GLOBAL WEAK SOLUTIONS IN THREE SPACE DIMENSIONS FOR ELECTRO-KINETIC FLOW PROCESSES

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ABSTRACT. For a Navier-Stokes-Nernst-Planck-Poisson system we construct global weak solutions in a three-dimensional bounded domain. A special feature of our approach is that we allow for non-constant diffusion coefficients which may vary from species to species as well as for L^2-initial data without any further constraints. Our approach is based on the intrinsic energy structure, Aubin-Simon compactness arguments, and maximal L^p-regularity.

1. Introduction

A very important phenomenon, which is related to electro-kinetic flow processes, is displayed by *electro-osmosis*. It describes the motion of an aqueous solution (with charged solutes) past a solid wall as a response to an electrical field and provides various powerful features which are employed by many different branches of industry, e.g. micro- and nano-electronics, filtration processes as well as separation and mixing techniques in analytical chemistry, [11, 12, 25, 29]. For more information on the applicability of electro-osmosis and more generally electro-kinetic effects, we refer to [10, 31, 36] and the references therein.

The important role being played by electro-osmosis motivates us to investigate the solvability question of a rather general 3D-model for electro-kinetic flow phenomena. This model describes the evolution of a dilute viscous solution with dissolved charged species, which is placed in a container $\Omega \subset \mathbb{R}^3$ with solid walls, the container being situated in an electrical field. The unknowns are the velocity field u, the pressure π, the species concentration c_i of species X_i and the electrical potential Φ. The following system of partial

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differential equations for the just mentioned unknowns will be examined.

$$\left.\begin{array}{rcll} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla\pi + \sum_j z_j c_j \nabla\Phi &=& 0, & t>0,\ x\in\Omega, \\ \operatorname{div} u &=& 0, & t>0,\ x\in\Omega, \\ u &=& 0, & t>0,\ x\in\partial\Omega, \\ u(0) &=& u^0, & x\in\Omega. \end{array}\right\}, \tag{1.1}$$

$$\left.\begin{array}{rcll} \partial_t c_i + \operatorname{div}\left(-d_i\nabla c_i - d_i z_i c_i \nabla\Phi + c_i u\right) &=& 0, & t>0,\ x\in\Omega, \\ \partial_\nu c_i + z_i c_i \partial_\nu\Phi &=& 0, & t>0,\ x\in\partial\Omega, \\ c_i(0) &=& c_i^0, & x\in\Omega. \end{array}\right\}, \tag{1.2}$$

$$\left.\begin{array}{rcll} -\Delta\Phi - \sum_j z_j c_j &=& \sigma, & t>0,\ x\in\Omega, \\ \partial_\nu\Phi + \tau\Phi &=& \xi, & t>0,\ x\in\partial\Omega. \end{array}\right\}. \tag{1.3}$$

System (1.1) represents the incompressible *Navier-Stokes* equations for the velocity field u and pressure π subject to no-slip boundary conditions, where the total density and the viscosity are set to 1 for simplicity. We denote the charge number of species X_i by $z_i(\in\mathbb{Z})$. By convention the electrical field is $-\nabla\Phi$, so the term $-\sum_j z_j c_j \nabla\Phi$ represents the Coulomb force exerted on the fluid by the (scaled) local charge distribution $\sum_j z_j c_j$, which is induced by the species mixture.

The evolution of the molar concentration c_i is modeled by system (1.2), a system of electro-diffusion-advection equations. They represent species mass balances with fluxes according to the *Nernst-Planck* equation. More precisely, the mass flux j_i of constituent X_i is composed of a Fickian diffusion term $-d_i\nabla c_i$ and an electro-migration term $-d_i z_i c_i \nabla\Phi$ due to the electrical field, where d_i denotes the diffusivity of species X_i. The total flux J_i is then the sum of mass flux j_i and the convective flux $c_i u$ from fluid movement. The absence of mass flux through the boundary is displayed by the so-called no-flux boundary condition; as a consequence total mass of each species is conserved.

Finally the electrical potential is determined via the *Poisson* problem (1.3), an elliptic boundary value problem which comes from Maxwell's equations of electro-statics. The right-hand side σ denotes the (given) charge distribution within Ω and again $\sum_j z_j c_j$ is the charge distribution resulting from the species mixture. By $\tau > 0$ we mean the (constant) capacity of the boundary and ξ is a given datum connected with an external electrical field. This boundary condition can be motivated by the well-known fact that electro-chemical double layers are generally present on the boundary, i.e. the boundary is expected to be charged, see [27]. Accordingly, we may consider the boundary $\partial\Omega$ locally as plate capacitor which results in a Robin boundary condition, see e.g. [10] for more details.

For a derivation of system (1.1)-(1.3) on the basis of fundamental mass and momentum balances and the system of Maxwell-Stefan equations for the determination of diffusive flux for species mass, we refer to [10]. Its applicability for the non-dilute case is discussed in [16].

For the description of electrolyte solutions the so-called *electro-neutrality* condition $\sum_i z_i c_i = 0$ has been used frequently in the mathematical literature, see e.g. [3, 11, 37]. However, this simplification seems not to be adequate when modeling electro-kinetic flows, since the electric force being exerted on the fluid vanishes and so no fluid motion can be expected, i.e., the Nernst-Planck system decouples from the Navier-Stokes equations.

In papers of P. Biler with W. Hebisch, T. Nadzieja and with J. Dolbeault [5, 7, 6], Y.S. Choi and R. Lui, see [13, 14], as well as in papers of H. Gajewski, A. Glitzky, K. Gröger and R. Hünlich, e.g. [17, 21, 18, 22, 23], the electro-neutrality condition is not employed. Instead, a system consisting of the Nernst-Planck equations and a Poisson equation, system (1.2)-(1.3) with $u = 0$ and $\sigma = 0$, is investigated. Thus, no fluid motion is taken into account and, hence, no momentum balance is accounted for. Employing suitable Lyapunov functionals, the existence of unique global solutions in two dimensions is proven and a careful analysis of the long-time behavior is provided. The situation in higher space dimensions is treated in the work [9], where global weak solutions are constructed.

In [10, 24, 30, 31] the afore mentioned system of Nernst-Planck equations and the Poisson equation is complemented by the Navier-Stokes system. However, apart from the first one, those works allow only for two oppositely charged species with equal and constant diffusivities, which simplifies the situation considerably, in general. From the physical point of view particularly the latter assumption is rather strong. In fact, the diffusivity of one species will in general depend on the full composition of the system, cf. [15]. To be more precise, the values of the diffusivities of individual constituents can differ significantly, up to orders of magnitude. Apart from different charge numbers in general, it is therefore significant to model each of the species concentrations. While in [30, 31] the electrostatic potential is assumed to satisfy homogeneous Neumann and Dirichlet conditions respectively, [24] accounts for mixed Dirichlet-Neumann boundary conditions, however, the connection of those setups to the possible occurrence of double layers at the boundary is not clear. In fact, e.g. homogeneous Neumann conditions even imply the boundary to remain electrically neutral, thus, boundary charges are even ruled out.

In [10] the Navier-Stokes-Nernst-Planck-Poisson system in the situation with N species with constant and possibly different diffusivities for different species is examined. It seems that so far [10] is the first work that considers the Navier-Stokes-Nernst-Planck-Poisson system in this generality. Local well-posedness in any space dimension and global well-posedness in two dimensions in terms of strong L^2-solutions are established and exponential convergence to unique steady states is proved. Corresponding results in three dimensions have only been shown under somewhat restrictive additional assumptions. These include:

(i) The initial data are small, regular and sufficiently close to the steady state solution, [30], see also [7].
(ii) Only two oppositely charged species with equal and constant diffusivities are present, [30, 31].
(iii) There is an a priori bound on c in $L^\infty(0,\infty;L^2(\Omega))$, [30], see also [14].

Existence of unique global-in-time solutions in three dimensions without any additional assumptions cannot be expected for two reasons. Firstly, as it is well-known, the question for global well-posedness for the Navier-Stokes system in 3D is unsolved up to the present day. Secondly, the nonlinearity in the Nernst-Planck equation - similarly to the Navier-Stokes case - prevents the derivation of suitable global (strong) estimates via the energy method. A sufficient criterion to guarantee those estimates is assumption (iii), for more details see [10, 14].

The goal of this paper is the construction of a global weak solution to system (1.1)-(1.3) without any of those restrictions. In other words, we allow for arbitrary L^2-initial data, we consider the N-species case, where the corresponding diffusivities are not restricted to being constant or the same for different species, and we do not suppose any a priori bounds.

