{u \in X : D^\alpha u \in X, \alpha \in \mathbb{N}^n_0, |\alpha| \leq 2, u|_{\partial\mathbb{R}^n_+} = 0\} \cap X_\sigma, \end{aligned}$$

where Δ_D denotes the Dirichlet Laplacian in X and $\alpha \in \mathbb{N}^n_0$ is a multi index. By a standard perturbation argument we will show afterwards that also

$$\begin{aligned} \mathbf{A}_E &:= \mathbf{A} + \Omega\mathbf{P}_+\mathbf{J}\mathbf{P}_+ + \mathbf{C}_E \\ D(\mathbf{A}_E) &= D(\mathbf{A}) \end{aligned}$$

is the generator of a holomorphic semigroup on X_σ.

Theorem 3.1. *The Stokes operator* $\mathbf{A}$ *is the generator of a bounded holomorphic semigroup on* X_σ*, which is strongly continuous if condition (2.1) is satisfied. For each* $\varphi_0 \in (0,\pi)$ *there is a* C_{φ_0} *such that we have the resolvent estimates*

$$\sum_{k=0}^{2} |\lambda|^{k/2} \|\nabla^{2-k}(\lambda + \mathbf{A})^{-1}\|_{\mathcal{L}(X)} \leq C_{\varphi_0}, \quad \lambda \in \Sigma_{\pi-\varphi_0}. \tag{3.3}$$

The proof of this result requires some preparations. First let us recall a suitable representation for the solution of the Stokes resolvent problem

$$(SRP)_{u_0,\lambda} \begin{cases} (\lambda-\Delta)u+\nabla p &= u_0 \text{ in } \mathbb{R}^n_+, \\ \nabla\cdot u &= 0 \text{ in } \mathbb{R}^n_+, \\ u &= 0 \text{ in } \mathbb{R}^{n-1}. \end{cases}$$

In [7] (see also [21]) it was shown that $u=(\lambda+\mathbf{A})^{-1}u_0$ can be represented as

$$u' = (\lambda-\Delta_D)^{-1}u_0' - R'v,$$
$$u^n = (\lambda-\Delta_D)^{-1}u_0^n + v,$$

where $R'=(R_1',\dots,R_{n-1}')$ and the Fourier transform of the remainder v is given by

$$\hat{v}(\xi',x_n) = \frac{\mathrm{e}^{-\omega(|\xi'|)x_n} - \mathrm{e}^{-|\xi'|x_n}}{\omega(|\xi'|)-|\xi'|}\int_0^\infty \mathrm{e}^{-\omega(|\xi'|)s}\hat{u}_0^n(\xi',s)\mathrm{d}s, \quad (\xi,x_n)\in\mathbb{R}^n_+,$$

where $\omega(|\xi'|)=\sqrt{\lambda+|\xi'|^2}$. Furthermore, the Fourier transform of the related pressure p is given as

$$\hat{p}(\xi',x_n) = \frac{i\xi'}{|\xi'|}\cdot\frac{\omega(|\xi'|)+|\xi'|}{\omega(|\xi'|)}\mathrm{e}^{-|\xi'|x_n}\int_0^\infty \mathrm{e}^{-\omega(|\xi'|)s}\hat{u}_0'(\xi',x_n)\mathrm{d}s, \quad (\xi,x_n)\in\mathbb{R}^n_+. \tag{3.4}$$

In order to estimate these formulae we follow the arguments in [21], i.e. we will prove

$$\|\nabla p\|_X \le C\|f\|_X. \tag{3.5}$$

Then, by plugging over ∇p to the right hand side of $(SRP)_{f,\lambda}$ it can be regarded as a resolvent problem for the Dirichlet-Laplacian with data $u_0-\nabla p$. The estimates for the solution of this problem, which are proved first, in combination with (3.5) then yields the assertion. The essential ingredient for estimating the formulae for u and p in [21] is the bounded H^∞-calculus for the tangential Poisson operator $|\nabla'|:=(-\Delta')^{1/2}=\mathcal{F}^{-1}[|\xi'|]\mathcal{F}$ on $L^q(\mathbb{R}^{n-1})$. The corresponding ingredient in the situation considered here will be the stronger property of an operator-valued bounded H^∞-calculus for $|\nabla'|$ on $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))$ as provided in Corollary 2.13. This is due to the fact that here we have to deal with E-valued spaces in contrast to [21].

As a further application of Proposition 2.10 we start with proving the desired resolvent estimates for the Dirichlet Laplacian Δ_D.

Proposition 3.2. *Let $\varphi_0\in(0,\pi)$. There is a $C_{\varphi_0}>0$ such that the Dirichlet Laplacian Δ_D with domain $D(\Delta_D)=\{u\in X : D^\alpha u\in X, \alpha\in\mathbb{N}_0^n, |\alpha|\le 2, u|_{\partial\mathbb{R}^n_+}=0\}$ admits the resolvent estimates*

$$\sum_{k=0}^2 |\lambda|^{k/2}\|\nabla^{2-k}(\lambda-\Delta_D)^{-1}\|_{\mathcal{L}(X)} \le C_{\varphi_0}, \quad \lambda\in\Sigma_{\pi-\varphi_0}.$$

If $X = \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ *is such that condition (2.1) is satisfied, then* Δ_D *is densely defined.*

Proof. Observe that the resolvent of Δ_D can be represented as

$$(\lambda - \Delta_D)^{-1} f = r(\lambda - \Delta)^{-1} e^- f \tag{3.6}$$

with r the restriction on $\mathbb{R}^n_+$, e^- as given in Definition 1.1, and $\Delta = \Delta_{\mathbb{R}^n}$ the Laplacian on $\mathbb{R}^n$. Therefore it is sufficient to prove the corresponding statements for Δ on the space $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}))$.

We have to estimate $D^\alpha(\lambda - \Delta)^{-1}$ for $\alpha \in \mathbb{N}^n$, $|\alpha| \leq 2$. For this purpose we write D^α as $D^\alpha = D^\beta \partial_n^k$ with $|\beta| + k = |\alpha| \leq 2$, where D^β contains tangential derivatives only. Next observe that the resolvent of Δ can be written as

$$(\lambda - \Delta)^{-1} = (\lambda + (-\Delta') - \Delta_n)^{-1},$$

where $-\Delta'$ denotes the tangential Laplacian, regarded as an operator in the $L^p(\mathbb{R})$-valued space $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R}))$, and Δ_n the Laplacian in the normal component, i.e. in $L^p(\mathbb{R})$. Hence we can rephrase $D^\alpha(\lambda - \Delta)^{-1}$ formally as

$$\begin{aligned} D^\alpha(\lambda - \Delta)^{-1} &= D^\beta \partial_n^k (\lambda + (-\Delta') - \Delta_n)^{-1} \\ &= D^\beta (-\Delta')^{-|\beta|/2} h_{\lambda,\beta,k}(-\Delta'), \end{aligned}$$

with

$$h_{\lambda,\beta,k}(-\Delta') := \partial_n^k (-\Delta')^{|\beta|/2} (\lambda + (-\Delta') - \Delta_n)^{-1}.$$

In the proof of Corollary 2.12 we already showed $R'_j \in \mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})))$ for the tangential Riesz operators $R'_j = \partial_j(-\Delta')^{-1/2}$, $j = 1, \ldots, n-1$. Hence we may estimate

$$\|D^\alpha(\lambda - \Delta)^{-1}\|_{\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})))} \leq C \|h_{\lambda,\beta,k}(-\Delta')\|_{\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})))}.$$

Now let $\phi \in (0, \varphi_0/2)$. Obviously

$$h_{\lambda,\beta,k} : \Sigma_\phi \to \mathcal{L}(L^p(\mathbb{R})), \quad \mu \mapsto h_{\lambda,\beta,k}(\mu) = \partial_n^k (\mu)^{|\beta|/2} (\lambda + \mu - \Delta_n)^{-1},$$

is holomorphic and we have

$$\begin{aligned} \sup_{\mu \in \Sigma_\phi} \|h_{\lambda,\beta,k}(\mu)\|_{\mathcal{L}(L^p(\mathbb{R}))} &\leq \sup_{\mu \in \Sigma_\phi} \frac{C(\varphi_0)|\mu|^{|\beta|/2}}{|\lambda + \mu|^{(2-k)/2}} \\ &\leq \frac{C(\varphi_0, \phi)}{|\lambda|^{(2-|\alpha|)/2}}, \quad \lambda \in \Sigma_{\pi - \varphi_0}, \ |\beta| + k = |\alpha| \leq 2, \end{aligned}$$

by well known results for Δ_n on $L^p(\mathbb{R})$, and since $\mathrm{Re}\sqrt{\lambda + \mu} \geq c_{\varphi_0}(\sqrt{|\lambda|} + \sqrt{|\mu|})$ for $\lambda \in \Sigma_{\pi - \varphi_0}$, $\mu \in \Sigma_\phi$, and a certain $c_{\varphi_0} > 0$, by our choice $\phi \in (0, \varphi_0/2)$. Let us remark, that this choice of ϕ is admissible, since we have $\phi^\infty_{Op}(-\Delta') = 0$ according to Proposition 2.10. Thus, we conclude

$$\|D^\alpha(\lambda - \Delta)^{-1}\|_{\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1}; L^p(\mathbb{R})))}$$

$$\le C\|h_{\lambda,\beta,k}(-\Delta')\|_{\mathcal{L}(\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R})))}$$
$$\le \frac{C_{\varphi_0}}{|\lambda|^{(2-|\alpha|)/2}}, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ |\alpha| \le 2,$$

which proves the first assertion.

Thanks to the item (3) of Lemma 2.4,

$$D(\Delta) = \{u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R})) : D^\alpha u \in \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R})), \alpha \in \mathbb{N}^3, |\alpha| \le 2\}$$

lies dense in $\dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R}))$, if condition (2.1) is satisfied. This implies

$$\lambda(\lambda-\Delta)^{-1}f \to f \quad \text{in } \dot{\mathcal{B}}^s_{r,q}(\mathbb{R}^{n-1};L^p(\mathbb{R})) \text{ if } \lambda \to \infty.$$

Thus, by (3.6) it follows that also

$$\lambda(\lambda-\Delta_D)^{-1}f \to re^-f = f \quad \text{in } X \text{ if } \lambda \to \infty,$$

which proves $D(\Delta_D)$ to be dense in X. □

With the above preparations in hand we can turn to the proof of the generation result for the Stokes operator.

Proof of Theorem 3.1

Regarding $(SRP)_{u_0,\lambda}$ as the problem

$$\begin{cases} (\lambda-\Delta)u = u_0 - \nabla p & \text{in } \mathbb{R}^n_+, \\ u = 0 & \text{on } \mathbb{R}^{n-1}, \end{cases}$$

Proposition 3.2 yields formally

$$\sum_{k=0}^{2} |\lambda|^{k/2}\|\nabla^{2-k}u\|_X \le C_{\varphi_0}\|u_0-\nabla p\|_X, \quad \lambda \in \Sigma_{\pi-\varphi_0}.$$

So, if we can show

$$\|\nabla p\|_X \le C_{\varphi_0}\|u_0\|_X, \quad \lambda \in \Sigma_{\pi-\varphi_0},$$

the resolvent estimates for $\mathbf{A}$ follow. But this is an immediate consequence of the next lemma for $\delta = 0$. To complete the proof it remains to show that $\mathbf{A}$ is densely defined in case that condition (2.1) is satisfied. To this end we write

$$\lambda(\lambda+\mathbf{A})^{-1}u_0 = \lambda(\lambda-\Delta_D)^{-1}(u_0 - S(\lambda)u_0), \tag{3.7}$$

where

$$S(\lambda)u_0 := \nabla p, \quad u_0 \in X, \tag{3.8}$$

and p is given by formula (3.4). So, if we can prove

$$S(\lambda)u_0 \to 0 \quad \text{in } X_\sigma \text{ if } \lambda \to \infty \tag{3.9}$$

for $u_0 \in X_\sigma$ we deduce in view of Proposition 3.2,

$$\lambda(\lambda + \mathbf{A})^{-1} f \to f \quad \text{in } X_\sigma \text{ if } \lambda \to \infty,$$

which yields the assertion. But (3.9) follows from the next lemma as well. □

For later purposes we state the estimate for the pressure term, i.e. for

$$S(\lambda)u_0 = \nabla p$$

in a more general form.

Lemma 3.3. *Let $\varphi_0 \in (0, \pi)$ and $\delta \in [0, 1/p']$, where $1 = \frac{1}{p} + \frac{1}{p'}$. Then there is a constant $C = C(\delta, \varphi_0)$ such that*

$$\| |\nabla'|^{-\delta} S(\lambda) \|_{\mathcal{L}(X)} \le \frac{C}{|\lambda|^{\delta/2}}, \quad \lambda \in \Sigma_{\pi - \varphi_0}. \tag{3.10}$$

Furthermore, if r, q, s, p fulfill condition (2.1), then

$$S(\lambda) f \to 0 \quad \text{in } X \text{ if } \lambda \to \infty$$

for $f \in X_\sigma$.