The key ingredient in our approach is the energy function V_0, defined by

$$V_0 := \frac{1}{2}\int_\Omega |u|^2 + \int_\Omega \sum_{i=1}^{N} c_i \log c_i + \frac{1}{2}\int_\Omega |\nabla\Phi|^2 + \frac{\tau}{2}\int_{\partial\Omega} |\Phi|^2. \tag{1.4}$$

It turns out that formally V_0 is a Lyapunov functional on system (1.1)-(1.3), cf. [10, 30]. In other words, whenever (u,π,c,Φ) solves system (1.1)-(1.3), its time derivative $\frac{\mathrm{d}}{\mathrm{d}t}V_0$, the dissipation rate, is nonpositive. By integrating the dissipation rate over 0 to T in time, we directly obtain one further natural a priori bound for the solution of (1.1)-(1.3). In general, no further energy estimates are at hand. However, we will show that those estimates - the bounds on V_0 and $\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}V_0$ - are sufficient to find a global weak solution.

This article is organised as follows. In Section 2 we state our notation and give a precise statement of our main result. Section 3 quotes to some analytical results which will be employed in the sequel. For the existence of a global weak solution to system (1.1)-(1.3) we construct global solutions for an approximate version of (1.1)-(1.3) in Section 4. This is done in several steps. First we consider the corresponding approximate Nernst-Planck-Poisson subsystem and show global well-posedness in Subsection 4.1. This will allow for finding unique local-in-time solutions for the full approximate system in Section 4.2. Then energy estimates on the basis of the function V_0, defined in (1.4), are carried out in Section 4.3. This implies that the just mentioned local-in-time solutions exist globally. Finally, Section 5 contains the proof of our main result, which is based on maximal L^p-regularity and relative compactness of approximate solutions.

2. Preliminaries and Main Result

Let us fix some notation. Throughout this work $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, denotes a domain with boundary $\partial\Omega$. Time-space cylinders are usually written as $Q_T = (0,T) \times \Omega$ and $\Sigma_T = (0,T) \times \partial\Omega$ for $T \in (0,\infty)$. Spaces of continuous functions will be denoted by $C(\Omega)$ in a standard way and we write $C_0^\infty(\Omega)$ for the space of smooth functions with compact support defined on Ω. Note that $C^\infty(\overline{\Omega}) = \{v|_\Omega : \ v \in C_0^\infty(\mathbb{R}^n)\}$, if Ω is a bounded domain. We write $L^p(\Omega)$ and $W^{m,p}(\Omega)$ for the usual Lebesgue and Sobolev spaces $1 \le p \le \infty$ and $m \in \mathbb{N}_0$. The space of smooth compactly supported solenoidal vector fields is denoted by $C_{0,\sigma}^\infty(\Omega)$. We write

$$L^p_\sigma(\Omega) := \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{L^p}}$$

for the space of solenoidal L^p-functions. For $s \in \mathbb{R}_+ \backslash \mathbb{N}_0$ we define the Sobolev-Slobodeckii spaces $W^s_p(\Omega) = (L^p(\Omega), W^{m,p}(\Omega))_{p,s/m}$ and the Bessel potential spaces $H^s_p(\Omega) = [L^p(\Omega), W^{m,p}(\Omega)]_{s/m}$, see e.g. [1, 35], where $m \in \mathbb{N}$ with $m > s$ and $(\cdot,\cdot)_{p,\theta}$ and $[\cdot,\cdot]_\theta$ denotes the real and the complex interpolation functor respectively, see [4]. We write $C(\Omega)^+, L^p(\Omega)^+$, etc. for positive cones of nonnegative functions. We define for $m \in \mathbb{N}$ and $1 < p < \infty$ the dual spaces $W^{-m,p}(\Omega) := (W_0^{m,p'}(\Omega))'$ and $W_0^{-m,p}(\Omega) := (W^{m,p'}(\Omega))'$, where $\frac{1}{p} + \frac{1}{p'} = 1$. We do not distinguish between spaces of scalar functions and spaces of vector fields, i.e. we write also $L^p(\Omega)$ for $L^p(\Omega)^n$, for example. For $r \ge 0$ we also set

$$W^{r,p}_{loc}(\Omega) := \{u : \Omega \to \mathbb{R}^n;\ u \in W^{r,p}(K) \text{ for every compact } K \subset \Omega\}.$$

For a Banach space X the corresponding X-valued function spaces are denoted by $C(\Omega, X)$, $L^p(\Omega, X)$, etc.

In view of the affinely linear character of the Poisson equation (1.3) we split the potential Φ into a c-independent part Φ_1 and a c-dependent part Φ_2, such that $\Phi = \Phi_1 + \Phi_2$ and

$$-\Delta\Phi_1 = \sigma \qquad \text{in } \Omega, \qquad \partial_\nu\Phi_1 + \tau\Phi_1 = \xi \quad \text{on } \partial\Omega, \tag{2.1}$$

$$-\Delta\Phi_2 = \sum_{j=1}^N z_j c_j \quad \text{in } \Omega, \qquad \partial_\nu\Phi_2 + \tau\Phi_2 = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

Note that (1.3) is equivalent to (2.1)-(2.2).

Having this notation at hand we are in position to formulate our main result.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded C^{4-}-domain. Let the data to problem (1.1)-(1.3) satisfy the following conditions:*

(a) $d_i \in L^\infty_{loc}(\mathbb{R}_+; L^\infty(\Omega))$ such that for all $T > 0$ there is a positive number $\overline{d}_T$ with

$$0 < \overline{d}_T^{-1} \le d_i(t,x) \le \overline{d}_T < \infty \qquad \text{for a.e. } (t,x) \in Q_T. \tag{2.3}$$

(b) $(u^0, c^0) \in L^2_\sigma(\Omega) \times L^2(\Omega)^+$

(c) $(\sigma, \xi) \in L^r(\Omega) \times W_r^{1-1/r}(\partial\Omega)$ for some $r > 3$.

Then there exist

$$(u,c) \in L^\infty(0,\infty; L^2_\sigma(\Omega) \times L^1(\Omega)^+),$$
$$\Phi = \Phi_1 + \Phi_2 \in W^{2,r}(\Omega) + L^\infty(0,\infty; W^{1,2}(\Omega)),$$
$$\pi = \partial_t \widehat{\pi} \quad \text{with} \quad \widehat{\pi} \in L^{4/3}_{loc}([0,\infty); L^2_{loc}(\Omega)),$$

such that (1.1)-(1.3) is satisfied in the following sense:

- *For all $T \in (0,\infty)$, $u \in L^2(0,T; W_0^{1,2}(\Omega))$, $c \in L^1(0,T; W^{1,\frac{3}{2}}(\Omega))$, $\Phi_2 \in L^1(0,T; W^{3,\frac{3}{2}}(\Omega)) \cap C([0,T]; L^p(\Omega))$, $p \in [1,6)$.*
- *The couple (u,c) satisfies (1.1)-(1.2) in the following sense: For all $\phi \in C^1([0,T]; C^\infty_{0,\sigma}(\Omega))$ and $\psi \in C^1([0,T], C^\infty(\Omega))$ with $\phi(T) = \psi(T) = 0$ we have*

$$\int_{Q_T} -u\partial_t\phi + \nabla u \cdot \nabla\phi + (u\cdot\nabla)u\cdot\phi = \int_\Omega u^0\phi(0) - \sum_{j=1}^N z_j \int_{Q_T} c_j \nabla\Phi\cdot\phi, \tag{2.4}$$

$$\int_{Q_T} -c_i\partial_t\psi + (d_i\nabla c_i + d_i z_i c_i \nabla\Phi - c_i u)\nabla\psi = \int_\Omega c_i^0\psi(0). \tag{2.5}$$

- *Φ_1 is the unique strong solution to (2.1) and Φ_2 is the unique strong solution to (2.2) in the sense that for a.e. $t \in \mathbb{R}_+$*

$$-\Delta\Phi_2 = \sum_{j=1}^N z_j c_j \quad \text{in } \Omega, \qquad \partial_\nu\Phi_2 + \tau\Phi_2 = 0 \quad \text{on } \partial\Omega, \tag{2.6}$$

  i.e. $\Phi = \Phi_1 + \Phi_2$ is a strong L^p-solution to (1.3).
- *π is an associated pressure, i.e. (1.1) is satisfied in a distributional sense.*

Remark 1. We give a remark why we restrict our main result to the physically relevant cases of dimension $n \le 3$. The crucial point for this restriction resides in the estimation of the term $\mathcal{J}_i^2$ in (4.27) which essentially exploits compactness of the embedding $W^{1,2}(\Omega) \hookrightarrow L^{8/3}(\partial\Omega)$. Already for dimension $n = 4$ this embedding becomes sharp, hence no compactness is available. To the authors it is not clear if then estimate (4.17) still holds true. This, however, essentially enters in the proof of compactness of the approximating sequence for the weak solution given through Theorem 2.1 (see Section 5). For the pure Nernst-Planck-Poisson system (i.e. without Navier-Stokes) the problem of global-in-time solvability in higher dimensions is already addressed in [9]. There the authors circumvent the appearance of boundary integrals of the form $\mathcal{J}_i^2$ by constructing spacially local solutions in the interior of Ω.

3. Tools for the proof

Because of the divergence-free condition and the no-slip boundary condition for the velocity field u we may reformulate (1.1) as an evolution equation with the help of the Stokes operator $A_S = -P\Delta$, where P is the Helmholtz projection, see e.g. [19, 33], and

$$\mathcal{D}(A_S) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \cap L^p_\sigma(\Omega).$$

More precisely, (1.1) is equivalent to solving

$$\left.\begin{array}{rcll} \partial_t u + A_S u + P(u \cdot \nabla)u + \sum_j z_j P(c_j \nabla\Phi) & = & 0, & t > 0, \\ u(0) & = & u^0. & \end{array}\right\}. \tag{3.1}$$

The associated pressure π can then be recovered by well-known methods, see [33].

Let us recall the following well-known properties of the Stokes operator, see e.g. [20, 28, 33, 34].

Proposition 1. *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded C^3-domain. Then $(-\infty, 0] \subset \rho(A_S)$ and A_S is sectorial, i.e., for every $\theta \in (0, \pi)$ there is $C = C(p, \varphi_0) > 0$ such that*

$$\|\lambda(\lambda + A_S)^{-1}\|_{\mathcal{L}(L^p_\sigma)} \le C \quad (\lambda \in \Sigma_\theta),$$

where $\Sigma_\theta = \{z \in \mathbb{C} \setminus \{0\} : \ |\arg z| < \theta\}$. Furthermore, A_S enjoys the property of maximal regularity, i.e. for any $f \in L^p(0, T; L^p_\sigma(\Omega))$ there is a solution $w \in W^{1,p}(0, T; L^p_\sigma(\Omega)) \cap L^p(0, T; \mathcal{D}(A_S))$ to the problem

$$\frac{d}{dt} w + A_S w = f, \quad t > 0, \quad w(0) = 0.$$

For its fractional powers it holds true that for $\alpha \in (0, 1)$,

$$\mathcal{D}(A^\alpha_{S,(p)}) = H_p^{2\alpha}(\Omega) \cap H^\alpha_{p,0}(\Omega) \cap L^p_\sigma(\Omega). \tag{3.2}$$

An Aubin-Simon compactness result will be employed frequently throughout this article. For convenience we state it here in the form of [32, Corollary 4].

Proposition 2. *Let $T \in (0, \infty)$ and X_0, Y, X_1 be Banach spaces such that X_0 is compactly embedded in Y and Y is continuously embedded in X_1.*

(i) *Let $1 \le p < \infty$. If F is bounded in $L^p(0,T;X_0)$ and $\frac{d}{dt}F := \{\frac{d}{dt}f : f \in F\}$ is bounded in $L^1(0,T;X_1)$, then F is relatively compact in $L^p(0,T;Y)$.*

(ii) *If F is bounded in $L^\infty(0,T;X_0)$ and $\frac{d}{dt}F$ is bounded in $L^r(0,T;X_1)$ for some $r > 1$, then F is relatively compact in $C([0,T];Y)$.*

4. Global well-posedness for an approximate problem

The goal of this section is to show the existence of global solutions to an approximate version of (1.1)-(1.3). As we will see, employing smoothing operators represented by resolvents of the Robin-Laplacian and the Stokes operator serve well for this purpose. Global well-posedness of the perturbed version is based on

(i) the Leray-Schauder fixed-point theorem for unique global weak solutions for the Nernst-Planck-Poisson subsystem, Section 4.1,

(ii) the contraction mapping principle in order to have unique local-in-time solutions for the full approximate system, Section 4.2,

(iii) energy estimates from a perturbed version of V_0, which show that the local-in-time solutions can be uniquely extended up to any finite time, Sections 4.3-4.4.

For the formulation of the just mentioned approximate version of (1.1)-(1.3) we introduce the following notation. For $\alpha \in (0,1)$ and $\varepsilon > 0$ we set $R_\varepsilon := (1+\varepsilon A_S)^{-1}$, $R_\varepsilon^\alpha := (1+\varepsilon A_S^\alpha)^{-1}$ (Throughout, no powers of R_ε will occur, so this should not cause any confusion.), $B_\varepsilon := 1 - \varepsilon\Delta_R$ and $S_\varepsilon := B_\varepsilon^{-1}$. Remark that due to $n = 3$ and Sobolev's embedding theorem we have

$$\|R_\varepsilon u\|_{L^\infty(\Omega)} \le C\|R_\varepsilon u\|_{W^{2,2}(\Omega)} \le C\|u\|_{L^2_\sigma(\Omega)}, \qquad u \in L^2_\sigma(\Omega). \tag{4.1}$$

For $1 < p < \infty$ we have

$$\|R_\varepsilon v\|_{L^p_\sigma(\Omega)} \le C\|v\|_{L^p_\sigma(\Omega)}, \quad \|S_\varepsilon w\|_{L^p(\Omega)} \le C\|w\|_{L^p(\Omega)}$$

for $v \in L^p_\sigma(\Omega),\ w \in L^p(\Omega)$, where here $C > 0$ does not depend on ε. For R_ε^α we observe that

$$\|R_\varepsilon^\alpha v\|_{L^p_\sigma(\Omega)} \le C\|v\|_{L^p_\sigma(\Omega)}, \qquad v \in L^p_\sigma(\Omega), \quad \alpha \in (0,1),$$

where $C > 0$ is also independent of $\varepsilon > 0$, see [2, Corollary 4.6.11].

With this notation at hand we build up an approximate version of (1.1)-(1.3) as follows.

$$\left.\begin{array}{rcll} \partial_t u + A_S u + P(R_\varepsilon u\cdot\nabla)u + \sum_j z_j R_\varepsilon^{1/2}(P(c_j\nabla\Phi)) &=& 0, & t>0,\\ u(0) &=& u^0. & \end{array}\right\}, \tag{4.2}$$

$$\left.\begin{array}{rcl}\partial_t c_i + \operatorname{div}(-d_i\nabla c_i - d_i z_i c_i \nabla\Phi + c_i R_\varepsilon^{1/2} u) &=& 0, \quad t>0,\ x\in\Omega,\\ \partial_\nu c_i + z_i c_i \partial_\nu \Phi &=& 0, \quad t>0,\ x\in\partial\Omega,\\ c_i(0) &=& c_i^0, \quad x\in\Omega.\end{array}\right\},\tag{4.3}$$

$$\left.\begin{array}{rcl}-\Delta\Phi_2 &=& S_\varepsilon \sum_j z_j c_j, \quad t>0,\ x\in\Omega,\\ \partial_\nu\Phi_2 + \tau\Phi_2 &=& 0, \qquad\qquad t>0,\ x\in\partial\Omega.\end{array}\right\},\tag{4.4}$$

where $\Phi = \Phi_1 + \Phi_2$ and Φ_1 solves (2.1).

Remark 2. The role of the smoothing terms in system (4.2)-(4.4) is not exclusively for regularity reasons. Indeed, it would be possible to construct a local-in-time solution without $R_\varepsilon^{1/2} u$ in (4.3). However, the regularization of $c_i u$ incorporated in (4.3) is present in accordance to the smoothing of the term $\sum_j z_j c_j \nabla\Phi$ in (4.2), see the proof of Lemma 4.3. In other words, it keeps the energy structure, which in turn allows us to extend the local solution.

4.1. **The Nernst-Planck-Poisson subsystem.** In order to construct a global solution to (4.2)-(4.4) we first investigate existence and uniqueness of global weak solutions to the following subsystem of Nernst-Planck-Poisson:

$$\left.\begin{array}{rcl}\partial_t c_i + \operatorname{div}(-d_i\nabla c_i + c_i v_i - d_i z_i c_i \nabla\Phi_2) &=& 0, \quad t>0,\ x\in\Omega,\\ -d_i\partial_\nu c_i + c_i v_i\cdot\nu - d_i z_i c_i \partial_\nu\Phi_2 &=& 0, \quad t>0,\ x\in\partial\Omega,\\ c_i(0) &=& c_i^0, \quad x\in\Omega.\end{array}\right\},\tag{4.5}$$

$$\left.\begin{array}{rcl}-\Delta\Phi_2 &=& S_\varepsilon(\sum_j z_j c_j), \quad t>0,\ x\in\Omega,\\ \partial_\nu\Phi_2 + \tau\Phi_2 &=& 0, \qquad\qquad\quad t>0,\ x\in\partial\Omega.\end{array}\right\},\tag{4.6}$$

where v_i is a given function.