Proof. Fix $\varphi_0 \in (0, \pi)$ and $\delta \in [0, 1/p']$. Let $\phi \in (0, \varphi_0/4)$ and define for $f \in L^p(\mathbb{R}_+)$,

$$\begin{aligned} (h_\lambda(z) f)(x_n) := &\left(1 + \frac{z}{\omega(z)}\right) z^{1-\delta} \mathrm{e}^{-z x_n} \\ &\times \int_0^\infty \mathrm{e}^{-\omega(z)s} f(s) \mathrm{d}s, \quad z \in \Sigma_\phi,\ x_n > 0. \end{aligned}$$

Then, by representation (3.4) we see that $|\nabla'|^{-\delta}(S(\lambda)u_0)^n$ can be written as

$$|\nabla'|^{-\delta}(S(\lambda)u_0)^n = -R' \cdot h_\lambda(|\nabla'|) u_0', \quad u_0 \in X.$$

We already know that $R' \in \mathcal{L}(X)$. Therefore, in view of Corollary 2.13, it remains to show that $h_\lambda \in H^\infty(\Sigma_\phi; \mathcal{L}(L^p(\mathbb{R}_+)))$ with the upper bound given in (3.10). But for $f \in L^p(\mathbb{R}_+)$ we have

$$\begin{aligned} \|h_\lambda(z) f\|_{L^p(\mathbb{R}_+)} &\le \left| \left(1 + \frac{z}{\omega(z)}\right) z^{1-\delta} \right| \|\mathrm{e}^{-z x_n}\|_{L^p(\mathbb{R}_+)} \int_0^\infty |\mathrm{e}^{-\omega(z)s} f(s)| \mathrm{d}s \\ &\le C \left| \left(1 + \frac{z}{\omega(z)}\right) z^{1-\delta} \right| |z|^{-1/p} \| \mathrm{e}^{-\omega(z)s} \|_{L^{p'}(\mathbb{R}_+)} \|f\|_{L^p(\mathbb{R}_+)} \\ &\le C \left(1 + \left| \frac{z}{\omega(z)} \right| \right) \left| \frac{z}{\omega(z)} \right|^{\frac{1}{p'} - \delta} \frac{1}{|\omega(z)|^\delta} \|f\|_{L^p(\mathbb{R}_+)}. \end{aligned}$$

Our choice $\phi \in (0, \varphi_0/4)$ (which is possible in view of $\phi^{\infty}_{|\nabla'|} = 0$) implies the existence of a $c_1 = c_1(\varphi_0) > 0$ such that $\mathrm{Re}\omega(z) \geq c_1(\sqrt{|\lambda|} + |z|)$ for $\lambda \in \Sigma_{\pi-\varphi_0}$, $z \in \Sigma_\phi$. Then, it easily follows

$$\left|\frac{z}{\omega(z)}\right| \leq C_{\varphi_0}, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ z \in \Sigma_\phi,$$

and

$$\frac{1}{|\omega(z)|^\delta} \leq \frac{C_{\varphi_0}}{|\lambda|^{\delta/2}}, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ z \in \Sigma_\phi.$$

Hence, since $\delta \in [0, 1/p']$, i.e. $\frac{1}{p'} - \delta > 0$,

$$\|h_\lambda(z)\|_{\mathcal{L}(L^p(\mathbb{R}_+))} \leq \frac{C_{\varphi_0}}{|\lambda|^{\delta/2}}, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ z \in \Sigma_\phi.$$

Employing Corollary 2.13 we finally may conclude

$$\begin{aligned}
\||\nabla'|^{-\delta}(S(\lambda)u_0)^n\|_X &= \|R' \cdot h_\lambda(|\nabla'|)u_0'\|_X \leq C\sum_{j=1}^{n-1}\|h_\lambda(|\nabla'|)u_0^j\|_X \\
&\leq (n-1)C\|h_\lambda\|_{L^\infty(\Sigma_\phi;\mathcal{L}(L^p(\mathbb{R}_+)))}\|u_0\|_X \\
&\leq C_{\varphi_0}|\lambda|^{-\delta/2}\|u_0\|_X, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ u_0 \in X_\sigma.
\end{aligned}$$

By the equality

$$i\xi'\hat{p} = \frac{i\xi'}{|\xi'|}|\xi|\hat{p} = -\frac{i\xi'}{|\xi'|}\partial_n\hat{p},$$

we have

$$|\nabla'|^{-\delta}(S(\lambda)u_0)' = -R'|\nabla'|^{-\delta}(S(\lambda)u_0)^n.$$

Again in view of $R' \in \mathcal{L}(X)$, we see that the corresponding estimate for $|\nabla'|^{-\delta}(S(\lambda)u_0)'$ is reduced to the just proved estimate for $|\nabla'|^{-\delta}(S(\lambda)u_0)^n$.

In order to see the second assertion note that for $\delta = 0$ the function h_λ can also be written in the form

$$(h_\lambda(z)f)(x_n) :=$$
$$\left(1 + \frac{z}{\omega(z)}\right)\omega(z)^{1/p'}z^{1-\frac{1}{p'}}\mathrm{e}^{-zx_n}\int_0^\infty \mathrm{e}^{-\omega(z)s}\left(\frac{z}{\omega(z)}\right)^{1/p'} f(s)\mathrm{d}s$$

for $z \in \Sigma_\phi$, $x_n > 0$. Consequently, by following the lines of the proof above we derive the estimate

$$\|S(\lambda)f\|_X \leq C\|(-\Delta')^{1/2p'}(\lambda - \Delta')^{-1/2p'}f\|_X.$$

The operator on the right hand side can be written as

$$(-\Delta')^\alpha(\lambda - \Delta')^{-\alpha} = (\lambda^\alpha + (-\Delta')^\alpha)(\lambda - \Delta')^{-\alpha}(-\Delta')^\alpha(\lambda^\alpha + (-\Delta')^\alpha)^{-1}$$

with $\alpha = 1/2p'$. Now, in view of Proposition 2.10, $(\lambda^\alpha + (-\Delta')^\alpha)(\lambda - \Delta')^{-\alpha}$ is bounded on X_σ even with an upper bound independent of λ. Moreover, since the sectoriality of $-\Delta'$ in X_σ implies also $(-\Delta')^\alpha$ to be sectorial in X_σ (with $\phi_{(-\Delta')^\alpha} = 0$), we have

$$(-\Delta')^\alpha(\sigma + (-\Delta')^\alpha)^{-1} f \to 0 \quad \text{if} \quad \sigma \to \infty$$

for $f \in X_\sigma$. Consequently

$$\|S(\lambda)f\|_{X_\sigma} \le C\|(-\Delta')^{1/2p'}(\lambda^{1/2p'} + (-\Delta')^{1/2p'})^{-1} f\|_{X_\sigma} \to 0 \quad \text{if} \quad \lambda \to \infty$$

for $f \in X_\sigma$. □

The boundedness of the operator $\mathbf{P}_+$ on X and of the Ekman spiral solution $\mathbf{U}^E$ now allows us to employ a standard perturbation argument for proving the generation result for the full linear operator $\mathbf{A}_E$. Here we give the detailed calculation, since we are also interested in the dependence on Ω and $\mathbf{U}^E$ of the shift of the growth bound of the semigroup $\mathrm{e}^{-t\mathbf{A}_E}$ and also since we would like to refer to some of the appearing formulas in the following sections.

Theorem 3.4. *Let $\varphi_0 \in (0, \pi/2]$. There are constants $K_1 = K_1(\varphi_0) > 0$, $K_2 = K_2(\varphi_0) \ge 1$ such that for $\omega_0 = \omega_0(\varphi_0) := 2K_2 \max\{1, [K_1(\Omega + \|\mathbf{U}^E\|_{1,\infty})]^2\}$ we have*

$$\Sigma_{\pi-\varphi_0} \subseteq \rho(-(\mathbf{A}_E + \omega_0))$$

and

$$\sum_{k=0}^{2} |\lambda|^{k/2} \|\nabla^{2-k}(\lambda + \mathbf{A}_E + \omega_0)^{-1}\|_{\mathcal{L}(X)} \le C_{\varphi_0}, \quad \lambda \in \Sigma_{\pi-\varphi_0},$$

for some $C_{\varphi_0} > 0$. Hence, $\mathbf{A}_E$ is the generator of a holomorphic semigroup on X_σ with growth bound $\omega_{\mathbf{A}_E} \le \omega_0(\pi/2)$. If s, r, q satisfy condition (2.1), this semigroup is strongly continuous.

Proof. Set $\mathbf{B} := \Omega\mathbf{P}_+\mathbf{J}\mathbf{P}_+ + \mathbf{C}_E$. For $\omega_0 > 0$ the resolvent of $\mathbf{A}_E + \omega_0 = \mathbf{A} + \mathbf{B} + \omega_0$ can be written as

$$(\lambda + (\omega_0 + \mathbf{A} + \mathbf{B}))^{-1} = (\lambda + \omega_0 + \mathbf{A})^{-1}[I + \mathbf{B}(\lambda + \omega_0 + \mathbf{A})^{-1}]^{-1}. \quad (3.11)$$

Next we estimate $\|\mathbf{B}(\lambda + \omega_0 + \mathbf{A})^{-1}\|_{\mathcal{L}(X)}$. Since $\mathbf{U}^E$ depends only on x_n we obtain

$$\begin{aligned}\|\mathbf{C}_E(\lambda + \omega_0 + \mathbf{A})^{-1}\|_{\mathcal{L}(X)} &\le C\Big(\|\mathbf{U}^E\|_\infty \|\nabla(\lambda + \omega_0 + \mathbf{A})^{-1}\|_{\mathcal{L}(X)} \\ &\qquad + \|\partial_n \mathbf{U}^E\|_\infty \|(\lambda + \omega_0 + \mathbf{A})^{-1}\|_{\mathcal{L}(X)}\Big) \\ &\le \frac{C_{\varphi_0}}{\sqrt{|\lambda + \omega_0|}} \|\mathbf{U}^E\|_{1,\infty}, \quad |\lambda + \omega_0| \ge 1,\end{aligned}$$

where we applied (3.3). This implies by the boundedness of $\mathbf{P}_+$ on X

$$
\begin{aligned}
&\|\mathbf{B}(\lambda+\omega_0+\mathbf{A})^{-1}\|_{\mathcal{L}(X)} \\
&\le C_{\varphi_0}\left(\Omega\|(\lambda+\omega_0+\mathbf{A})^{-1}\|_{\mathcal{L}(X)}+\frac{1}{\sqrt{|\lambda+\omega_0|}}\|\mathbf{U}^E\|_{1,\infty}\right) \\
&\le \frac{K_1}{\sqrt{|\lambda+\omega_0|}}(\Omega+\|\mathbf{U}^E\|_{1,\infty}), \quad |\lambda+\omega_0|\ge 1,
\end{aligned}
\tag{3.12}
$$

where $K_1 = K_1(\varphi_0)$ depends on upper bounds for $\|\mathbf{P}_+\|_{\mathcal{L}(X)}$ and $\|\lambda(\lambda+\mathbf{A})^{-1}\|_{\mathcal{L}(X)}$ only. Note that there is a constant $K_2 = K_2(\varphi_0) \ge 1$ such that $|\lambda+\omega_0| \ge K_2^{-1}\omega_0$ for all $\lambda \in \Sigma_{\pi-\varphi_0}$ and $\omega_0 > 0$. Now we set $\omega_0 := 2K_2 \max\{1, [K_1(\Omega+\|\mathbf{U}^E\|_{1,\infty})]^2\}$. Then we may employ the Neumann series obtaining

$$
\begin{aligned}
&\|\nabla^k(\lambda+(\omega_0+\mathbf{A}+\mathbf{B}))^{-1}\|_{\mathcal{L}(X)} \\
&\le \|\nabla^k(\lambda+\omega_0+\mathbf{A})^{-1}\|_{\mathcal{L}(X)}\|[I+\mathbf{B}(\lambda+\omega_0+\mathbf{A})^{-1}]^{-1}\|_{\mathcal{L}(X)} \\
&\le \frac{C}{|\lambda+\omega_0|^{(2-k)/2}}\sum_{j=0}^{\infty}\left[\left(\frac{\omega_0}{2K_2|\lambda+\omega_0|}\right)^{1/2}\right]^j \\
&\le \frac{C}{|\lambda+\omega_0|^{(2-k)/2}}\frac{1}{1-(1/2)^{1/2}} \\
&\le \frac{C}{|\lambda+\omega_0|^{(2-k)/2}}, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ k \in \{0,1,2\},
\end{aligned}
$$

where we applied again estimate (3.3) for the Stokes operator $\mathbf{A}$.

The assertion about the strong continuity is obvious, since $D(\mathbf{A}_E) = D(\mathbf{A})$. □

4. Estimates for the nonlinear term

In this section we estimate the nonlinear term by utilizing the linear estimates obtained in the last section. The goal of this section is Proposition 4.5. To prove this result we will employ the Neumann series representation obtained in the proof of Theorem 3.4. The difficulty is that the normal derivative terms combined with Riesz operators as $\mathbf{P}_+\partial_n f$ cannot be estimated in the same way as the corresponding terms involving tangential derivatives. As mentioned in the Introduction, we overcome this difficulty by using a certain splitting of $\mathbf{P}_+\partial_n f$ as it will be introduced in (4.5). We start with some basic estimates for the semigroup associated to the Laplacian. By $\mathcal{BUC}^1(\mathbb{R}^N;\mathsf{E})$ for an arbitrary Banach space E and $N \in \mathbb{N}$ we denote the space of all E-valued $\mathcal{BUC}$ functions whose first derivatives belong to $\mathcal{BUC}$.

Lemma 4.1. *Assume $n \geq 2$, $1 < q \leq p < \infty$, $\delta \in (0, 1/2]$. Then the following four estimates hold for $f \in \mathcal{BUC}^1(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)) \cap \mathcal{BUC}(\mathbb{R}^{n-1}; W^{1,q}(\mathbb{R}_+))$ such that $f|_{\partial\mathbb{R}^n_+} = 0$.*

(1) $$||e^{t\Delta_D}\partial_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \leq C(\delta) t^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \left(||f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} + ||\nabla' f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} \right)$$
for $j = 1, ..., n-1$ and any $t > 0$.
(2) $||e^{t\Delta_D}\partial_n f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \leq C t^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} ||f||_{L^\infty(\mathbb{R}^{n-1};W^{1,q}(\mathbb{R}_+))}$
for any $t > 0$.
(3) $||\partial_j e^{t\Delta_D} f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \leq C t^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} ||f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}$
for $j = 1, ..., n-1$ and any $t > 0$.
(4) $||\partial_n e^{t\Delta_D} f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \leq C t^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} ||f||_{L^\infty(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}$
for any $t > 0$.

The constants $C(\delta), C > 0$ do not depend on f.

Remark 4.2. (a) Note that the trace $f|_{\partial\mathbb{R}^n_+}$ always makes sense, due to the fact that $f(x', \cdot) \in W^{1,q}(\mathbb{R}_+) \hookrightarrow \mathrm{BC}(\overline{\mathbb{R}_+})$ and the trace operator acts on the normal component only. Note also that here and in the sequel we need the assumption $f|_{\partial\mathbb{R}^n_+} = 0$ in order to ensure the validity of

$$E\partial_n f = \partial_n \widetilde{E} f. \tag{4.1}$$

Here $E = \mathrm{diag}[e^+, e^+, e^-]$ and $\widetilde{E} = \mathrm{diag}[e^-, e^-, e^+]$, see Definition 1.1.
(b) In the normal derivative case we cannot expect regularizing effect since the normal derivative ∂_n acts on the third component (L^p-part). Hence we cannot replace the estimates (2) and (4) by $\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; W^{1,q}(\mathbb{R}_+)) \to \dot{B}^0_{\infty,1}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ and $\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)) \to \dot{B}^0_{\infty,1}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$, respectively.
(c) Since ∂_j and $e^{t\Delta_D}$ commute for $j = 1, \ldots, n-1$, the estimates (1) and (3) remain true for $\partial_j e^{t\Delta_D} f$ and $e^{t\Delta_D}\partial_j f$, respectively.
(d) The properties of the Dirichlet Laplacian Δ_D we use in the proof of Lemma 4.1 are known also for the Neumann Laplacian Δ_N. Hence all assertions of Lemma 4.1 are valid for Δ_N as well.
(e) Combining (c) and the fact that $\partial_n e^{t\Delta_D} f = e^{t\Delta_N}\partial_n f$ and $e^{t\Delta_D}\partial_n f = \partial_n e^{t\Delta_N} f$, the estimates (2) and (4) are still true for $\partial_n e^{t\Delta_D} f$ and $e^{t\Delta_D}\partial_n f$, respectively.

Proof. (1) Since ∂_j for $1 \leq j \leq n-1$, $-\Delta'$, and $e^{t\Delta'}$ act only on the tangential direction we have

$$||e^{t\Delta_D}\partial_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} = \sum_{k=-\infty}^{\infty} || \, |\phi_k * e^{t\Delta_D}\partial_j f|_{L^p(\mathbb{R}_+)} ||_{L^\infty(\mathbb{R}^{n-1})}$$

$$= \sum_{k=-\infty}^{\infty} || \, |(-\Delta')^{\delta} e^{t\Delta'} \phi_k * e^{t\Delta_{D,n}} \partial_j (-\Delta')^{-\delta} f|_p ||_\infty,$$

where we used the splitting $\Delta_D = \Delta' + \Delta_{D,n}$ and $\Delta_{D,n}$ denotes the Dirichlet Laplacian in the normal component. Note that $\phi_k = \phi_k(x_1, x_2)$ for all $k \in \mathbb{Z}$, i.e. the convolution is with respect to the first two components. Multiplying $1 = \sum_{l \in \mathbb{Z}} \phi_l *$, it follows from $e^{t\Delta_{D,n}}(\phi_l * f) = \phi_l * e^{t\Delta_{D,n}} f$ that

$$\begin{aligned}
&||e^{t\Delta_D} \partial_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))} \\
&= \sum_{k,l \in \mathbb{Z}, |k-l| \le 2} || \, |e^{t\Delta'} \phi_k * \phi_l * e^{t\Delta_{D,n}} \partial_j f|_p ||_\infty \\
&= \sum_{k,l \in \mathbb{Z}, |k-l| \le 2} || \, |(-\Delta')^{\delta} e^{t\Delta'} \phi_k * e^{t\Delta_{D,n}} (\phi_l * \partial_j (-\Delta')^{-\delta} f)|_p ||_\infty.
\end{aligned}$$

Then vector-valued Young's inequality yields (see page 13 in [1])

$$\begin{aligned}
&||e^{t\Delta_D} \partial_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))} \\
&\le \sum_{k,l \in \mathbb{Z}, |k-l| \le 2} \big(||(-\Delta')^{\delta} e^{t\Delta'} \phi_k||_{L^1(\mathbb{R}^{n-1})} \\
&\qquad \times ||e^{t\Delta_{D,n}} (\phi_l * \partial_j (-\Delta')^{-\delta} f)||_{L^\infty(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))} \big).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&||e^{t\Delta_D} \partial_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))} \\
&\le \sup_{l \in \mathbb{Z}} ||e^{t\Delta_{D,n}} (\phi_l * \partial_j (-\Delta')^{-\delta} f)||_{L^\infty(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))} \\
&\qquad \times \sum_{|k-l| \le 2} ||(-\Delta')^{\delta} e^{t\Delta'} \phi_k||_{L^1(\mathbb{R}^{n-1})} \\
&= 5 \sup_{l \in \mathbb{Z}} ||e^{t\Delta_{D,n}} (\phi_l * \partial_j (-\Delta')^{-\delta} f)||_{L^\infty(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))} \\
&\qquad \times \sum_{k \in \mathbb{Z}} ||(-\Delta')^{\delta} e^{t\Delta'} \phi_k||_{L^1(\mathbb{R}^{n-1})}.
\end{aligned}$$

The $L^p - L^q$-estimate of the operator $e^{t\Delta_{D,n}}$ yields

$$\begin{aligned}
&||e^{t\Delta_{D,n}} (\phi_l * g)||_{L^\infty(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))} = || \, |e^{t\Delta_{D,n}} (\phi_l * g)|_p ||_\infty \\
&\le C t^{-\frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} || \, |\phi_l * g|_q ||_\infty \\
&= C t^{-\frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} ||\phi_l * g||_{L^\infty(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+))}. \qquad (4.2)
\end{aligned}$$

On the other hand, it follows for $\delta > 0$ from Lemma A.1 (1) that

$$\begin{aligned}
\sum_{k \in \mathbb{Z}} ||(-\Delta')^{\delta} e^{t\Delta'} \phi_k||_{L^1(\mathbb{R}^{n-1}; \mathbb{R})} &= \sum_{k \in \mathbb{Z}} ||(-\Delta')^{\delta} G_t(x') * \phi_k||_{L^1(\mathbb{R}^{n-1}; \mathbb{R})} \\
&= ||(-\Delta')^{\delta} G_t(x')||_{\dot{B}^0_{1,1}(\mathbb{R}^{n-1})} \le C t^{-\delta}.
\end{aligned}$$

Thus we conclude by applying (4.2) for $g = \partial_j(-\Delta')^{-\delta} f$ that

$$\begin{aligned} &||e^{t\Delta_D}\partial_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \\ &\leq Ct^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \sup_{l\in\mathbb{Z}} ||\phi_l * \partial_j(-\Delta')^{-\delta} f||_{L^\infty(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} \qquad (4.3) \\ &= Ct^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} ||\partial_j(-\Delta')^{-1/2}(-\Delta')^{\frac{1}{2}-\delta} f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} \\ &\leq Ct^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} ||(-\Delta')^{\frac{1}{2}-\delta} f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}, \end{aligned}$$

where we used $R'_j \in \mathcal{L}(\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+)))$ for the tangential Riesz operator $R'_j = \partial_j(-\Delta')^{-1/2}$ in the last inequality. By Proposition 2.10 the operator $(-\Delta')^{\frac{1}{2}-\delta}(1-\Delta')^{-1/2}$ is bounded on $\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))$ for $\delta \in [0,\frac{1}{2}]$. Moreover, by general results for fractional powers of sectorial operators we know that the norms $\|(1-\Delta')^{1/2}\cdot\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}$ and $\|\cdot\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} + \|(-\Delta')^{1/2}\cdot\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}$ are equivalent. This implies

$$\begin{aligned} &||(-\Delta')^{\frac{1}{2}-\delta} f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} \\ &= ||(-\Delta')^{\frac{1}{2}-\delta}(1-\Delta')^{-1/2}(1-\Delta')^{1/2} f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} \\ &\leq C\left(\|f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} + \|(-\Delta')^{1/2} f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}\right). \end{aligned}$$

Combining this with (4.3) it remains to show

$$\|(-\Delta')^{1/2} f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))} \leq C\|\nabla' f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}.$$

But this estimate follows easily from the representation $(-\Delta')^{1/2} = \sum_{j=1}^{n-1} R'_j\partial_j$ by applying once again $R'_j \in \mathcal{L}(\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+)))$.

For (2), since $e^{t\Delta_D} = e^{t\Delta'}e^{t\Delta_{D,n}}$ and $e^{t\Delta_{D,n}}\partial_n = \partial_n e^{t\Delta_{N,n}}$, with $\Delta_{N,n}$ the Neumann Laplacian in the normal component, we see

$$\begin{aligned} ||e^{t\Delta_D}\partial_n f||_{L^\infty(\mathbb{R}^2;L^p(\mathbb{R}_+))} &= ||\partial_n e^{t\Delta_{N,n}} e^{t\Delta'} f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \\ &= ||\, |\partial_n e^{t\Delta_{N,n}} e^{t\Delta'} f|_{L^p(\mathbb{R}_+)}||_{L^\infty(\mathbb{R}^{n-1})}. \end{aligned}$$

The $L^p - L^q$-estimate of the Neumann Laplacian $e^{t\Delta_{N,n}}$ yields

$$||e^{t\Delta_D}\partial_n f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \leq Ct^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} ||\, |e^{t\Delta'} f|_{L^q(\mathbb{R}_+)}||_{L^\infty(\mathbb{R}^{n-1})}.$$

Since $e^{t\Delta'}$ is a positive operator, we get

$$\begin{aligned} &||e^{t\Delta_D}\partial_n f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \leq Ct^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} ||\, e^{t\Delta'}|f|_{L^q(\mathbb{R}_+)}||_{L^\infty(\mathbb{R}^{n-1})} \\ &\leq Ct^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} ||f||_{L^\infty(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))},\ t>0. \qquad (4.4) \end{aligned}$$

On the other hand we can directly estimate

$$||e^{t\Delta_D}\partial_n f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))} \leq Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} ||\partial_n f||_{L^\infty(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}$$

$$\leq Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}||f||_{L^\infty(\mathbb{R}^{n-1};W^{1,q}(\mathbb{R}_+))},\ t>0.$$

Combining the above two estimates results

$$||e^{t\Delta_D}\partial_n f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))}\leq Ct^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}||f||_{L^\infty(\mathbb{R}^{n-1};W^{1,q}(\mathbb{R}_+))},\ t>0.$$

The remaining two inequalities (3) and (4) are consequences of estimates we already derived above. Indeed, setting $\delta=1/2$ and having in mind that $R'_j\in\mathcal{L}(\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+)))$, it follows easily from (4.3) that

$$\begin{aligned}||\partial_j e^{t\Delta_D}f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))}&=||e^{t\Delta_D}\partial_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))}\\&\leq Ct^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}||f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}\end{aligned}$$

for $j=1,...,n-1$ and $t>0$. Furthermore, in the same way as we obtained (4.4) we can show (4) in view of $\partial_n \mathrm{e}^{-t\Delta_D}=\mathrm{e}^{-t\Delta_N}\partial_n$ and by interchanging the roles of Δ_D and Δ_N. This completes the proof. □

From Lemma 4.1 we will derive estimates for terms of the form $\partial_j e^{t\Delta_D}\mathbf{P}_+f$ for $1\leq j\leq n$. The crucial term here is the one with normal derivative ∂_n. The idea is to split $\mathbf{P}_+\partial_n$ and $\partial_n\mathbf{P}_+$ into a normal derivative term without Riesz operators and terms including only tangential derivatives and Riesz operators.

Lemma 4.3. *Let $n\geq 2$, $1<q\leq p<\infty$, $\delta\in(0,1/2]$, and $f\in\mathcal{BUC}^1(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))\cap\mathcal{BUC}(\mathbb{R}^{n-1};W^{1,q}(\mathbb{R}_+))$ such that $f|_{\partial\mathbb{R}^n_+}=0$. Then there are constants $C_\delta, C>0$ independent of f such that for $j=1,\dots,n-1$ and $k=1,\dots,n$ we have*

$$\begin{aligned}(1)\qquad &||e^{t\Delta_D}\mathbf{P}_+\partial_j f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))}\\&\leq C_\delta t^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}||f||_{W^{1,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))},\ t>0,\\(2)\qquad &||e^{t\Delta_D}\mathbf{P}_+\partial_n f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))}\\&\leq C_\delta t^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\Bigg(||f||_{L^\infty(\mathbb{R}^{n-1};W^{1,q}(\mathbb{R}_+))}\\&\qquad+\|f\|_{W^{1,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}\Bigg),\ t>0,\\(3)\qquad &||\partial_k e^{t\Delta_D}\mathbf{P}_+f||_{L^\infty(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))}\\&\leq Ct^{-\frac{1}{2}-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}||f||_{L^\infty(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))},\ t>0.\end{aligned}$$

Proof. In the case that $1\leq j\leq n-1$ we have $\mathbf{P}_+\partial_j f=\partial_j\mathbf{P}_+f$. Moreover, by Corollary 2.12 the operator $\mathbf{P}_+$ is bounded on $\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))$. Hence, Lemma 4.1 (1) implies for $t>0$,

$$\begin{aligned}&||e^{t\Delta_D}\mathbf{P}_+\partial_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1};L^p(\mathbb{R}_+))}\\&\leq C_\delta t^{-\delta-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\left(||f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}+||\nabla' f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1};L^q(\mathbb{R}_+))}\right),\end{aligned}$$

yielding (1) in virtue of Lemma 2.3.