Lemma 4.1. *Let $\Omega\subset\mathbb{R}^3$ be a bounded C^{4-}-domain, $T\in(0,\infty)$, $s>3$ and let the following conditions hold true:*

- $d_i \in C(\overline{Q_T};(0,\infty))$ *satisfying (2.3) with* $\nabla d_i \in L^\infty(0,T;L^s(\Omega))$.
- $v_i \in L^\infty(0,T;L^s(\Omega))$.
- $c^0 \in L^\infty(\Omega)^+$.

Then for the problem (4.5), (4.6) there exist a unique vector of concentrations $c\in L^\infty(Q_T)^+\cap L^2(0,T;W^{1,2}(\Omega))$ with $\partial_t c\in L^2(0,T;W_0^{-1,2}(\Omega))$ and a $\Phi_2\in C([0,T],W^{4,2}(\Omega))$, such that (4.5) is satisfied in the sense that for all $\psi\in C^\infty(\overline{Q_T})$ such that $\psi(T)=0$,

$$\int_{Q_T} -c_i\partial_t\psi + (d_i\nabla c_i + c_i v_i + d_i z_i c_i\nabla\Phi_2)\nabla\psi = \int_\Omega c_i^0\psi(0), \qquad i=1,\ldots,N,\tag{4.7}$$

and (4.6) is satisfied in a pointwise sense.

Proof. Existence. We will use Leray-Schauder's fixed-point theorem. To this end let us fix $3 < r < \infty$, set $X := L^\infty(0,T;W^{1,r}(\Omega))$, and let $\Phi_2 \in X$. According to [26, Theorem 5.1 on page 170], there exists a unique (non-negative) $c \in L^\infty(Q_T)^+ \cap L^2(0,T;W^{1,2}(\Omega))$ with $\partial_t c \in L^2\big(0,T;W_0^{-1,2}(\Omega)\big)$, satisfying (4.5) with data Φ_2 in the sense of (4.7), see also [8]. Moreover, $c \in C([0,T];L^2(\Omega))$, so we can define $\widehat{\Phi}_2 \in C([0,T];W^{4,2}(\Omega))$ as the solution of (4.6) with data c. Note that we have $W^{4,2}(\Omega) \hookrightarrow C^1(\overline{\Omega}) \hookrightarrow W^{1,r}(\Omega)$, which is even compact. Therefore we can define

$$\mathcal{T} : X \to X,\ \Phi_2 \mapsto \widehat{\Phi}_2.$$

Let us show that $\mathcal{T}$ maps bounded sets into relatively compact ones. To this end suppose $(\Phi_2^m)_{m\in\mathbb{N}}$ is a bounded sequence in X. Let c^m be the solution to (4.5) with data Φ_2^m and let $\widehat{\Phi}_2^m = \mathcal{T}\Phi_2^m$. Using [8], $\partial_t c^m$ is bounded in $L^2(0,T;W_0^{-1,2}(\Omega))$ and c^m is bounded in $L^\infty(Q_T)$. Differentiating (4.6) in time $\partial_t\widehat{\Phi}_2^m$ is bounded in $L^2(0,T;W^{3,2}(\Omega))$. Since c^m is also bounded in $L^\infty(Q_T)$, using elliptic L^p-regularity, $\widehat{\Phi}_2^m$ is bounded in $L^\infty(0,T;W^{4,p}(\Omega))$ for any $p < \infty$. Choosing $p = 2$ we already observed that the embedding $W^{4,2}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ is compact. Then, using Aubin-Simon, see Proposition 2, the set $\big\{\widehat{\Phi}_2^m, m \in \mathbb{N}\big\}$ is relatively compact in X.

To prove continuity of $\mathcal{T}$, let $\Phi_2^m \to \Phi_2$ in X. As a consequence $\big\{\widehat{\Phi}_2^m = \mathcal{T}\Phi_2^m,\ m \in \mathbb{N}\big\}$ is relatively compact in X. Let $\widehat{\Phi}_2$ be a limit point. Similarly as before, the estimates mentioned above guarantee that $(c^m)_{m\in\mathbb{N}}$ is bounded in $L^\infty(Q_T)$ and relatively compact in $L^2(Q_T)$ by Aubin-Simon. Therefore, we may extract a subsequence that converges a.e. and in any $L^p(Q_T)$ for $p < \infty$ to a limit c, and such that $\nabla c^m \to \nabla c$ weakly in $L^2(Q_T)$. Then we pass to the limit $m \to \infty$ in (4.7) and using uniqueness for linear parabolic equations from [26], c is *the* solution of (4.5) with data Φ_2. Then we pass to the limit $m \to \infty$ in equation (4.6), so that $\Phi_2^m \to \Phi_2$, where Φ_2 is the solution of (4.6) with data c, which yields $\widehat{\Phi}_2 = \mathcal{T}\Phi_2$. Thus, the only possible limit point for $(\mathcal{T}\Phi_2^m)_{m\in\mathbb{N}}$ is $\mathcal{T}\Phi_2$ and $(\mathcal{T}\Phi_2^m)_{m\in\mathbb{N}}$ lies in a compact subset of X. This implies $\mathcal{T}\Phi_2^m \to \mathcal{T}\Phi_2$, whence the continuity of $\mathcal{T}$.

Now, let $\lambda \in [0,1]$ and $\Phi_2^\lambda = \lambda\mathcal{T}\Phi_2^\lambda$ with $\Phi_2^\lambda \in X$. Let us denote the solution to (4.5) with data Φ_2^λ by c^λ. By integration of (4.5) on Q_t for any $t \in (0,T)$, we have

$$\int_\Omega c_i^\lambda(t) = \int_\Omega c_i^0, \qquad i = 1,\ldots,N. \tag{4.8}$$

Nonnegativity of c^λ implies that $(c^\lambda)_\lambda$ is bounded in $L^\infty(0,T;L^1(\Omega))$. Using elliptic L^1- and L^p regularity theory, for equation (4.6), $(\Phi_2^\lambda)_\lambda$ is bounded in $L^\infty(0,T;W^{3,q}(\Omega))$ for any $q \in [1,\frac{3}{2})$, since Φ_2^λ satisfies (4.6) with data

$\lambda S_\varepsilon \sum_j z_j c_j^\lambda$ instead of $S_\varepsilon \sum_j z_j c_j$. Thus, $(\Phi_2^\lambda)_\lambda$ is bounded in

$$L^\infty(0,T;W^{3,1}(\Omega)) \hookrightarrow L^\infty(0,T;W^{1,r}(\Omega)).$$

Therefore, the set $\{\Phi_2 \in X : \ \exists \lambda \in [0,1] : \ \Phi_2 = \lambda \mathcal{T}\Phi_2\}$ is bounded in X. By Leray-Schauder's fixed-point theorem, $\mathcal{T}$ has a fixed point Φ_2. Thus, the couple (c, Φ_2) satisfies (4.5)-(4.6) in the sense as claimed and by construction we have $\Phi_2 \in C([0,T]; W^{4,2}(\Omega))$.

Uniqueness. Let $(c, \Phi_2), (\widehat{c}, \widehat{\Phi}_2)$ be two solutions to (4.5)-(4.6). Then we have

$$\int_{Q_T} -(c_i - \widehat{c}_i)\psi_t + \nabla\psi \times \\ \Big(d_i \nabla(c_i - \widehat{c}_i) + (c_i - \widehat{c}_i)v_i + d_i z_i\Big((c_i - \widehat{c}_i)\nabla\Phi_2 + \widehat{c}_i \nabla(\Phi_2 - \widehat{\Phi}_2)\Big)\Big) = 0 \tag{4.9}$$

for all $\psi \in C^\infty(\overline{Q_T})$ with $\psi(T) = 0$. The diffusivities d_i are bounded on Q_T from below and from above by (2.3) and from the proof of existence we know that the $L^\infty(L^r(\Omega))$-norms of $\nabla\Phi_2, \nabla\widehat{\Phi}_2$ are finite for some $r > 3$. We also have $v \in L^r(\Omega)$. So testing (4.9) formally with $\psi = (c_i - \widehat{c}_i)\chi_{(0,t_0)}$, $t_0 \in (0,T)$, by employing the Hölder and the Young inequality as well as the Sobolev embedding it is straight forward to derive an estimate as

$$\|(c_i - \widehat{c}_i)(t_0)\|^2_{L^2(\Omega)} + \overline{d}_T^{-1} \int_{Q_{t_0}} |\nabla(c_i - \widehat{c}_i)|^2 \\ \leq \frac{\overline{d}_T^{-1}}{2} \int_{Q_{t_0}} |\nabla(c_i - \widehat{c}_i)|^2 + C \int_{Q_{t_0}} |c_i - \widehat{c}_i|^2.$$

Since $(c_i - \widehat{c}_i)(0) = 0$, Gronwall's inequality implies $c_i = \widehat{c}_i$, whence uniqueness. It is possible to make this computation rigorous e.g. by choosing test functions $\psi_h(t) := \frac{1}{2h}\int_{t-h}^{t+h} \chi_{(0,t_0)}(c_i - \widehat{c}_i)(s)ds$, $h > 0$ small, and passing to the limit $h \to 0$. □

Remark 3. We will frequently refer to (4.8) (without parameter λ) as *mass conservation.*

4.2. **Local-in-time solutions for the approximate problem.** Let $T_0 > 0$ and assume the following stronger conditions on the data to problem (4.2)-(4.4):

- (A1) $d_i \in C\big(\overline{Q_{T_0}}; (0,\infty)\big)$ satisfying (2.3) with $\nabla d_i \in L^\infty(0,T_0; L^s(\Omega))$ for some $s > 3$.
- (A2) $(u^0, c^0) \in \mathcal{D}(A_S^{1/2}) \times L^\infty(\Omega)^+$,

where, unless stated otherwise, from now on A_S denotes the L^2-realization of the Stokes operator.