In order to see (2) first recall that $\mathbf{P}_+ = r\mathbf{P}E$ by (1.17) and (4.1). Now we use the following splitting of $\mathbf{P}_+\partial_n f$:

$$\begin{aligned}
\mathbf{P}_+\partial_n f &= r\mathbf{P}E\partial_n f = r\mathbf{P}\partial_n \widetilde{E}f \\
&= r\partial_n((\widetilde{E}f)', 0) + \sum_{j=1}^{n-1} r\partial_j R(R_n(\widetilde{E}f)^j) + \\
&\quad r(\nabla', 0)R_n^2(\widetilde{E}f)^n - \sum_{j=1}^{n-1} r\partial_j R_j R_n(\widetilde{E}f)^n e_n \\
&= \partial_n \mathbf{Q}_0 f + \sum_{j=1}^{n-1} \partial_j \mathbf{Q}_j f \qquad (4.5) \\
&=: I + II,
\end{aligned}$$

where the operators $\mathbf{Q}_j$, $j = 0, \dots, n-1$, are defined by

$$\mathbf{Q}_0 g = r((\widetilde{E}g)', 0) = (g', 0), \qquad (4.6)$$

$$\mathbf{Q}_j g = rR(R_n(\widetilde{E}g)^j) + rR_n^2(\widetilde{E}g)^n e_j - rR_j R_n(\widetilde{E}g)^n e_n. \qquad (4.7)$$

Here we denote by e_j the unit vector whose j-th component is 1 and $Rh = (R_1 h, \dots, R_n h)$ for scalar function h. To derive (4.5) we also used the facts that

$$\begin{aligned}
&\partial_n R_j = \partial_n \partial_j (-\Delta)^{-1/2} = \partial_j \partial_n (-\Delta)^{-1/2} = \partial_j R_n, \quad 1 \le j \le n, \\
&R_n^2 = -\mathrm{I} - \sum_{j=1}^{n-1} R_j^2,
\end{aligned}$$

where I denotes the identity operator. By the boundedness of $r, \widetilde{E}$ and in view of $R \in \mathcal{L}(\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R})))$ this implies that $\mathbf{Q}_j \in \mathcal{L}(\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)))$, $j = 1, \dots, n-1$. Applying Lemma 4.1 (2) to I and Lemma 4.1 (1) to II then yields

$$\begin{aligned}
&||e^{t\Delta_D}\mathbf{P}_+\partial_n f||_{L^\infty(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))} \\
&\le ||e^{t\Delta_D}\partial_n \mathbf{Q}_0 f||_{L^\infty(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))} + \sum_{j=1}^{n-1} ||e^{t\Delta_D}\partial_j \mathbf{Q}_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))} \\
&\le C_\delta t^{-\delta - \frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} \Bigg[||\mathbf{Q}_0 f||_{L^\infty(\mathbb{R}^{n-1}; W^{1,q}(\mathbb{R}_+))} \\
&\quad + \sum_{j=1}^{n-1} \Big(||\mathbf{Q}_j f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+))} + ||\mathbf{Q}_j \nabla' f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+))} \Big) \Bigg] \\
&\le C_\delta t^{-\delta - \frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} \left(||f||_{L^\infty(\mathbb{R}^{n-1}; W^{1,q}(\mathbb{R}_+))} + \|f\|_{W^{1,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+))} \right).
\end{aligned}$$

For $1 \leq k \leq n-1$ inequality (3) is an immediate consequence of Lemma 4.1 (3), $\mathbf{P}_+ \in \mathcal{L}(\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)))$, and Lemma 2.3. In order to see (3) for $k = n$ we use $\partial_n e^{t\Delta_D}\mathbf{P}_+ f = e^{t\Delta_N}\partial_n \mathbf{P}_+ f$ and the splitting

$$\begin{aligned}\partial_n \mathbf{P}_+ f &= \partial_n (f', 0) + \sum_{j=1}^{n-1} \partial_j rR(R_n(Ef)^j) + (\nabla', 0) rR_n^2(Ef)^n \\ &\qquad - \sum_{j=1}^{n-1} \partial_j rR_j R_n (Ef)^n e_n \\ &= \partial_n \mathbf{Q}_0 f + \sum_{j=1}^{n-1} \partial_j \widetilde{\mathbf{Q}}_j f,\end{aligned}$$

with $\widetilde{\mathbf{Q}}_j$, $j = 1, \dots, n-1$, as defined in (4.7) with $\widetilde{E}$ replaced by E. After applying $e^{t\Delta_N}$, the first term of the right hand side can be represented by $\partial_n e^{t\Delta_D}(f', 0)$, using $e^{t\Delta_N}\partial_n = \partial_n e^{t\Delta_D}$ again. Hence, Lemma 4.1 (4) yields the desired estimate for this term. Employing Lemma 4.1 (3), Remark 4.2 (b), and $\widetilde{\mathbf{Q}}_j \in \mathcal{L}(\dot{B}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+)))$, $j = 1, \dots, n-1$, we obtain for the other terms

$$\begin{aligned}||e^{t\Delta_N} \sum_{j=1}^{n-1} \partial_j \widetilde{\mathbf{Q}}_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+))} &= ||\partial_j e^{t\Delta_N} \sum_{j=1}^{n-1} \widetilde{\mathbf{Q}}_j f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+))} \\ \leq Ct^{-\frac{1}{2} - \frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} ||f||_{\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^{n-1}; L^q(\mathbb{R}_+))}, \ t > 0, &\end{aligned}$$

which proves (3) in view of Lemma 2.3. □

Lemma 4.4. *Let $\varphi_0 \in (0, \pi)$, $2 < p < \infty$, and $\mathbf{B} = \Omega \mathbf{P}_+ \mathbf{J} \mathbf{P}_+ + \mathbf{C}_E$. There is a $C = C_{\varphi_0} > 0$ such that*

$$(1) \ \ \|(\lambda - \Delta_D)^{-1} S(\lambda) f\|_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \leq \frac{C}{|\lambda|^{1 - \frac{1}{2p}}} \|f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})}$$

$$(2) \ \ \|\nabla(\lambda - \Delta_D)^{-1} S(\lambda) f\|_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \leq \frac{C}{|\lambda|^{\frac{1}{2} - \frac{1}{2p}}} \|f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})}$$

$$(3) \ \ \|\mathbf{B}(\lambda - \Delta_D)^{-1} S(\lambda) f\|_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \leq \frac{C}{|\lambda|^{\frac{1}{2} - \frac{1}{2p}}} \|f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})}$$

for $\lambda \in \Sigma_{\pi - \varphi_0}$, $|\lambda| \geq 1$, $f \in \dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})$.

Proof. We write the operator under consideration as

$$\nabla^k (\lambda - \Delta_D)^{-1} S(\lambda) = (-\Delta')^{1/4p'} \nabla^k (\lambda - \Delta_D)^{-1} (-\Delta')^{-1/4p'} S(\lambda)$$

for $k = 0, 1$. Observe that

$$\|\nabla^k \mathrm{e}^{-t\Delta_D}\|_{\mathcal{L}(\dot{\mathcal{B}}^0_{\infty,1}(L^p))} \leq Ct^{-\frac{k}{2}}, \quad t > 0,$$

due to Proposition 3.2. By Lemma A.1 (2) with $\alpha = 1/4p'$ and the $L^p - L^q$-estimates for $\Delta_{D,n}$ we therefore deduce

$$\begin{aligned}
&\|\nabla^k(-\Delta')^{1/4p'}\mathrm{e}^{-t\Delta_D}f\|_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \\
&= \|\nabla^k\mathrm{e}^{-t\Delta_D/2}(-\Delta')^{1/4p'}\mathrm{e}^{-t\Delta'/2}e^{-t\Delta_{D,n}/2}f\|_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \\
&\le Ct^{-\frac{k}{2}-\frac{1}{4p'}}\|e^{-t\Delta_{D,n}/2}f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^p)} \\
&\le Ct^{-\frac{k}{2}-\frac{1}{4p'}-\frac{1}{2p}}\|f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})}, \quad t>0,\ k=0,1.
\end{aligned}$$

This implies for the resolvent

$$\|\nabla^k(-\Delta')^{1/4p'}(\lambda-\Delta_D)^{-1}f\|_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \le \frac{C}{|\lambda|^{1-\frac{k}{2}-\frac{1}{4p'}-\frac{1}{2p}}}\|f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})}$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$, $k=0,1$. In combination with Lemma 3.3 this yields

$$\begin{aligned}
&\|\nabla^k(\lambda-\Delta_D)^{-1}S(\lambda)f\|_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \\
&\le \frac{C}{|\lambda|^{1-\frac{k}{2}-\frac{1}{4p'}-\frac{1}{2p}}}\|(-\Delta')^{-1/4p'}S(\lambda)f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})} \\
&\le \frac{C}{|\lambda|^{1-\frac{k}{2}-\frac{1}{2p}}}\|f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})}
\end{aligned}$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$, $k=0,1$. This shows (1) and (2). In view of the boundedness of $\mathbf{P}_+$ in $\dot{\mathcal{B}}^0_{\infty,1}(L^p)$ and the function $\mathbf{U}^E$, relation (3) is an easy consequence of (1) and (2). □

The next proposition contains the crucial estimates that allow us to construct solutions in the space $\mathcal{BUC}(\mathbb{R}^2;L^p(\mathbb{R}_+))$. Note again that due to the fact mentioned in Remark 4.2 (a) we are not able to carry out the iteration in the space $\dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2;L^p(\mathbb{R}_+))$.

Proposition 4.5. *Let $2<p<\infty$, $\varphi_0 \in (0,\pi/2)$, $\delta \in (0,1/2]$, and $\omega_0 = \omega_0(\varphi_0)$ as in Theorem 3.4. There exist $C = C(\varphi_0,\delta) > 0$ and $\omega_1 \ge \omega_0$ such that*

$$\begin{aligned}
&\| \nabla^\ell e^{-t(\mathbf{A}_E+\omega_1)}\mathbf{P}_+\partial_j f\|_{L^\infty(\mathbb{R}^2;L^p(\mathbb{R}_+))} \\
&\le Ct^{-\frac{\ell}{2}-\frac{1}{2p}-\delta(1-\ell)}\big(\|f\|_{W^{1,\infty}(\mathbb{R}^2;L^{p/2}(\mathbb{R}_+))} + \|f\|_{L^\infty(\mathbb{R}^2;W^{1,p/2}(\mathbb{R}_+))}\big) \qquad (4.8)
\end{aligned}$$

for $t>0$, $\ell=0,1$, $j=1,2,3$, and
$f \in \mathcal{BUC}^1(\mathbb{R}^2;L^{p/2}(\mathbb{R}_+)) \cap \mathcal{BUC}(\mathbb{R}^2;W^{1,p/2}(\mathbb{R}_+))$ with $f|_{\partial\mathbb{R}^n_+} = 0$.

Proof. For simplicity we omit the $\mathbb{R}$ notation in the spaces, i.e. we write $W^{1,\infty}(L^p) = W^{1,\infty}(\mathbb{R}^2;L^p(\mathbb{R}_+))$, $L^\infty(L^p) = L^\infty(\mathbb{R}^2;L^p(\mathbb{R}_+))$ and so on in the sequel. We will prove the corresponding estimates for the resolvent of $\mathbf{A}_E$, i.e.

$$\|\nabla^\ell(\lambda+\omega_1+\mathbf{A}_E)^{-1}\mathbf{P}_+\partial_j f\|_{L^\infty(L^p)}$$

$$\leq \frac{C}{|\lambda|^{1-\frac{\ell}{2}-\frac{1}{2p}-\delta(1-\ell)}}\left(\|f\|_{W^{1,\infty}(L^{p/2})}+\|f\|_{L^\infty(W^{1,p/2})}\right) \tag{4.9}$$

for $j=1,2,3$, $\ell=0,1$, $\lambda\in\Sigma_{\pi-\varphi_0}$, and $f\in\mathcal{BUC}^1(L^{p/2})$. Then (4.8) easily follows by the representation

$$\mathrm{e}^{-t(\mathbf{A}_E+\omega_1)}=\frac{1}{2\pi i}\int_\Gamma \mathrm{e}^{-t\lambda}(\lambda+\omega_1+\mathbf{A}_E)^{-1}\mathrm{d}\lambda.$$

Now fix $\varphi_0\in(0,\pi/2)$ and set $\mu:=\lambda+\omega_1$. We already pointed out that the resolvent of the Stokes operator $\mathbf{A}$ can be written as

$$(\mu+\mathbf{A})^{-1}=(\mu-\Delta_D)^{-1}(I-S(\mu)), \tag{4.10}$$

where $S(\mu)$ is defined as in (3.8). Thus, according to (3.11) the resolvent of $\mathbf{A}_E$ is represented as

$$(\mu+\mathbf{A}_E)^{-1}=(\mu-\Delta_D)^{-1}(I-S(\mu))[I+\mathbf{B}(\mu+\mathbf{A})^{-1}]^{-1}.$$

Let us first consider the easier case of tangential derivatives. Since ∂_j commutes with $\mathbf{P}_+$ and all parts of $\mathbf{A}_E$ for $j=1,2$, in this case we have

$$\begin{aligned}
&\|(\mu+\mathbf{A}_E)^{-1}\mathbf{P}_+\partial_j f\|_{L^\infty(L^p)}\\
&=C\|(\mu-\Delta_D)^{-1}\partial_j(I-S(\mu))[I+\mathbf{B}(\mu+\mathbf{A})^{-1}]^{-1}\mathbf{P}_+f\|_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)}\\
&\leq \frac{C}{|\mu|^{1-\frac{1}{2p}-\delta}}\bigg(\|(I-S(\mu))[I+\mathbf{B}(\mu+\mathbf{A})^{-1}]^{-1}\mathbf{P}_+f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})}\\
&\quad+\|(I-S(\mu))[I+\mathbf{B}(\mu+\mathbf{A})^{-1}]^{-1}\mathbf{P}_+\nabla' f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})}\bigg)\\
&\leq \frac{C}{|\mu|^{1-\frac{1}{2p}-\delta}}\left(\|f\|_{L^\infty(L^{p/2})}+\|\nabla' f\|_{L^\infty(L^{p/2})}\right)\\
&\leq \frac{C}{|\mu|^{1-\frac{1}{2p}-\delta}}\|f\|_{W^{1,\infty}(L^{p/2})},
\end{aligned} \tag{4.11}$$

where we applied Lemma 4.1 (1) as well as the boundedness of $S(\mu)$, $[I+\mathbf{B}(\mu+\mathbf{A})^{-1}]^{-1}$, and $\mathbf{P}_+$ in the space $\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})$ given by Lemma 3.3, by $\omega_1\geq\omega_0$ and our choice of ω_0 (see Theorem 3.4), and Corollary 2.12, respectively. Applying Lemma 4.1 (3) instead of Lemma 4.1 (1) we can obtain in an analogous way

$$\|\partial_i(\mu+\mathbf{A}_E)^{-1}\mathbf{P}_+\partial_j f\|_{L^\infty(L^p)}\leq \frac{C_{\varphi_0}}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}}\|\partial_j f\|_{L^\infty(L^{p/2})}$$

for $i=1,2$ and $j=1,2,3$. Since

$$\begin{aligned}
&||\partial_j f||_{L^\infty(L^{p/2})}\leq ||f||_{W^{1,\infty}(L^{p/2})} \qquad \text{if } j=1,2,\\
&||\partial_3 f||_{L^\infty(L^{p/2})}\leq ||f||_{L^\infty(W^{1,p/2})},
\end{aligned}$$

we conclude

$$\begin{aligned}&\|\partial_i(\mu+\mathbf{A}_E)^{-1}\mathbf{P}_+\partial_j f\|_{L^\infty(L^p)}\\&\qquad\le \frac{C_{\varphi_0}}{|\mu|^{\frac12-\frac1{2p}}}\left(\|f\|_{W^{1,\infty}(L^{p/2})}+\|f\|_{L^\infty(W^{1,p/2})}\right).\end{aligned}\tag{4.12}$$

The case of normal derivatives is more involved. Here we employ the Neumann series and use the representation of the form

$$(\mu+\mathbf{A}_E)^{-1}=\sum_{k=0}^{\infty}(\mu+\mathbf{A})^{-1}[\mathbf{B}(\mu+\mathbf{A})^{-1}]^k.$$

In order to estimate this expression we need

Lemma 4.6. *There are constants $K=K(\varphi_0)>0$ and $\omega_1\ge\omega_0$ such that*

$$\begin{aligned}&\|(\mu+\mathbf{A})^{-1}[\mathbf{B}(\mu+\mathbf{A})^{-1}]^k\mathbf{P}_+\partial_3 f\|_{L^\infty(L^p)}\\&\le \frac{K}{|\mu|^{1-\frac1{2p}-\delta}}\left(\frac{1}{\sqrt2}\right)^k\left(\|f\|_{W^{1,\infty}(L^{p/2})}+\|f\|_{L^\infty(W^{1,p/2})}\right)\end{aligned}\tag{4.13}$$

and

$$\begin{aligned}&\|\partial_3(\mu+\mathbf{A})^{-1}[\mathbf{B}(\mu+\mathbf{A})^{-1}]^k\mathbf{P}_+\partial_j f\|_{L^\infty(L^p)}\\&\le \frac{K}{|\mu|^{\frac12-\frac1{2p}}}\left(\frac{1}{\sqrt2}\right)^k\left(\|f\|_{W^{1,\infty}(L^{p/2})}+\|f\|_{L^\infty(W^{1,p/2})}\right)\end{aligned}\tag{4.14}$$

for all $\mu-\omega_1\in\Sigma_{\pi-\varphi_0}$, $j=1,2,3$, $k=0,1,2,\ldots$, and $f\in\mathcal{BUC}^1(L^{p/2})\cap\mathcal{BUC}(W^{1,p/2})$ with $f|_{\partial\mathbb{R}^n_+}=0$.

Before proving Lemma 4.6 let us complete the proof of Proposition 4.5 first. From (4.13) we immediately conclude

$$\begin{aligned}&\|(\mu+\mathbf{A}_E)^{-1}\mathbf{P}_+\partial_3 f\|_{L^\infty(L^p)}\\&\le\sum_{k=0}^{\infty}\frac{C_{\varphi_0}}{|\mu|^{1-\frac1{2p}-\delta}}\left(\frac{1}{\sqrt2}\right)^k\left(\|f\|_{W^{1,\infty}(L^{p/2})}+\|f\|_{L^\infty(W^{1,p/2})}\right)\\&\le\frac{C_{\varphi_0}}{|\mu|^{1-\frac1{2p}-\delta}}\left(\|f\|_{W^{1,\infty}(L^{p/2})}+\|f\|_{L^\infty(W^{1,p/2})}\right).\end{aligned}$$

On the other hand (4.14) implies

$$\begin{aligned}&\|\partial_3(\mu+\mathbf{A}_E)^{-1}\mathbf{P}_+\partial_j f\|_{L^\infty(L^p)}\\&\le\sum_{k=0}^{\infty}\frac{C_{\varphi_0}}{|\mu|^{\frac12-\frac1{2p}}}\left(\frac{1}{\sqrt2}\right)^k\left(\|f\|_{W^{1,\infty}(L^{p/2})}+\|f\|_{L^\infty(W^{1,p/2})}\right)\\&\le\frac{C_{\varphi_0}}{|\mu|^{\frac12-\frac1{2p}}}\left(\|f\|_{W^{1,\infty}(L^{p/2})}+\|f\|_{L^\infty(W^{1,p/2})}\right)\end{aligned}$$

for $j = 1,2,3$. Combining these two inequalities with (4.11) and (4.12), estimate (4.9) and thus the assertion of Proposition 4.5 follows. □

Proof. (of Lemma 4.6)
We apply induction over k. For $k = 0$ an application of Lemma 4.3 (2) and Lemma 4.4 (1) implies

$$\begin{aligned} &\|(\mu+\mathbf{A})^{-1}\mathbf{P}_+\partial_3 f\|_{L^\infty(L^p)} = \|(\mu-\Delta_D)^{-1}(I-S(\mu))\mathbf{P}_+\partial_3 f\|_{L^\infty(L^p)} \\ &\le C\Bigg[\frac{1}{|\mu|^{1-\frac{1}{2p}-\delta}}\left(\|f\|_{W^{1,\infty}(L^{p/2})} + \|f\|_{L^\infty(W^{1,p/2})}\right) \\ &\quad + \frac{1}{|\mu|^{1-\frac{1}{2p}}}\|\partial_3 f\|_{L^\infty(L^{p/2})}\Bigg] \\ &\le \frac{K}{|\mu|^{1-\frac{1}{2p}-\delta}}\left(\|f\|_{W^{1,\infty}(L^{p/2})} + \|f\|_{L^\infty(W^{1,p/2})}\right) \end{aligned}$$

for $\mu - \omega_1 \in \Sigma_{\pi-\varphi_0}$ and some $K > 0$. Instead, if we employ Lemma 4.3 (3) and Lemma 4.4 (2), we obtain completely analogous

$$\begin{aligned} \|\partial_3(\mu+\mathbf{A})^{-1}\mathbf{P}_+\partial_j f\|_{L^\infty(L^p)} &\le \frac{K}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}}\|\partial_j f\|_{L^\infty(L^{p/2})} \\ &\le \frac{K}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}}\left(\|f\|_{W^{1,\infty}(L^{p/2})} + \|f\|_{L^\infty(W^{1,p/2})}\right) \end{aligned}$$

for $\mu - \omega_1 \in \Sigma_{\pi-\varphi_0}$ and $j = 1,2,3$. For the step $k \longrightarrow k+1$ we first consider again (4.13). Note that we have to make sure that the constant K, while doing this step, is not increasing. We compute by using (4.10)

$$\begin{aligned} &\ (\mu+\mathbf{A})^{-1}[\mathbf{B}(\mu+\mathbf{A})^{-1}]^{k+1}\mathbf{P}_+\partial_3 f \\ =&\ (\mu+\mathbf{A})^{-1}[\mathbf{B}(\mu+\mathbf{A})^{-1}]^{k}\mathbf{B}(\mu-\Delta_D)^{-1}\mathbf{P}_+\partial_3 f \\ &\ -(\mu+\mathbf{A})^{-1}[\mathbf{B}(\mu+\mathbf{A})^{-1}]^{k}\mathbf{B}(\mu-\Delta_D)^{-1}S(\mu)\mathbf{P}_+\partial_3 f \\ =:&\ I_1 + I_2. \end{aligned}$$

We start by estimating I_2. According to (3.12) we have $\|\mathbf{B}(\mu+\mathbf{A})^{-1}\|_{\mathcal{L}(\dot{\mathcal{B}}^0_{\infty,1}(L^p))} \le 1/\sqrt{2}$. Consequently

$$\begin{aligned} \|I_2\|_{L^\infty(L^p)} &\le C\|I_2\|_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \\ &\le \frac{C}{|\mu|}\left(\frac{1}{\sqrt{2}}\right)^k \|\mathbf{B}(\mu-\Delta_D)^{-1}S(\mu)\mathbf{P}_+\partial_3 f\|_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)}. \end{aligned}$$

By using Lemma 4.4 (3) we can continue the calculation, concluding

$$\begin{aligned} \|I_2\|_{L^\infty(L^p)} &\le \frac{C}{\sqrt{|\mu|}}\frac{1}{|\mu|^{1-\frac{1}{2p}}}\left(\frac{1}{\sqrt{2}}\right)^k \|\mathbf{P}_+\partial_3 f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})} \\ &\le \frac{C}{\sqrt{|\mu|}}\frac{1}{|\mu|^{1-\frac{1}{2p}}}\left(\frac{1}{\sqrt{2}}\right)^k \|\partial_3 f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})} \end{aligned}$$

$$\leq \frac{C_1}{\sqrt{|\mu|}} \frac{1}{|\mu|^{1-\frac{1}{2p}}} \left(\frac{1}{\sqrt{2}}\right)^k \|f\|_{L^\infty(W^{1,p/2})}, \quad \mu - \omega_1 \in \Sigma_{\pi-\varphi_0}. \quad (4.15)$$

Here, we used the boundedness of $\mathbf{P}_+$ in $\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})$ and Lemma 2.3. The term I_1 we split in further two parts by recalling $\mathbf{B} = \mathbf{C}_E + \Omega\mathbf{P}_+\mathbf{J}$. Note that in order to avoid another splitting, we write here $\mathbf{P}_+\mathbf{J}$ instead of $\mathbf{P}_+\mathbf{J}\mathbf{P}_+$. This is possible, since $\mathbf{P}_+\mathbf{J}\mathbf{P}_+$ is a priori applied on solenoidal fields only. This yields

$$\begin{aligned} I_1 &= (\mu + \mathbf{A})^{-1}[\mathbf{B}(\mu + \mathbf{A})^{-1}]^k \mathbf{C}_E(\mu - \Delta_D)^{-1}\mathbf{P}_+\partial_3 f \\ &\quad + \Omega(\mu + \mathbf{A})^{-1}[\mathbf{B}(\mu + \mathbf{A})^{-1}]^k \mathbf{P}_+\mathbf{J}(\mu - \Delta_D)^{-1}\mathbf{P}_+\partial_3 f \\ &=: I_{11} + I_{12}. \end{aligned}$$

In order to estimate I_{11}, note that the Ekman part $\mathbf{C}_E$ can be written as

$$\begin{aligned} \mathbf{C}_E v &= \mathbf{P}_+(\mathbf{U}^E \cdot \nabla)v + \mathbf{P}_+ v_3 \partial_3 \mathbf{U}^E \\ &= \mathbf{P}_+ \sum_{j=1}^{2} \mathbf{U}^E_j \partial_j v + \mathbf{P}_+\mathbf{U}^E(\nabla' \cdot v') + \mathbf{P}_+\partial_3(\mathbf{U}^E v_3), \qquad (4.16) \end{aligned}$$

if $\operatorname{div} v = 0$. Observe that we can employ this representation before our splitting in I_1 and I_2, since then this divergence condition is fulfilled. Now consider the first term of $\mathbf{C}_E$ which contains tangential derivatives only. Since $\mathbf{U}^E$ depends only on x_3, we can pull out its supremum obtaining

$$\begin{aligned} &\|\mathbf{P}_+ \sum_{j=1}^{2} \mathbf{U}^E_j \partial_j(\mu - \Delta_D)^{-1}\mathbf{P}_+\partial_3 f\|_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \\ &\leq C\|\mathbf{U}^E\|_\infty \sum_{j=1}^{2} \|\partial_j(\mu - \Delta_D)^{-1}\mathbf{P}_+\partial_3 f\|_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \\ &\leq \frac{C}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} \|\mathbf{U}^E\|_\infty \|\mathbf{P}_+\partial_3 f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})} \\ &\leq \frac{C}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} \|\partial_3 f\|_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^{p/2})} \leq \frac{C}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} \|f\|_{L^\infty(W^{1,p/2})}, \end{aligned}$$

where we used Lemma 4.1 (3) in the second inequality. Hence, by $\dot{\mathcal{B}}^0_{\infty,1}(L^p) \hookrightarrow L^\infty(L^p)$ we have for the first term of (4.16)

$$\begin{aligned} &\|(\mu + \mathbf{A})^{-1}[\mathbf{B}(\mu + \mathbf{A})^{-1}]^k \mathbf{P}_+ \sum_{j=1}^{2} \mathbf{U}^E_j \partial_j(\mu - \Delta_D)^{-1}\mathbf{P}_+\partial_3 f\|_{L^\infty(L^p)} \\ &\leq \frac{C}{|\mu|} \left(\frac{1}{\sqrt{2}}\right)^k \|\mathbf{P}_+ \sum_{j=1}^{2} \mathbf{U}^E_j \partial_j(\mu - \Delta_D)^{-1}\mathbf{P}_+\partial_3 f\|_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \\ &\leq \frac{C_2}{\sqrt{|\mu|}} \frac{1}{|\mu|^{1-\frac{1}{2p}}} \left(\frac{1}{\sqrt{2}}\right)^k \|f\|_{L^\infty(W^{1,p/2})}. \end{aligned}$$

We omit the details for the second part of (4.16). As it also contains only tangential derivatives, we deduce in a completely analogous way

$$\begin{aligned}&\|(\mu+\mathbf{A})^{-1}[\mathbf{B}(\mu+\mathbf{A})^{-1}]^k\mathbf{P}_+\mathbf{U}^E(\nabla'\cdot((\mu-\Delta_D)^{-1}\mathbf{P}_+\partial_3 f)')\|_{L^\infty(L^p)}\\ \leq\ &\frac{C_3}{\sqrt{|\mu|}}\frac{1}{|\mu|^{1-\frac{1}{2p}}}\left(\frac{1}{\sqrt{2}}\right)^k\|f\|_{L^\infty(W^{1,p/2})}.\end{aligned}$$

For the third addend of (4.16) we conclude by applying the assumption of induction

$$\begin{aligned}&\|(\mu+\mathbf{A})^{-1}[\mathbf{B}(\mu+\mathbf{A})^{-1}]^k\mathbf{P}_+\partial_3\left(\mathbf{U}^E[(\mu-\Delta_D)^{-1}\mathbf{P}_+\partial_3 f\cdot e_3]\right)\|_{L^\infty(L^p)}\\ \leq\ &\frac{K}{|\mu|^{1-\frac{1}{2p}-\delta}}\left(\frac{1}{\sqrt{2}}\right)^k\Big(\|\mathbf{U}^E[(\mu-\Delta_D)^{-1}\mathbf{P}_+\partial_3 f\cdot e_3]\|_{L^\infty(W^{1,p/2})}\\ &+\|\mathbf{U}^E[(\mu-\Delta_D)^{-1}\mathbf{P}_+\partial_3 f\cdot e_3]\|_{W^{1,\infty}(L^{p/2})}\Big)\\ \leq\ &\frac{C_4}{\sqrt{|\mu|}}\frac{1}{|\mu|^{1-\frac{1}{2p}-\delta}}\left(\frac{1}{\sqrt{2}}\right)^k\left(\|f\|_{W^{1,\infty}(L^{p/2})}+\|f\|_{L^\infty(W^{1,p/2})}\right),\end{aligned}$$

where we used Lemma 4.3 (2) in the last estimate and set $e_3=(0,0,1)$.