Let the Banach space X_T be defined by

$$X_T := C([0,T]; \mathcal{D}(A_S^{1/2})).$$

Note that $\mathcal{D}(A_S^{1/2}) = W_0^{1,2}(\Omega) \cap L^2_\sigma(\Omega)$ up to equivalent norms. We equip X_T with the norm $\|\cdot\|_T$ given by

$$\|u\|_T := \sup_{t\in[0,T]} \Big(\|u(t)\|_{L^2} + \|A_S^{1/2} u(t)\|_{L^2} \Big), \qquad u \in X_T.$$

Let $u \in X_T$. Note that $u \in L^\infty(0,T;L^6(\Omega))$ by Sobolev's embedding theorem and that $\nabla\Phi_1 \in L^\infty(0,T;L^\infty(\Omega))$. Setting $v_i = -d_i z_i \nabla\Phi_1 + R_\varepsilon^{1/2} u$ in Lemma 4.1 there is a unique weak solution $(c,\Phi_2) \in Y_T$,

$$Y_T := \big(L^2(0,T;W^{1,2}(\Omega)) \cap L^\infty(Q_T)\big) \times L^\infty(0,T;W^{4,2}(\Omega)),$$

to problem (4.3)-(4.4). Therefore we may define operators $F_\varepsilon, G_\varepsilon$ by

$$F_\varepsilon u := P(R_\varepsilon u \cdot \nabla) u, \qquad u \in X_T,$$

$$G_\varepsilon u := \sum_{j=1}^N z_j R_\varepsilon^{1/2}(P(c_j \nabla \Phi)), \qquad u \in X_T,$$

with $\Phi = \Phi_1 + \Phi_2$ as before. We now construct approximate solutions to system (4.2)-(4.4) by solving the variation of constants formula for (4.2)

$$u(t) = e^{-tA_S} u^0 - \int_0^t e^{-(t-s)A_S} \Big(F_\varepsilon u(s) + G_\varepsilon u(s) \Big) ds, \quad t > 0. \tag{4.10}$$

Lemma 4.2. *Suppose conditions (A1), (A2) hold true. Let $\varepsilon > 0$ and set*

$$Z(M,T) := \{u \in X_T;\ u(0) = u^0, \|u\|_T \le M\}.$$

Then there are $M > 0$, $T \in (0,T_0)$ such that problem (4.2)-(4.4) has a unique solution $(u,c,\Phi_2) \in Z(M,T) \times Y_T$, in the following sense:

- *u is a mild solution to (4.2), i.e. it satisfies (4.10).*
- *c is a weak solution to (4.3), i.e. for all $\psi \in C^\infty(\overline{Q_T})$ with $\psi(T) = 0$ it holds true that*

  $$\int_{Q_T} -c_i \partial_t \psi + (d_i \nabla c_i + d_i z_i c_i \nabla\Phi - c_i R_\varepsilon^{1/2} u) \nabla\psi = \int_\Omega c_i^0 \psi(0),$$

  where $\Phi = \Phi_1 + \Phi_2$.
- *Φ is a strong solution to (4.4).*

Proof. For the proof it is sufficient to show that (4.10) has a unique solution. For this purpose let Γ_ε be defined as

$$(\Gamma_\varepsilon u)(t) := e^{-tA_S} u^0 - \int_0^t e^{-(t-s)A_S} \Big(F_\varepsilon u(s) + G_\varepsilon u(s) \Big) ds, \qquad u \in X_T.$$

We will show that Γ_ε is a contraction on $Z(M,T)$, if $M,T>0$ are chosen in an appropriate way.

Self-map property. Let $u \in Z(M,T)$. For the term $F_\varepsilon u$ we estimate with Hölder's inequality, Sobolev's embedding theorem and Young's inequality

$$\begin{aligned}\|F_\varepsilon u(t)\|_{L^2} &\le \|(R_\varepsilon u(t)\cdot\nabla)u(t)\|_{L^2} \le \|R_\varepsilon u(t)\|_{L^\infty}\|\nabla u(t)\|_{L^2}\\ &\le C\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \le C\Big(\|u(t)\|_{L^2}^2 + \|A_S^{1/2}u(t)\|_{L^2}^2\Big). \end{aligned} \tag{4.11}$$

For G_ε we need a priori estimates for Φ_2 and c. Checking the proof of Lemma 4.1 the $L^\infty(Q_T)$-norms of c and of $\nabla\Phi = \nabla\Phi_1 + \nabla\Phi_2$ can be estimated by a constant which depends on T_0 and $\|u\|_T$. By the fact that $\|u\|_T \le M$ it hence depends only on M and T_0. This yields

$$\|G_\varepsilon u(t)\|_{L^\infty(\Omega)} \le C(M,T_0), \text{ for a.e. } t\in(0,T). \tag{4.12}$$

With the help of relations (4.11)-(4.12) we may estimate

$$\begin{aligned}
&\|\Gamma_\varepsilon u(t)\|_{L^2}\\
&\quad\le \|u^0\|_{L^2} + \int_0^t \|e^{-(t-s)A_S}\|_{\mathcal{L}(L^2_\sigma)}\Big(\|F_\varepsilon u(s)\|_{L^2} + \|G_\varepsilon u(s)\|_{L^2}\Big)ds\\
&\quad\le \|u^0\|_{L^2} + C\left(\int_0^t\Big(\|u(s)\|_{L^2}^2 + \|A_S^{1/2}u(s)\|_{L^2}^2 + C(M,T_0)\Big)\,ds\right)\\
&\quad\le \|u^0\|_{L^2} + CT\Big(M^2 + C(M,T_0)\Big),\\
&\|A_S^{1/2}\Gamma_\varepsilon u(t)\|_{L^2}\\
&\quad\le \|A_S^{1/2}u^0\|_{L^2} + \int_0^t \|A_S^{1/2}e^{-(t-s)A_S}\|_{\mathcal{L}(L^2_\sigma)}\Big(\|F_\varepsilon u(s)\|_{L^2} + \|G_\varepsilon u(s)\|_{L^2}\Big)ds\\
&\quad\le \|A_S^{1/2}u^0\|_{L^2} + C\int_0^t \frac{1}{\sqrt{t-s}}\Big(M^2 + C(M,T_0)\Big)ds\\
&\quad\le \|A_S^{1/2}u^0\|_{L^2} + C\sqrt{T}\Big(M^2 + C(M,T_0)\Big).
\end{aligned}$$

Choosing first $M>0$ large enough such that

$$\|u^0\|_{L^2} + \|A_S^{1/2}u^0\|_{L^2} \le \frac{M}{2}$$

and then $T>0$ small enough such that

$$C(T+\sqrt{T})\Big(M^2 + C(M,T_0)\Big) \le \frac{M}{2}$$

ensures Γ_ε to be a self-map.