In order to see the estimate for I_{12} we split $\mathbf{P}_+\partial_3 f$ as in (4.5) and get

$$\begin{aligned}&\frac{1}{\Omega}||I_{12}||_{L^\infty(L^p)}\\ \leq\ &||(\mu+\mathbf{A})^{-1}[\mathbf{B}(\mu+\mathbf{A})^{-1}]^k\mathbf{P}_+\mathbf{J}(\mu-\Delta_D)^{-1}\partial_3\mathbf{Q}_0 f||_{L^\infty(L^p)}\\ &+\sum_{j=1}^{2}||(\mu+\mathbf{A})^{-1}[\mathbf{B}(\mu+\mathbf{A})^{-1}]^k\mathbf{P}_+\mathbf{J}(\mu-\Delta_D)^{-1}\partial_j\mathbf{Q}_j f||_{L^\infty(L^p)}\\ =:\ &A_0+\sum_{j=1}^{2}A_j.\end{aligned}$$

For $j=1,2$ we obtain

$$\begin{aligned}A_j&\leq||(\mu+\mathbf{A})^{-1}[\mathbf{B}(\mu+\mathbf{A})^{-1}]^k\mathbf{P}_+\mathbf{J}(\mu-\Delta_D)^{-1}\partial_j\mathbf{Q}_j f||_{\dot{B}^0_{\infty,1}(L^p)}\\ &\leq C\frac{1}{|\mu|}(\frac{1}{\sqrt{2}})^k||(\mu-\Delta_D)^{-1}\partial_j\mathbf{Q}_j f||_{\dot{B}^0_{\infty,1}(L^p)}.\end{aligned}$$

Applying Lemma 4.1 (1) and the boundedness of $\mathbf{Q}_j$ on $\dot{B}^0_{\infty,\infty}(L^{p/2})$ yields

$$\begin{aligned}A_j&\leq C\frac{1}{|\mu|}(\frac{1}{\sqrt{2}})^k\frac{1}{|\mu|^{1-\frac{1}{2p}-\delta}}\left(||\mathbf{Q}_j f||_{\dot{B}^0_{\infty,\infty}(L^{p/2})}+||\mathbf{Q}_j\nabla' f||_{\dot{B}^0_{\infty,\infty}(L^{p/2})}\right)\\ &\leq\frac{C_6}{|\mu|}(\frac{1}{\sqrt{2}})^k\frac{1}{|\mu|^{1-\frac{1}{2p}-\delta}}||f||_{W^{1,\infty}(L^{p/2})}.\end{aligned}$$

For A_0 we use

$$\mathbf{J}(\mu-\Delta_D)^{-1}\partial_3\mathbf{Q}_0 f=\partial_3\mathbf{J}(\mu-\Delta_N)^{-1}\mathbf{Q}_0 f \qquad (4.17)$$

to get by the assumption of induction that

$$A_0 \leq K \frac{1}{|\mu|^{1-\frac{1}{2p}-\delta}} (\frac{1}{\sqrt{2}})^k \Big(||\mathbf{J}(\mu - \Delta_N)^{-1}\mathbf{Q}_0 f||_{L^\infty(W^{1,p/2})} \\ + ||\mathbf{J}(\mu - \Delta_N)^{-1}\mathbf{Q}_0 f||_{W^{1,\infty}(L^{p/2})} \Big).$$

By definition (4.6) obviously $\mathbf{Q}_0$ is bounded in $L^\infty(W^{1,p/2})$. Applying well-known estimates for the Laplacian with Neumann boundary conditions then yields

$$A_0 \leq K \frac{1}{|\mu|^{1-\frac{1}{2p}-\delta}} (\frac{1}{\sqrt{2}})^k \frac{1}{|\mu|} C \left(||f||_{L^\infty(W^{1,p/2})} + \|f\|_{W^{1,\infty}(L^{p/2})} \right) \\ \leq \frac{C_7}{|\mu|} \frac{1}{|\mu|^{1-\frac{1}{2p}-\delta}} (\frac{1}{\sqrt{2}})^k \left(\|f\|_{L^\infty(W^{1,p/2})} + \|f\|_{W^{1,\infty}(L^{p/2})} \right).$$

Thus, setting $C_{\max} := \max\{C_1, \dots, C_7\}$ and choosing ω_1 large enough, we can achieve $\sqrt{|\mu|} = \sqrt{|\lambda + \omega_1|} \geq 8C_{\max}\sqrt{2}/K$, $\lambda \in \Sigma_{\pi-\varphi_0}$. Consequently,

$$\|\partial_3(\mu + \mathbf{A})^{-1}[\mathbf{B}(\mu + \mathbf{A})^{-1}]^{k+1}\mathbf{P}_+\partial_j f\|_{L^\infty(L^p)} \\ \leq \frac{K}{|\mu|^{1-\frac{1}{2p}-\delta}} \left(\frac{1}{\sqrt{2}}\right)^{k+1} \left(\|f\|_{L^\infty(W^{1,p/2})} + \|f\|_{W^{1,\infty}(L^{p/2})} \right)$$

for $\mu - \omega_1 \in \Sigma_{\pi-\varphi_0}$ and (4.13) is proved.

Since it is very similar, we will be brief in details in demonstrating the step $k \longrightarrow k+1$ for (4.14). We use again the splitting

$$\partial_3(\mu + \mathbf{A})^{-1}[\mathbf{B}(\mu + \mathbf{A})^{-1}]^{k+1}\mathbf{P}_+\partial_j f \\ = \partial_3(\mu + \mathbf{A})^{-1}[\mathbf{B}(\mu + \mathbf{A})^{-1}]^{k}\mathbf{B}(\mu - \Delta_D)^{-1}\mathbf{P}_+\partial_j f \\ - \partial_3(\mu + \mathbf{A})^{-1}[\mathbf{B}(\mu + \mathbf{A})^{-1}]^{k}\mathbf{B}(\mu - \Delta_D)^{-1}S(\mu)\mathbf{P}_+\partial_j f \\ =: J_1 + J_2$$

and deduce in the same way as in (4.15)

$$\|J_2\|_{L^\infty(L^p)} \leq \frac{C_1}{\sqrt{|\mu|}} \frac{1}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} \left(\frac{1}{\sqrt{2}}\right)^k \|\partial_j f\|_{L^\infty(L^{p/2})} \\ \leq \frac{C_1}{\sqrt{|\mu|}} \frac{1}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} \left(\frac{1}{\sqrt{2}}\right)^k \left(\|f\|_{W^{1,\infty}(L^{p/2})} + \|f\|_{L^\infty(W^{1,p/2})} \right)$$

for $\mu - \omega_1 \in \Sigma_{\pi-\varphi_0}$ and $j = 1, 2, 3$. The term J_1 we split again as

$$J_1 = \partial_3(\mu + \mathbf{A})^{-1}[\mathbf{B}(\mu + \mathbf{A})^{-1}]^{k}\mathbf{C}_E(\mu - \Delta_D)^{-1}\mathbf{P}_+\partial_j f \\ + \Omega\partial_3(\mu + \mathbf{A})^{-1}[\mathbf{B}(\mu + \mathbf{A})^{-1}]^{k}\mathbf{P}_+\mathbf{J}(\mu - \Delta_D)^{-1}\mathbf{P}_+\partial_j f \\ =: J_{11} + J_{12}.$$

In this case we can use for $\mathbf{C}_E$ the representation

$$\mathbf{C}_E v = \mathbf{P}_+ \mathrm{div}(v \otimes \mathbf{U}^E + \mathbf{U}^E \otimes v).$$

Then we can apply directly the assumption of induction, which implies

$$\begin{aligned}
&\|J_{11}\|_{L^\infty(L^{p/2})} \\
&\le \frac{K}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} \left(\frac{1}{\sqrt{2}}\right)^k \|\mathbf{U}^E\|_{1,\infty} \Big(\|(\mu - \Delta_D)^{-1}\mathbf{P}_+ \partial_j f\|_{W^{1,\infty}(L^{p/2})} \\
&\quad + \|(\mu - \Delta_D)^{-1}\mathbf{P}_+ \partial_j f\|_{L^\infty(W^{1,p/2})} \Big) \\
&\le \frac{C_2}{\sqrt{|\mu|}} \frac{1}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} \left(\frac{1}{\sqrt{2}}\right)^k \left(\|f\|_{W^{1,\infty}(L^{p/2})} + \|f\|_{L^\infty(W^{1,p/2})} \right)
\end{aligned}$$

for $\quad \mu - \omega_1 \in \Sigma_{\pi - \varphi_0}$.

We turn to the term J_{12}. In the same way as we applied Lemma 4.1, the boundedness of $\mathbf{P}_+$, and Lemma 2.3 several times above, the cases $j = 1, 2$ can be handled. Therefore we restrict our considerations to the case $j = 3$. Splitting $\mathbf{P}_+ \partial_3 f$ yields

$$\begin{aligned}
&\frac{1}{\Omega} ||J_{12}||_{L^\infty(L^p)} \\
&\le \|\partial_3 (\mu + \mathbf{A})^{-1} [\mathbf{B}(\mu + \mathbf{A})^{-1}]^k \mathbf{P}_+ \mathbf{J}(\mu - \Delta_D)^{-1} \partial_3 \mathbf{Q}_0 f\|_{L^\infty(L^p)} \\
&\quad + \sum_{j=1}^{2} \|\partial_3 (\mu + \mathbf{A})^{-1} [\mathbf{B}(\mu + \mathbf{A})^{-1}]^k \mathbf{P}_+ \mathbf{J}(\mu - \Delta_D)^{-1} \partial_j \mathbf{Q}_j f\|_{L^\infty(L^p)} \\
&=: B_0 + \sum_{j=1}^{2} B_j.
\end{aligned}$$

By (4.17) and the assumption of induction we obtain analogously to the estimate for A_0 that

$$\begin{aligned}
B_0 &\le \frac{K}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} \left(\frac{1}{\sqrt{2}}\right)^k \Big(||\mathbf{J}(\mu - \Delta_N)^{-1} \mathbf{Q}_0 f\|_{W^{1,\infty}(L^{p/2})} \\
&\quad + ||\mathbf{J}(\mu - \Delta_N)^{-1} \mathbf{Q}_0 f\|_{L^\infty(W^{1,p/2})} \Big) \\
&\le \frac{C_3}{|\mu|} \frac{1}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} \left(\frac{1}{\sqrt{2}}\right)^k \left(||f\|_{W^{1,\infty}(L^{p/2})} + ||f\|_{L^\infty(W^{1,p/2})} \right).
\end{aligned}$$

Furthermore, similarly to the estimates for A_j, $j = 1, 2$, we deduce

$$B_j \le \frac{C}{\sqrt{|\mu|}} \left(\frac{1}{\sqrt{2}}\right)^k ||\mathbf{P}_+ \mathbf{J}(\mu - \Delta_D)^{-1} \partial_j \mathbf{Q}_j f\|_{\dot{B}^0_{\infty,1}(L^p)}$$

$$
\begin{aligned}
&\le \frac{C}{\sqrt{|\mu|}} \left(\frac{1}{\sqrt{2}}\right)^k ||(\mu - \Delta_D)^{-1}\partial_j \mathbf{Q}_j f||_{\dot{B}^0_{\infty,1}(L^p)} \\
&\le \frac{C}{\sqrt{|\mu|}} \left(\frac{1}{\sqrt{2}}\right)^k \frac{1}{|\mu|^{1-\frac{1}{2p}}} ||\mathbf{Q}_j f||_{\dot{B}^0_{\infty,\infty}(L^{p/2})} \\
&\le \frac{C_4}{|\mu|} \left(\frac{1}{\sqrt{2}}\right)^k \frac{1}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} ||f||_{L^\infty(L^{p/2})},
\end{aligned}
$$

where we simply applied Lemma 4.1 (3) instead of Lemma 4.1 (1) this time.

Summarizing, and by choosing ω_1 large enough we can again achieve $\sqrt{|\mu|} \ge 5C_{\max}\sqrt{2}/K$, $\lambda = \mu - \omega_1 \in \Sigma_{\pi-\varphi_0}$, where $C_{\max} := \max\{C_1, \ldots, C_4\}$. Consequently,

$$
\begin{aligned}
&\|\partial_3(\mu + \mathbf{A})^{-1}[\mathbf{B}(\mu + \mathbf{A})^{-1}]^{k+1}\mathbf{P}_+\partial_j f\|_{L^\infty(L^p)} \\
\le &\frac{K}{|\mu|^{\frac{1}{2}-\frac{1}{2p}}} \left(\frac{1}{\sqrt{2}}\right)^{k+1} \left(\|f\|_{W^{1,\infty}(L^{p/2})} + \|f\|_{L^\infty(W^{1,p/2})}\right)
\end{aligned}
$$

for $\mu - \omega_1 \in \Sigma_{\pi-\varphi_0}$, $j = 1, 2, 3$, and the lemma is proved. □

5. Nonlinear problem - local existence

In this section we prove Theorem 1.2.

For $v_0 \in \dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ choose $K > 2C_{\varphi_0} e^{\omega_1} ||v_0||_{\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^2;L^p(\mathbb{R}_+))}$. Here C_{φ_0} is the constant appearing in Theorem 3.4.
Put $Y := L^\infty((0,T); \mathcal{BUC}(\mathbb{R}^2; L^p(\mathbb{R}_+)))$. For $T > 0$ let

$$
X_{T,K} = \{v(x,t) \in Y;\ t^{1/2}\nabla v(x,t) \in Y,\ \mathrm{div} v = 0,\ v|_{\partial\mathbb{R}^n_+} = 0,\ ||v||_{X_T} < K\},
$$

where

$$
||v||_{X_T} := \sup_{0\le t\le T} ||v||_{L^\infty(L^p)}(t) + \sup_{0\le t\le T} t^{1/2}||\nabla v||_{L^\infty(L^p)}(t).
$$

Next define F by

$$
(Fv)(t) := e^{-t\mathbf{A}_E} v_0 + N(v,v)(t) \qquad \text{for } v \in X_{T,K},
$$

where

$$
N(v,w)(t) := -\int_0^t e^{-(t-s)\mathbf{A}_E} \mathbf{P}_+ \mathrm{div}(v \otimes w)(s) ds.
$$

To apply the contraction mapping principle we prepare the following estimates derived from Proposition 4.5.

Lemma 5.1. *Let p, φ_0, δ and $\omega_0 = \omega_0(\varphi_0)$ as in Proposition 4.5. There exists a constant $C = C(\varphi_0, \delta, p) > 0$ such that*

$$||N(v,w)(t)||_{L^\infty(L^p)} \le Ce^{t\omega_1}\{M_1 t^{1-\frac{1}{2p}-\delta} + M_2 t^{-\frac{1}{2p}-\delta+\frac{1}{2}}\}, \qquad (5.1)$$

$$||\nabla N(v,w)(t)||_{L^\infty(L^p)} \le Ce^{t\omega_1}\{M_1 t^{\frac{1}{2}-\frac{1}{2p}} + M_2 t^{-\frac{1}{2p}}\} \qquad (5.2)$$

for $t > 0$ and all $v, w \in X_{T,K}$. Here, $M_1 = M_v M_w$ and $M_2 = M_v' M_w + M_v M_w'$, where

$$M_u := \sup_{0\le s\le T} ||u||_{L^\infty(L^p)}(s) \quad \text{and} \quad M_u' := \sup_{0\le s\le T} s^{1/2}||\nabla u||_{L^\infty(L^p)}(s)$$

for $u \in X_{T,K}$.

Proof. We have by Proposition 4.5

$$||N(v,w)(t)||_{L^\infty(L^p)} = ||-\int_0^t e^{-(t-s)\mathbf{A}_E}\mathbf{P}_+\mathrm{div}(v\otimes w)(s)ds||_{L^\infty(L^p)}$$
$$\le e^{t\omega_1}\int_0^t ||e^{-(t-s)(\mathbf{A}_E+\omega_1)}\mathbf{P}_+\mathrm{div}(v\otimes w)(s)||_{L^\infty(L^p)}ds$$
$$\le C_{\varphi_0,\delta,p}e^{t\omega_1}\int_0^t (t-s)^{-\frac{1}{2p}-\delta}(||v\otimes w||_{W^{1,\infty}(L^{p/2})} + ||v\otimes w||_{L^\infty(W^{1,p/2})})(s)ds.$$

Since

$$\left(||v\otimes w||_{W^{1,\infty}(L^{p/2})} + ||v\otimes w||_{L^\infty(W^{1,p/2})}\right)(s)$$
$$\le C\left(||v\otimes w||_{L^\infty(L^{p/2})} + ||\nabla'(v\otimes w)||_{L^\infty(L^{p/2})} + ||\partial_3(v\otimes w)||_{L^\infty(L^{p/2})}\right)(s)$$
$$\le C\Bigg(||v||_{L^\infty(L^p)}||w||_{L^\infty(L^p)} + ||\nabla v||_{L^\infty(L^p)}||w||_{L^\infty(L^p)} + ||v||_{L^\infty(L^p)}||\nabla w||_{L^\infty(L^p)}\Bigg)(s),$$

we get

$$||N(v,w)(t)||_{L^\infty(L^p)} \le C_{\varphi_0,\delta,p}e^{t\omega_1}\Bigg\{M_vM_w\int_0^t (t-s)^{-\frac{1}{2p}-\delta}ds + (M_v'M_w + M_vM_w')\int_0^t (t-s)^{-\frac{1}{2p}-\delta}s^{-1/2}ds\Bigg\},$$

which implies (5.1). For (5.2) we similarly estimate by Proposition 4.5

$$||\nabla N(v,w)(t)||_{L^\infty(L^p)}$$
$$\le e^{t\omega_1}\int_0^t ||\nabla e^{-(t-s)(\mathbf{A}_E+\omega_1)}\mathbf{P}_+\mathrm{div}(v\otimes w)(s)||_{L^\infty(L^p)}ds$$
$$\le C_{\varphi_0,\delta,p}e^{t\omega_1}\int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2p}}(||v\otimes w||_{W^{1,\infty}(L^{p/2})}$$

$$
\begin{aligned}
&+||v \otimes w||_{L^\infty(W^{1,p/2})})(s)ds \\
&\leq C_{\varphi_0,\delta,p} e^{t\omega_1}\{M_v M_w \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2p}} ds \\
&\quad + (M_v' M_w + M_v M_w') \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2p}} s^{-1/2} ds\}.
\end{aligned}
$$

□

Finally we turn to the proof of our main result.

Proof of Theorem 1.2
First we show that $F : X_{T,K} \to X_{T,K}$ for $K > 2C_{\varphi_0} e^{\omega_1} ||v_0||_{\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^2;L^p(\mathbb{R}_+))}$ and $T > 0$ small enough. Assume $v \in X_{T,K}$. Applying Theorem 3.4, Lemma 5.1, and Lemma 2.3 results for $t > 0$

$$
\begin{aligned}
&||Fv||_{L^\infty(L^p)} \leq ||e^{-t\mathbf{A}_E} v_0||_{L^\infty(L^p)} + ||N(v,v)(t)||_{L^\infty(L^p)} \\
&\leq ||e^{-t\mathbf{A}_E} v_0||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} + ||N(v,v)(t)||_{L^\infty(L^p)} \\
&\leq C_{\varphi_0} e^{t\omega_1} ||v_0||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} + C_{\varphi_0,\delta,p} e^{t\omega_1} K^2 (t^{1-\frac{1}{2p}-\delta} + t^{-\frac{1}{2p}-\delta+\frac{1}{2}}).
\end{aligned}
$$

Here, we used the fact that the constants M_1 and M_2 in Lemma 5.1 are not larger than $(M_v + M_v')(M_w + M_w') = K^2$. Similarly we obtain

$$
\begin{aligned}
t^{1/2} ||\nabla Fv||_{L^\infty(L^p)} \leq\ & C_{\varphi_0} e^{t\omega_1} ||v_0||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \\
&+ C_{\varphi_0,\delta,p} e^{t\omega_1} K^2 t^{1/2} (t^{\frac{1}{2}-\frac{1}{2p}} + t^{-\frac{1}{2p}}).
\end{aligned}
$$

Our assumptions $2 < p < \infty$ and $\delta \in (0, \frac{1}{4})$ imply the powers of t to be positive. Consequently,

$$
\begin{aligned}
||Fv||_{X_T} &\leq 2C_{\varphi_0} e^{T\omega_1} ||v_0||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} + 4C_{\varphi_0,\delta,p} e^{T\omega_1} K^2 T^{-\frac{1}{2p}-\delta+\frac{1}{2}} \\
&\leq 2C_{\varphi_0} e^{\omega_1} ||v_0||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} + 4C_{\varphi_0,\delta,p} e^{\omega_1} K^2 T^{-\frac{1}{2p}-\delta+\frac{1}{2}} \\
&\leq K,
\end{aligned}
$$

for small enough $T > 0$. More precisely we have to demand

$$
T < \min \left\{ 1, \left(\frac{K - 2C_{\varphi_0} e^{\omega_1} ||v_0||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)}}{4C_{\varphi_0,\delta,p} e^{\omega_1} K^2} \right)^{\frac{1}{-\frac{1}{2p}-\delta+\frac{1}{2}}} \right\}. \tag{5.3}
$$

Clearly, $\operatorname{div} Fv = 0$ and $Fv|_{\partial\mathbb{R}^n_+} = 0$. Thus we have proved $Fv \in X_{T,K}$.

Next we show that F is a contraction. Let $v, w \in X_{T,K}$. Noting that $Fv - Fw = N(v,v)(t) - N(w,w)(t) = N(v, v-w)(t) - N(v-w, w)(t)$, we get by Lemma 5.1

$$
\begin{aligned}
&||Fv - Fw||_{L^\infty(L^p)} \\
&\leq ||N(v, v-w)(t)||_{L^\infty(L^p)} + ||N(v-w, w)(t)||_{L^\infty(L^p)}
\end{aligned}
$$

$$
\begin{aligned}
&\le C_{\varphi_0,\delta,p} e^{t\omega_1}\{M_v M_{v-w} t^{1-\frac{1}{2p}-\delta} + (M_v' M_{v-w} + M_v M_{v-w}')t^{-\frac{1}{2p}-\delta+\frac{1}{2}} \\
&\qquad + M_{v-w} M_w t^{1-\frac{1}{2p}-\delta} + (M_{v-w}' M_w + M_{v-w} M_w')t^{-\frac{1}{2p}-\delta+\frac{1}{2}}\} \\
&\le 2C_{\varphi_0,\delta,p} e^{t\omega_1} K ||v-w||_{X_T}(t^{1-\frac{1}{2p}-\delta} + t^{-\frac{1}{2p}-\delta+\frac{1}{2}}),
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&t^{1/2}||\nabla Fv - \nabla Fw||_{L^\infty(L^p)} \\
&\le t^{1/2}||\nabla N(v, v-w)(t)||_{L^\infty(L^p)} + t^{1/2}||\nabla N(v-w, w)(t)||_{L^\infty(L^p)} \\
&\le 2C_{\varphi_0,\delta,p} e^{t\omega_1} K ||v-w||_{X_T} t^{1/2}(t^{\frac{1}{2}-\frac{1}{2p}} + t^{-\frac{1}{2p}}).
\end{aligned}
$$

This yields

$$
||Fv - Fw||_{X_T} \le 8C_{\varphi_0,\delta,p} e^{\omega_1} K T^{-\frac{1}{2p}-\delta+\frac{1}{2}} ||v-w||_{X_T}.
$$

Hence the operator F is a contraction if

$$
T < \min\left\{1, \left(\frac{1}{8C_{\varphi_0,\delta,p} e^{\omega_1} K}\right)^{\frac{1}{-\frac{1}{2p}-\delta+\frac{1}{2}}}\right\}. \tag{5.4}
$$

The contraction mapping theorem now implies unique existence of a mild solution v of (1.19) such that $v \in X_{T,K}$ for $K > 2C_{\varphi_0} e^{\omega_1} ||v_0||_{\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^2;L^p(\mathbb{R}_+))}$ and $T > 0$ satisfying (5.3) and (5.4). It is easy to see the time interval determined by (5.3) and (5.4) is largest, if we choose $K = 4C_{\varphi_0,\delta,p} e^{\omega_1} \times ||v_0||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)}$, which gives the lower estimate for existence time (Remark 1.3).

Finally, let us show that the local-in-time solution v obtained above is continuous in time. By the strong continuity of $t \mapsto \mathrm{e}^{-t\mathbf{A}_E}$ on X_σ it follows immediately

$$
\|v(t) - v_0\|_{\mathcal{BUC}(\mathbb{R}^2;L^p(\mathbb{R}_+))} \to 0 \quad \text{if} \quad t \to 0. \tag{5.5}
$$

This follows easily by using the representation $v(t) = Fv(t)$. In order to see that $t \mapsto v(t)$ is continuous on the entire existence interval we have to prove the continuity of $t \mapsto Fu(t)$. To this end let $t > t_0 > 0$ and consider

$$
\begin{aligned}
&\|Fv(t) - Fv(t_0)\|_{\mathcal{BUC}(L^p)} \\
&\le \|(\mathrm{e}^{-t\mathbf{A_E}} - \mathrm{e}^{-t_0\mathbf{A_E}})v_0\|_{\mathcal{BUC}(L^p)} \\
&\quad + \int_0^{t_0} \|(\mathrm{e}^{-(t-s)\mathbf{A_E}} - \mathrm{e}^{-(t_0-s)\mathbf{A_E}})\mathbf{P}_+ \mathrm{div}(v(s)\otimes v(s))\|_{\mathcal{BUC}(L^p)} \mathrm{d}s \\
&\quad + \int_{t_0}^{t} \|\mathrm{e}^{-(t-s)\mathbf{A_E}}\mathbf{P}_+ \mathrm{div}(v(s)\otimes v(s))\|_{\mathcal{BUC}(L^p)} \mathrm{d}s \\
&= I_1(t) + I_2(t) + I_3(t).
\end{aligned} \tag{5.6}
$$

Here the problem becomes clear. Obviously, $I_1(t), I_3(t) \to 0$ if $t \to t_0$. But although we know the continuity of $t \mapsto \mathrm{e}^{-tA_E}$ in X_σ it is a priori not clear, whether $I_2(t) \to 0$ if $t \to t_0$, since we construct solutions in $\mathcal{BUC}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ and we may not assume that $\{\mathrm{e}^{-tA_E}\}_{t\geq 0}$ is strongly continuous in that space. We do not even expect e^{-tA_E} to be bounded in $\mathcal{BUC}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ in general. Nevertheless, we have the estimates in Proposition 4.5, which show that in this situation it is better to deal with the operator

$$T(t) := \mathrm{e}^{-t(\omega_1+\mathbf{A}_E)}\mathbf{P}_+\partial_j, \quad j = 1, 2, 3, \ t > 0.$$

For this operator the desired continuity can be proved.

Lemma 5.2. *Let* $2 < p < \infty$. *Then*

$$(t \mapsto T(t)f), (t \mapsto \sqrt{t}\nabla T(t)f) \in C((0,\infty); \mathcal{BUC}(\mathbb{R}^2; L^p(\mathbb{R}_+)))$$

for $f \in \mathcal{BUC}^1(\mathbb{R}^2; W^{1,p/2}(\mathbb{R}_+))$.

Proof. Observe that by Theorem 3.4 the operator $\mathbf{A}_E$ is the generator of a holomorphic semigroup on $\dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^2; L^p(\mathbb{R}_+))$. Since $\mathbf{P}_+\partial_j f \in \dot{\mathcal{B}}^0_{\infty,\infty}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ if $f \in \mathcal{BUC}^1(\mathbb{R}^2; W^{1,p/2}(\mathbb{R}_+))$, we therefore may write for $\omega_1 > 0$ large enough

$$T(t)f = \frac{1}{2\pi i}\int_\Gamma \mathrm{e}^{-t\lambda}(\lambda - (\omega_1 + \mathbf{A}_E))^{-1}\mathbf{P}_+\partial_j f \mathrm{d}\lambda, \quad j = 1, 2, 3, \ t > 0,$$

with $\Gamma := \{r\mathrm{e}^{i\theta}; \infty > r \geq 0\} \cup \{r\mathrm{e}^{-i\theta}; 0 \leq r < \infty\}$ and a proper $\theta \in (0, \pi/2)$. Applying estimate (4.9) this yields for $t, t_0 > 0$,

$$\begin{aligned}
&\|\nabla^\ell(T(t) - T(t_0))f\|_{\mathcal{BUC}(L^p)} \\
&\leq C\int_0^\infty |\mathrm{e}^{-t\lambda} - \mathrm{e}^{-t_0\lambda}|\|\nabla^\ell(\lambda - (\omega_1 + \mathbf{A}_E))^{-1}\mathbf{P}_+\partial_j f\|_{\mathcal{BUC}(L^p)}\mathrm{d}r \\
&\leq C\int_0^\infty \frac{|\mathrm{e}^{-t\lambda} - \mathrm{e}^{-t_0\lambda}|}{r^{1-\frac{\ell}{2}-\frac{1}{2p}-\delta(1-\ell)}}\mathrm{d}r\|f\|_{\mathcal{BUC}(W^{1,p/2})\cap\mathcal{BUC}^1(L^{p/2})}
\end{aligned}$$

for $\ell = 0, 1$, $\delta \in (0, 1/4)$, and $\lambda = r\mathrm{e}^{\pm i\theta}$. Note that $\sigma := 1 - \frac{\ell}{2} - \frac{1}{2p} - \delta(1-\ell) < 1$. Furthermore,

$$\frac{|\mathrm{e}^{-t\lambda} - \mathrm{e}^{-t_0\lambda}|}{r^\sigma} \leq \frac{|\mathrm{e}^{-|t-t_0|\lambda} - 1|\mathrm{e}^{-\min\{t,t_0\}r\cos\theta}}{r^\sigma} \leq \frac{2\mathrm{e}^{-\min\{t,t_0\}r\cos\theta}}{r^\sigma}$$

and for each $r \in (0, \infty)$

$$\frac{|\mathrm{e}^{-t\lambda} - \mathrm{e}^{-t_0\lambda}|}{r^\sigma} \to 0 \quad \text{if} \quad t \to t_0.$$

Lebesgue's dominated convergence theorem then results

$$\|\nabla^\ell(T(t) - T(t_0))f\|_{\mathcal{BUC}(L^p)} \to 0 \quad \text{if} \quad t \to t_0$$

for $\ell = 0, 1$ and $f \in \mathcal{BUC}(\mathbb{R}^2; W^{1,p/2}(\mathbb{R}_+)) \cap \mathcal{BUC}^1(L^{p/2})$, which yields the assertion. □

Now we turn to the proof of the continuity of our solutions. Lemma 5.2 implies that $I_2(t) \to 0$ if $t \to t_0$ in (5.6). Hence $t \mapsto Fu(t)$ is continuous and by similar arguments we see that also $t \mapsto \sqrt{t}\nabla Fu(t)$ is continuous on $(0, \infty)$ with values in $\mathcal{BUC}(\mathbb{R}^2; L^p(\mathbb{R}_+))$.