Contraction property. Let $u^1, u^2 \in Z(M,T)$. We estimate

$$\begin{aligned}
&\|F_\varepsilon u^1(t) - F_\varepsilon u^2(t)\|_{L^2}\\
&\le \big\|(R_\varepsilon u^2(t)\cdot\nabla)\big(u^1(t)-u^2(t)\big)\big\|_{L^2} + \big\|\big((R_\varepsilon(u^1(t)-u^2(t)))\cdot\nabla\big)u^1(t)\big\|_{L^2}\\
&\le \big\|R_\varepsilon u^2(t)\big\|_{L^\infty}\big\|\nabla\big(u^1(t)-u^2(t)\big)\big\|_{L^2} + \big\|R_\varepsilon\big(u^1(t)-u^2(t)\big)\big\|_{L^\infty}\big\|\nabla u^1(t)\big\|_{L^2}
\end{aligned}$$

$$\leq C(M)\big\|u^1(t)-u^2(t)\big\|_T.$$

For u^1, u^2 let (c^1, Φ_2^1) and (c^2, Φ_2^2) be the corresponding solutions to (4.3)-(4.4) with data u^1, u^2 respectively. Setting $v^k = R_\varepsilon^{1/2} u^k$, $k = 1, 2$, it holds true that

$$\begin{aligned}\partial_t(c_i^1-c_i^2)+\operatorname{div}\big(&-d_i\nabla(c_i^1-c_i^2)-d_iz_ic_i^2\nabla(\Phi_2^1-\Phi_2^2)\\&-d_iz_i(c_i^1-c_i^2)(\nabla\Phi_1+\nabla\Phi_2^1)+(c_i^1-c_i^2)v^1+c_i^2(v^1-v^2)\big)=0\end{aligned}$$

in a weak sense. Recall that from (4.4) we can estimate $\|\nabla(\Phi_2^1-\Phi_2^2)\|_{L^2} \leq C\|c^1-c^2\|_{L^2}$ by elliptic regularity. Since the $L^\infty(Q_T)$-norms of $c^k, \nabla\Phi_2^k$ are controlled by a constant depending only on M, T_0 and since we have (4.1) at our disposal, multiplication with $c_i^1-c_i^2$ and integration over Ω employing integration by parts yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c_i^1-c_i^2\|_{L^2}^2+\overline{d}_T^{-1}\|\nabla(c_i^1-c_i^2)\|_{L^2}^2\\&\leq C(M,T_0)\|c^1-c^2\|_{L^2}\|\nabla(c_i^1-c_i^2)\|_{L^2}\\&\quad+C(M,T_0)\|c_i^1-c_i^2\|_{L^2}\|\nabla(c_i^1-c_i^2)\|_{L^2}\\&\quad+C(M,T_0)\|v^1-v^2\|_{L^2}\|\nabla(c_i^1-c_i^2)\|_{L^2}.\end{aligned}$$

Taking into account that $\|v^1-v^2\|_{L^2} \leq C\|u^1-u^2\|_{L^2}$ and employing Young's inequality we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|c_i^1-c_i^2\|_{L^2}^2\leq C\big(\overline{d}_T,M,T_0\big)\Big(\|c_i^1-c_i^2\|_{L^2}^2+\|c^1-c^2\|_{L^2}^2+\|u^1-u^2\|_{L^2}^2\Big),$$

hence summation over i and Gronwall's inequality result in

$$\|(c^1-c^2)(t)\|_{L^2}^2\leq C\big(\overline{d}_T,M,T_0\big)\|u^1-u^2\|_T\exp\Big(C\big(\overline{d}_T,M,T_0\big)\cdot T\Big)$$

for $t \in (0, T)$. Thus, we infer for the G_ε-term

$$\begin{aligned}&\|G_\varepsilon u^1(t)-G_\varepsilon u^2(t)\|_{L^2}\\&\leq C\bigg(\Big\|\sum_{j=1}^N(c_j^1-c_j^2)(t)\nabla(\Phi_1+\Phi_2^1)(t)\Big\|_{L^2}+\Big\|\sum_{j=1}^N c_j^2(t)\nabla(\Phi_2^1-\Phi_2^2)(t)\Big\|_{L^2}\bigg)\\&\leq C\big(\overline{d}_T,M,T_0\big)\|u^1-u^2\|_T\exp\Big(C\big(\overline{d}_T,M,T_0\big)\cdot T\Big)\\&\qquad+C\big(\overline{d}_T,M,T_0\big)\|(c^1-c^2)(t)\|_{L^2}\\&\leq C\big(\overline{d}_T,M,T_0\big)\exp\Big(C\big(\overline{d}_T,M,T_0\big)T\Big)\|u^1-u^2\|_T,\end{aligned}$$

where we again used elliptic regularity in the second step in order to estimate $\|\nabla(\Phi_2^1-\Phi_2^2)\|_{L^2}$. Finally we deduce the contraction property:

$$\begin{aligned}\|\Gamma_\varepsilon u^1-\Gamma_\varepsilon u^2\|_T\leq&\int_0^t\|F_\varepsilon u^1(s)-F_\varepsilon u^2(s)\|_{L^2}+\|G_\varepsilon u^1(s)-G_\varepsilon u^2(s)\|_{L^2}ds\\&+C\int_0^t\frac{1}{\sqrt{t-s}}\Big(\|F_\varepsilon u^1(t)-F_\varepsilon u^2(t)\|_{L^2}+\|G_\varepsilon u^1(t)-G_\varepsilon u^2(t)\|_{L^2}\Big)ds\end{aligned}$$

$$\leq C(\overline{d}_T, M, T_0)\Big(T+\sqrt{T}\Big)\Big(1+\exp\Big(C(\overline{d}_T, M, T_0)\cdot T\Big)\Big)\|u^1-u^2\|_T.$$

With possibly smaller choice of $T>0$ such that

$$C(\overline{d}_T, M, T_0)(T+\sqrt{T})\Big(1+\exp\Big(C(\overline{d}_T, M, T_0)\cdot T\Big)\Big)\leq \frac{1}{2},$$

the map Γ_ε becomes a contraction on $Z(M,T)$. Thus, there is a unique fixed point. □

4.3. **Energy estimates.** We will work with the auxiliary variable Ψ, defined by

$$\Psi = S_\varepsilon\Big(\sum_{j=1}^N z_j c_j\Big), \tag{4.13}$$

which is equivalent to

$$(1-\varepsilon\Delta)\Psi = \sum_{j=1}^N z_j c_j \qquad \text{in } \Omega, \qquad \partial_\nu \Psi + \tau\Psi = 0, \quad \text{on } \partial\Omega.$$

Hence we have $-\Delta\Phi_2 = \Psi$ with the approximate potential Φ_2 solving (4.6). In the following, we derive a priori estimates that assure the local solution (u, c, Φ_2) from Lemma 4.2 to exist globally on the one hand. On the other hand, they will allow us to pass to the limit $\varepsilon \to 0$ in equations (4.3)-(4.4). As already mentioned above, the function V_0, defined by

$$V_0(t) = \frac{1}{2}\int_\Omega |u|^2 + \int_\Omega \sum_{i=1}^N c_i \log c_i + \frac{1}{2}\int_\Omega |\nabla\Phi|^2 + \frac{\tau}{2}\int_{\partial\Omega}\Phi^2,$$

is a Lyapunov functional for system (1.1)-(1.3), see e.g. [10]. In the subsequent lemma, we show that our approximation procedure does preserve this energetic structure:

Lemma 4.3. *Let $(u^0, c^0) \in L^2_\sigma(\Omega)\times L^2(\Omega)^+$ and suppose that (u, c, Φ_2) satisfies (4.2)-(4.4) in the sense of Lemma 4.2 on the maximal time interval $[0, T_{max})$, let Ψ be defined as in (4.13) and set*

$$V(t) = \frac{1}{2}\int_\Omega |u|^2 + \int_\Omega \sum_{i=1}^N c_i \log c_i + \frac{1}{2}\int_\Omega |\nabla\Phi|^2 + \frac{\tau}{2}\int_{\partial\Omega}\Phi^2$$
$$+\,\varepsilon\left(\frac{1}{2}\int_\Omega |\Psi|^2 + \int_\Omega \sigma\Psi + \int_{\partial\Omega}\xi\Psi\right).$$

Then

(i)

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = -\int_\Omega |\nabla u|^2 - \int_\Omega \sum_{i=1}^N \frac{1}{d_i c_i}|d_i\nabla c_i + d_i z_i c_i \nabla\Phi|^2 \leq 0. \tag{4.14}$$

(ii) There is $C > 0$ (independent of t and ε), such that for almost all $t \in (0,T)$ we have

$$\|\Psi(t)\|_{L^{3/2}(\Omega)} + \|\Psi(t)\|_{L^1(\partial\Omega)} \leq \frac{C}{\varepsilon}. \tag{4.15}$$

(iii) There is $C > 0$ (independent of T and ε), such that

$$\int_{Q_T} |\nabla u|^2 + \int_{Q_T} \sum_{i=1}^N \frac{1}{d_i c_i} |d_i \nabla c_i + d_i z_i c_i \nabla \Phi|^2 \leq C. \tag{4.16}$$

(iv) There is a constant $C = C(T) > 0$ not depending on $\varepsilon > 0$ such that

$$\int_{Q_T} |\nabla u|^2 + \int_{Q_T} \sum_{i=1}^N \frac{|\nabla c_i|^2}{c_i} + c_i |\nabla \Phi|^2 + |\Delta \Phi|^2 \leq C. \tag{4.17}$$

Proof. We only provide a formal proof, Remark 4 will indicate how the computations can be made rigorous. Set $j_i = -d_i \nabla c_i - d_i z_i c_i \nabla \Phi$ and $J_i = j_i + c_i R_\varepsilon^{1/2} u$. We compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \sum_{i=1}^N c_i \log c_i = \int_\Omega \sum_{i=1}^N (\log c_i + 1) \operatorname{div}(d_i \nabla c_i + d_i z_i c_i \nabla \Phi - c_i R_\varepsilon^{1/2} u) \\
&\quad = \int_\Omega \sum_{i=1}^N J_i \frac{\nabla c_i}{c_i} = -\int_\Omega \sum_{i=1}^N \frac{1}{d_i c_i} J_i (j_i + d_i z_i c_i \nabla \Phi) \\
&\quad = -\int_\Omega \sum_{i=1}^N \frac{1}{d_i c_i} |j_i|^2 + \int_\Omega \sum_{i=1}^N z_i c_i \nabla \Phi \cdot R_\varepsilon^{1/2} u + \int_\Omega (R_\varepsilon^{1/2} u) \cdot \nabla c_i \\
&\qquad - \int_\Omega \sum_{i=1}^N z_i J_i \cdot \nabla \Phi \\
&\quad = -\int_\Omega \sum_{i=1}^N \frac{1}{d_i c_i} |j_i|^2 + \int_\Omega \sum_{i=1}^N z_i R_\varepsilon^{1/2} (P(c_i \nabla \Phi)) \cdot u - \int_\Omega \sum_{i=1}^N z_i J_i \cdot \nabla \Phi
\end{aligned} \tag{4.18}$$

by the divergence-free condition of u and self-adjointness of A_S. Forming the dual pairing of (4.2) with u we determine

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^2}^2 = -\int_\Omega |\nabla u|^2 - \int_\Omega \sum_{i=1}^N z_i R_\varepsilon^{1/2} (P(c_i \nabla \Phi)) \cdot u, \tag{4.19}$$

since $\langle (R_\varepsilon u \cdot \nabla) u, u \rangle_{L^2} = 0$. For the last term in (4.18) it is easy to see that, using integration by parts,

$$-\int_\Omega \sum_{i=1}^N z_i J_i \cdot \nabla \Phi = \int_\Omega \sum_{i=1}^N z_i (\operatorname{div} J_i) \Phi = -\int_\Omega \partial_t \Big(\sum_{i=1}^N z_i c_i \Big) \Phi$$

$$= -\int_\Omega (B_\varepsilon \Psi_t)\cdot(\Phi_1+\Phi_2). \tag{4.20}$$

For the Φ_1-term we obtain with integration by parts and exploiting the boundary condition in (2.1) and the fact that $\Psi_t = -\Delta\Phi_{2,t} = -\Delta\Phi_t$,

$$\begin{aligned}\int_\Omega [B_\varepsilon\Psi_t]\Phi_1 &= -\int_\Omega \Delta\Phi_t\cdot\Phi_1 - \varepsilon\int_\Omega \Delta\Psi_t\Phi_1\\ &= \int_\Omega \nabla\Phi_t\cdot\nabla\Phi_1 + \tau\int_{\partial\Omega}\Phi_t\Phi_1 + \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega \sigma\Psi + \int_{\partial\Omega}\xi\Psi\right].\end{aligned} \tag{4.21}$$

With a similar computation we arrive at

$$\int_\Omega [B_\varepsilon\Psi_t]\Phi_2 = \int_\Omega \nabla\Phi_t\cdot\nabla\Phi_2 + \tau\int_{\partial\Omega}\Phi_t\Phi_2 + \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Psi|^2 \tag{4.22}$$

for the Φ_2-term. Thus, combining (4.20)-(4.22) we conclude

$$\begin{aligned}&-\int_\Omega \sum_{i=1}^N z_i J_i\cdot\nabla\Phi\\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_\Omega |\nabla\Phi|^2 + \frac{\tau}{2}\int_{\partial\Omega}\Phi^2 + \varepsilon\Big(\int_\Omega |\Psi|^2 + \int_\Omega \sigma\Psi + \int_{\partial\Omega}\xi\Psi\Big)\right].\end{aligned} \tag{4.23}$$

Thus, (4.14) is a consequence of (4.18), (4.19), and (4.23). Relation (4.15) is easily seen by mass conservation and elliptic L^1-regularity. From (4.14) and (4.15) V is bounded from above and from below independently of ε, hence (4.16) follows.

Let us show (4.17). Because of $\nabla\Phi_1 \in C(\overline{\Omega})$, mass conservation, and the fact that $\Delta\Phi_1 = -\sigma \in L^2(\Omega)$, it is sufficient to show

$$\int_{Q_T} \frac{|\nabla c_i|^2}{c_i} + c_i|\nabla\Phi_2|^2 + |\Delta\Phi_2|^2 \le C. \tag{4.24}$$

Recall that $4|\nabla\sqrt{c_i}|^2 = \frac{|\nabla c_i|^2}{c_i}$ and $0 < \overline{d}_T^{-1} \le d_i \le \overline{d}_T < \infty$ in Q_T. So after expanding the square in (4.16) we get

$$\int_{Q_T}\sum_{i=1}^N \big(4|\nabla\sqrt{c_i}|^2 + z_i^2 c_i|\nabla\Phi_2|^2 + 2z_i\nabla c_i\cdot\nabla\Phi_2\big) \le C. \tag{4.25}$$

In this situation after integrating by parts the third term on the left-hand side we need to deal with a boundary integral:

$$\int_{Q_T}\sum_{i=1}^N z_i\nabla c_i\cdot\nabla\Phi_2 = -\int_{Q_T}\sum_{i=1}^N z_i c_i\Delta\Phi_2 + \sum_{i=1}^N\int_{\Sigma_T} z_i c_i\partial_\nu\Phi_2 =: \mathcal{J}^1 + \sum_{i=1}^N \mathcal{J}_i^2. \tag{4.26}$$

Recalling the boundary conditions for Φ_2 integral $\mathcal{J}_i^2$ can be rephrased by

$$\mathcal{J}_i^2 = -\int_{\Sigma_T}\tau z_i c_i\Phi_2.$$

From (*i*) it is known that $\|\Phi\|_{L^\infty(W^{1,2}(\Omega))}$ can be estimated by a constant independent of ε. The same holds true for Φ_2 because of $\Phi_1 \in C^1(\overline{\Omega})$. So employing Hölder's inequality there is $C > 0$ depending on T such that

$$\begin{aligned}\mathcal{J}_i^2 &\geq -C \int_{\Sigma_T} |c_i \Phi_2| \geq -\|\sqrt{c_i}\|^2_{L^2(L^{8/3}(\partial\Omega))} \|\Phi_2\|_{L^\infty(L^4(\partial\Omega))} \\ &\geq -C\|\sqrt{c_i}\|^2_{L^2(L^{8/3}(\partial\Omega))} \|\Phi_2\|_{L^\infty(W^{1,2}(\Omega))} \\ &\geq -\|\nabla\sqrt{c_i}\|^2_{L^2(Q_T)} - C\|\sqrt{c_i}\|^2_{L^2(Q_T)} \geq -\|\nabla\sqrt{c_i}\|^2_{L^2(Q_T)} - C. \end{aligned} \tag{4.27}$$

In the third step we utilized the compactness of the map $W^{1,2}(\Omega) \to L^{8/3}(\partial\Omega)$ in order to deduce for $\delta > 0$

$$\|v\|_{L^{8/3}(\partial\Omega)} \leq \delta\|\nabla v\|_{L^2(\Omega)} + C_\delta\|v\|_{L^2(\Omega)}, \qquad v \in W^{1,2}(\Omega). \tag{4.28}$$

Finally replacing $\sum_i z_i c_i$ in $\mathcal{J}^1$ with the help of (4.13), we obtain, again using integration by parts, that

$$\begin{aligned}\mathcal{J}^1 &= \int_{Q_T} B_\varepsilon(\Delta\Phi_2) \cdot \Delta\Phi_2 = \int_{Q_T} |\Delta\Phi_2|^2 - \varepsilon \int_{Q_T} \Delta^2\Phi_2\Delta\Phi_2 \\ &= \int_{Q_T} |\Delta\Phi_2|^2 + \varepsilon \int_{Q_T} |\nabla(\Delta\Phi_2)|^2 + \varepsilon\tau \int_{\Sigma_T} |\Delta\Phi_2|^2. \end{aligned} \tag{4.29}$$

Thus, (4.25)-(4.29) imply (4.17).

□

Remark 4. Since $x \mapsto x\log x$ is not differentiable in 0 we cannot differentiate $\int_\Omega c_i \log c_i$ directly. However, the formal calculation in the proof of Lemma 4.3 can be made rigorous, if $\int c_i \log c_i$ is replaced by $\int (c_i + \delta)\log(c_i + \delta)$, $\delta > 0$; passage to the limit $\delta \to 0$ then yields (4.14); see also [10, Proof of Lemma 3.7].