For the continuity of first order derivatives of our solution v at $t = 0$, it is easy to see that the nonlinear term $t^{1/2}||\nabla N(v, v)(t)||_{\mathcal{BUC}(L^p)}$ goes to 0 as $t \downarrow 0$, thanks to (5.2). However, it seems difficult to show that the linear term $t^{1/2}||\partial_3 e^{-t\mathbf{A}_E} v_0||_{\mathcal{BUC}(L^p)}$ tends to 0 as $t \downarrow 0$, since ∂_n and $e^{-t\mathbf{A}_E}$ do not commute. This is the reason why we show (1.23) only for tangential derivatives. For (1.23) we claim that for any $f \in \dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ there exists a sequence $\{f_\varepsilon\}_{\varepsilon>0} \subset \dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ such that

$$||f_\varepsilon - f||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \to 0 \quad \text{as } \varepsilon \downarrow 0, \qquad \text{and} \tag{5.7}$$

$$||\nabla' f_\varepsilon||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \le C\varepsilon^{-1/2}||f||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)}. \tag{5.8}$$

In fact, we may set $f_\varepsilon = e^{\varepsilon\Delta'} f$. Then (5.7) is clear and (5.8) follows from Lemma A.1 (2). The commutativity of the tangential derivatives and the semigroup $\mathrm{e}^{-t\mathbf{A}_E}$ implies that

$$\begin{aligned}
&t^{1/2}||\nabla' \mathrm{e}^{-t\mathbf{A}_E} v_0||_{\mathcal{BUC}(L^p)} \le t^{1/2}||\nabla' \mathrm{e}^{-t\mathbf{A}_E} v_0||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \\
&\le t^{1/2}||\nabla' \mathrm{e}^{-t\mathbf{A}_E}(v_0 - v_0^\varepsilon)||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} + t^{1/2}||\nabla' \mathrm{e}^{-t\mathbf{A}_E} v_0^\varepsilon||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \\
&\le Ct^{1/2}t^{-1/2}||v_0 - v_0^\varepsilon||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} + t^{1/2}||\mathrm{e}^{-t\mathbf{A}_E} \nabla' v_0^\varepsilon||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \\
&\le C||v_0 - v_0^\varepsilon||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} + t^{1/2}||\nabla' v_0^\varepsilon||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)} \\
&\le C\varepsilon + Ct^{1/2}\varepsilon^{-1/2}||f||_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^p)}.
\end{aligned}$$

Since $||\cdot||_{\dot{\mathcal{B}}^0_{\infty,\infty}(L^p)} \le ||\cdot||_{L^\infty(L^p)} \le ||\cdot||_{\dot{\mathcal{B}}^0_{\infty,1}(L^p)}$ is finite (Lemma 2.3), we send $t \downarrow 0$ to see that the RHS tends to 0 after putting $\varepsilon = t^{1/2}$. The proof of Theorem 1.2 is thus complete. □

Appendix A. Appendix

Appendix A.1. Heat kernel estimates

We claim the following estimate which was used in Lemma 4.1 and Lemma 4.4. The proof can be found in [10].

Lemma A.1. *Let $n \in \mathbb{N}$ and $\alpha > 0$. Then there exists $C_\alpha = C(\alpha) > 0$ such that*
(1) $||(-\Delta)^\alpha G_t(x)||_{\dot{B}^0_{1,1}(\mathbb{R}^n)} \le Ct^{-\alpha}$,
(2) $||(-\Delta)^\alpha e^{t\Delta} f||_{\dot{B}^0_{\infty,1}(\mathbb{R}^n)} \le Ct^{-\alpha}||f||_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)}$
for $t > 0$ and $f \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$.

Appendix A.2. Characterization of $\dot{B}^0_{\infty,0,\sigma}(L^p)$

It is well-known, that in the L^p-case, $1 < p < \infty$, there is the characterization

$$L^p_\sigma(\mathbb{R}^n_+) = \mathbf{P}_+(L^p(\mathbb{R}^n_+)) = \{u \in L^p(\mathbb{R}^n_+) : \mathrm{div} u = 0, u \cdot \nu|_{\partial\mathbb{R}^n_+} = 0\}, \quad (A.1)$$

where ν denotes the outer normal and the trace is understood in the sense of the generalized Gauss theorem. More precisely, this result yields a bounded operator $\gamma_\nu : E^p(\mathbb{R}^n_+) \to W^{-1/p}(\partial\mathbb{R}^n_+)$, where $E^p(\mathbb{R}^n_+) = \{u \in L^p(\mathbb{R}^n_+) : \mathrm{div} u \in L^p(\mathbb{R}^n_+)\}$ equipped with $\|u\|_p + \|\mathrm{div} u\|_p$, such that $\gamma_\nu u = u \cdot \nu|_{\partial\mathbb{R}^n_+}$ for smooth u. As it is also well-known, this fact remains true for a wide class of domains $\Omega \subset \mathbb{R}^n$, as for instance bounded or exterior domains of class C^1 (see e.g. [8], [9] for the details). Since in certain applications it can be helpful to have characterization (A.1), instead of having the very implicit definition (2.13) only, we would like to briefly explain how one can obtain such a characterization for the space $\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ as defined in (2.13), and which is the most important one in this note. So we intend to show

Lemma A.2. *Let $1 < p < \infty$.*

$$\begin{aligned} &\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+)) \\ &= \{u \in \dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+)) : \mathrm{div} u = 0, u \cdot \nu|_{\partial\mathbb{R}^n_+} = 0\} \end{aligned} \quad (A.2)$$

Proof. Of course, here we first have to give a meaning to the trace $u \cdot \nu|_{\partial\mathbb{R}^n_+} = 0$. It is not clear how to define a global trace operator γ_ν acting on $\dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$, since this space contains certain nondecaying functions. However, at least it can be defined locally. For this purpose we consider K in the class

$$\mathcal{M} := \{K \subset \mathbb{R}^n_+ : K \text{ bounded and of class } C^1, \mu(\partial K \cap \partial\mathbb{R}^n_+) \neq 0\},$$

where μ denotes the boundary measure on $\partial\mathbb{R}^n_+$. Since the restriction $u|_K$ of a function $u \in \dot{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$ satisfying $\mathrm{div} u = 0$ in $\mathbb{R}^n_+$ belongs to $E^p(K)$, the trace $\gamma_{\nu,K} u|_K$ is well defined as we explained above. And in that sense we can give a meaning to $u \cdot \nu|_{\partial\mathbb{R}^n_+} = 0$ for $u \in \dot{\mathcal{B}}^s_{r,q,\sigma}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$. To be precise we set

$$u \cdot \nu|_{\partial\mathbb{R}^n_+} = 0 \quad :\Longleftrightarrow \quad (\gamma_{\nu,K}(u|_K))|_{\partial K \cap \partial\mathbb{R}^n_+} = 0 \text{ for all } K \in \mathcal{M}.$$

Now, if $u \in \dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$, obviously $\mathrm{div} u = 0$, and since $(\mathbf{P}Eu)^n$ is an odd function we obtain by using representation (1.17) that $(\gamma_{\nu,K}(\mathbf{P}_+ u|_K))|_{\partial K \cap \partial\mathbb{R}^n_+} = 0$ for all $K \in \mathcal{M}$. Thus "$\subseteq$" is proved in (A.2). In order the see the inverse inclusion we use again representation (1.17). Observe that in view of $u \cdot \nu|_{\partial\mathbb{R}^n_+} = 0$ a straight forward calculation shows that $\partial_n e^- u^n = e^+ \partial_n u^n$ with e^+, e^- as given in Definition 1.1. This implies $\mathrm{div} Eu = e^+ \mathrm{div} u = 0$. Hence $\mathbf{P}Eu = Eu$, which yields $\mathbf{P}_+ u = r\mathbf{P}Eu = rEu = u$, and we conclude $u \in \dot{\mathcal{B}}^0_{\infty,1,\sigma}(\mathbb{R}^{n-1}; L^p(\mathbb{R}_+))$. $\square$

References

1. H. Amann: Operator-Valued Fourier Multipliers, Vector-Valued Besov Spaces, and Applications. *Math. Nachr.* **186**, 5–56 (1997)
2. H. Amann: Vector-Valued Distributions and Fourier Multipliers. *http://www.math.unizh.ch/amann/books.html*, 2003
3. J. Bergh and J. Löfström: *Interpolation Spaces. An Introduction.* Springer-Verlag, 1976
4. M. Cowling, I. Doust, A. McIntosh, and A. Yagi: Banach space operators with a bounded H^∞ functional calculus. *J. Austral. Math. Soc.* **60**, 51–89 (1996)
5. M.J.P. Cullen: New mathematical developments in atmosphere and ocean dynamics, and their application to computer simulation. *Large-Scale Atmosphere-Ocean Dynamics. Analytical Methods and Numerical Models, Volume I, edited by John Norbury and Ian Roulstone*, Cambridge University Press, 2002
6. R. Denk, M. Hieber, J. Prüss: R-Boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, **166**, viii-114 (2003)
7. W. Desch, M. Hieber, J. Prüss: L^p-Theory of the Stokes equation in a half-space. *J. Evol. Equ.* **1**, 115–142 (2001)
8. D. Fujiwara and H. Morimoto: An L^r theorem of the Helmholtz decomposition of vector fields. *J. Fac. Sci. Univ. Tokyo* **24**, Sect. IA, 685–700 (1977)
9. G.P. Galdi: *An Introduction to the Mathematical Theory of the Navier-Stokes Equations Volume I, Linearized Steady Problems.* Springer Verlag, 1994
10. Y. Giga, K. Inui, A. Mahalov and S. Matsui: Navier-Stokes Equations in a Rotating Frame in $\mathbb{R}^3$ with Initial Data Nondecreasing at Infinity, *Hokkaido Math. J.*, to appear.
11. H.P. Greenspan: *The Theory of Rotating Fluids.* Cambridge University Press, 1968
12. P. Grisvard: Commutativité de deux foncteurs d'interpolation et applications. *J. Math. Pures Appl.*, **45**, 143–290 (1966)
13. R. Haller: Poissonkern Abschätzungen und beschränkter H^∞-Kalkül, PhD Thesis, TU Darmstadt, *Wissenschaftlicher Verlag, Berlin* (2003)
14. J.R. Holton: *An Introduction to Dynamic Meteorology.* Academic Press, 1992
15. N. Kalton and L. Weis: The H^∞-calculus and sums of closed operators. *Math. Ann.* **321** 319–345 (2001)
16. H. Kozono and M. Yamazaki: Semilinear heat equations and Navier-Stokes equation with distributions in new function spaces as initial data. *Commun. Partial Differ. Equations* **19** No. 5-6, 959–1014 (1994)
17. A. Majda: *Introduction to PDE's and Waves for the Atmosphere and Ocean.* Courant Lecture Notes in Mathematics, 2003.
18. T. Muramatu: Besov spaces and Sobolev spaces of generalized functions defined on a general region. *Publ. R.I.M.S. Kyoto Univ.* **9**, 325–396 (1974)
19. J. Pedlosky: *Geophysical Fluid Dynamics, 2nd edition.* Springer–Verlag, 1987
20. J. Saal: Robin Boundary Conditions and Bounded H^∞-Calculus for the Stokes Operator. PhD Thesis, TU Darmstadt, *Logos Verlag, Berlin*, 2003.
21. J. Saal: Stokes and Navier-Stokes equations with Robin boundary conditions in a half-space. *J. Math. Fluid Mech.* to appear.
22. O. Sawada: The Navier-Stokes flow with linearly growing initial velocity in the whole space. *Bol. Soc. Paran. Mat.* **3**, 22 no. 2, 75–96 (2004)
23. O. Sawada and Y. Taniuchi: On the Boussinesq flow with nondecaying initial data. *Funkcialaj Ekvacioj* **47**, no. 2, 225–250 (2003)
24. Y. Shimizu: L^∞-Estimate of first-order space derivatives of Stokes flow in a half space. *Funkcialaj Ekvacioj* **42** 291–309 (1999)

25. V. A. Solonnikov: On nonstationary Stokes problem and Navier-Stokes problem in a half-space with initial data nondecreasing at infinity. *Journal of Mathematical Sciences* **114**, No. 5, 1726–1740 (2003)
26. E.M. Stein: *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press, 1993
27. Y. Taniuchi: Remarks on global solvability of 2-D Boussinesq equations with non-decaying initial data, preprint.
28. H. Triebel: *Theory of Function Spaces*. Monographs in Mathematics, **78**, Basel-Boston-Stuttgart: Birkhäuser Verlag, 1992
29. S. Ukai: A solution formula for the Stokes equation in $\mathbf{R}_3^+$. *Comm. on pure and Appl. Math.* **11** 611–621 (1987)

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