4.4. **Global-in-time solutions for the approximate problem.** Here we will see that Lemma 4.3 allows us to show that the local-in-time solutions to (4.2)-(4.4) are global.

Lemma 4.4. *Suppose conditions (A1), (A2) hold true. Then for every $\varepsilon > 0$ the local solution (u, c, Φ_2) to system (4.2)-(4.4) from Lemma 4.2 can be uniquely extended up to any finite time $T \in (0, \infty)$.*

Proof. For the proof $C > 0$ always is a positive constant only depending on ε, T and the initial data.

Lemma 4.3 (*i*) and (*iii*) imply that u can be bounded in $L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; \mathcal{D}(A_S^{1/2}))$ by a constant $C > 0$. From mass conservation and Lemma 4.3 (*iv*) it holds true that there is $C > 0$ that controls $\sqrt{c_i}$ in $L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega))$. Sobolev's embedding theorem and Hölder's inequality imply that c_i is bounded in $L^2(0,T; L^{3/2}(\Omega))$. From mass conservation und elliptic L^1-regularity we deduce that $\nabla\Phi_2$ is bounded in $L^\infty(0,T;W^{2,p}(\Omega))$, $p\in[1,\frac{3}{2})$, therefore $\nabla\Phi_2\in L^\infty(0,T;L^q(\Omega))$ for every $q<\infty$. Recall that $\nabla\Phi_1\in L^\infty(\Omega)$ and that $W^{1,\frac{6}{5}}(\Omega)\hookrightarrow L^2(\Omega)$. For the L^p-realization $A_{S,(p)}$ of the Stokes operator we have $\mathcal{D}(A_{S,(p)}^{1/2})=W_0^{1,p}(\Omega)\cap L^p_\sigma(\Omega)$, $1<p<\infty$. So therefore

$$\begin{aligned}\|R_\varepsilon^{1/2}(P(c_j\nabla\Phi))\|_{L^2(Q_T)}&\le C\|R_\varepsilon^{1/2}(P(c_j\nabla\Phi))\|_{L^2(W^{1,\frac{6}{5}})}\le C\|c_j\nabla\Phi\|_{L^2(L^{6/5})}\\&\le C\|c_j\|_{L^2(L^{3/2})}\|\nabla\Phi\|_{L^\infty(L^6)}\le C.\end{aligned}\tag{4.30}$$

From maximal regularity of A_S we deduce that u is a solution to (4.2). Multiplying this equation by A_Su, integrating over Ω, using the fact that $\langle u_t,A_Su\rangle_{L^2}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{L^2}$, and employing Young's inequality results in

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2_{L^2}+\|A_Su\|^2_{L^2}\\&\qquad\le\int_\Omega|(R_\varepsilon u\cdot\nabla)u\cdot A_Su|+\int_\Omega\Big|\sum_{j=1}^N z_jR_\varepsilon^{1/2}(P(c_j\nabla\Phi))\cdot A_Su\Big|\\&\qquad\le\|A_Su\|^2_{L^2}+C\bigg(\|\nabla u\|^2_{L^2}+\sum_{j=1}^N\|R_\varepsilon^{1/2}(P(c_j\nabla\Phi))\|^2_{L^2}\bigg).\end{aligned}$$

Note that in the last step we used Young's inequality and $\|u(t)\|_{L^2}\le C$ independently of t due to Lemma 4.3 in order to estimate $\|R_\varepsilon u\|_{L^\infty}\le C$. Integration over t, $t\in(0,T)$, yields

$$\|\nabla u(t)\|^2_{L^2}\le\|\nabla u^0\|^2_{L^2}+C\bigg(\|\nabla u\|^2_{L^2(Q_T)}+\sum_{j=1}^N\|R_\varepsilon^{1/2}(P(c_j\nabla\Phi))\|^2_{L^2(Q_T)}\bigg).$$

From Lemma 4.3 *(iv)* we have $\|\nabla u\|_{L^2(Q_T)}<\infty$ for $T<\infty$. So together with (4.30) we deduce $\|u\|_T\le C$ for $T<\infty$, hence the solution is global. Uniqueness follows from uniqueness of local solutions by Lemma 4.2. $\square$

5. Proof of the Main Result

Let $T>0$. For $m\in\mathbb{N}$ let $d_i^m\in C(\overline{Q_T},(0,\infty))$ with $\nabla d_i^m\in L^\infty(0,T;L^s(\Omega))$ for some $s>3$ and $d_i^m(t,x)\to d_i(t,x)$ almost everywhere. Moreover, let $(\varepsilon^m)_{m\in\mathbb{N}}$ be a sequence of positive numbers with $\varepsilon^m\to0$ as $m\to\infty$, set $u^{0m}:=R_{\varepsilon^m}u^0\in\mathcal{D}(A_S)\subset\mathcal{D}(A_S^{1/2})$, and let $c^{0m}\in L^\infty(\Omega)^+$ such that $c^{0m}\to c^0$ in $L^2(\Omega)$. From Lemma 4.2 and Lemma 4.4 there is a unique solution (u^m,c^m,Φ_2^m) to problem (4.2)-(4.4) on Q_T with parameter ε^m subject to the data d_i^m, c^{0m}, u^{0m}. We write $\Phi^m=\Phi_1+\Phi_2^m$, $j_i^m=-d_i^m\nabla c_i^m-d_i^mz_ic_i^m\nabla\Phi^m$, and $J_i^m=j_i^m+c_i^mR_{\varepsilon^m}^{1/2}u^m$.
Lemma 4.3 implies that

$$u^m\text{ is bounded in }Y=L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;\mathcal{D}(A_S^{1/2})),\tag{5.1}$$

so there exists $u \in Y$ such that

$$u^m \to u \text{ weakly-* in } L^\infty(0,T;L^2_\sigma(\Omega)) \text{ and weakly in } L^2\big(0,T;\mathcal{D}(A_S^{1/2})\big).$$

We will show that $(u^m)_m$ is relatively compact in $L^2(Q_T)$. To this end we set

$$\begin{aligned} u^m(t) &= e^{-tA_S}u^{0m} - \int_0^t e^{-(t-s)A_S}\Big(F_{\varepsilon^m}u^m(s) + G_{\varepsilon^m}u^m(s)\Big)ds \\ &=: w_1^m(t) + w_2^m(t), \quad t \in [0,T]. \end{aligned}$$

Clearly $w_1^m(\cdot) \to e^{-\cdot A_S}u^0$ in $L^2(Q_T)$. Note that $w = w_1^m$ is a solution to

$$\partial_t w(t) + A_S w(t) = 0, \qquad w(0) = u^{0m}.$$

Forming the dual pairing with w and integrating by parts shows that $(w_1^m)_{m\in\mathbb{N}}$ is bounded in Y, since u^{0m} is bounded in $L^2_\sigma(\Omega)$. As a consequence, w_2^m is also bounded in Y. In order to apply maximal regularity arguments for w_2^m we estimate $F_{\varepsilon^m}u^m$ and $G_{\varepsilon^m}u^m$ independently of m. Employing Hölder's inequality, $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ by Sobolev's embedding theorem, and Poincaré's inequality we compute

$$\begin{aligned} \|F_{\varepsilon^m}u^m(t)\|_{L^{5/4}(\Omega)} &\le \|R_{\varepsilon^m}u^m(t)\|_{L^{10/3}}\|\nabla u^m(t)\|_{L^2} \\ &\le C\|u^m(t)\|_{L^{10/3}}\|\nabla u^m(t)\|_{L^2} \\ &\le C\|u^m(t)\|_{L^2}^{2/5}\|u^m(t)\|_{L^6}^{3/5}\|\nabla u^m(t)\|_{L^2} \\ &\le C'\|\nabla u^m(t)\|_{L^2}^{8/5}. \end{aligned}$$

Thus

$$\|F_{\varepsilon^m}u^m\|_{L^{5/4}(Q_T)} \le C \text{ independently of } m. \tag{5.2}$$

From Lemma 4.3 and mass conservation the sequence $\sqrt{c_i^m}$ is bounded in $L^2(0,T;W^{1,2}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ independently of m. Employing Hölder's inequality and Sobolev's embedding theorem we have

$$L^2(0,T;W^{1,2}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \hookrightarrow L^{10/3}(Q_T). \tag{5.3}$$

For $G_{\varepsilon^m}u^m$ we estimate using Hölder's inequality and Lemma 4.3 *(iv)*

$$\begin{aligned} \|G_{\varepsilon^m}u^m\|_{L^{5/4}(Q_T)} &= \|\sum_{j=1}^N z_j R_\varepsilon^{1/2}(P(c_j^m\nabla\Phi^m))\|_{L^{5/4}(Q_T)} \\ &\le C\sum_{j=1}^N \|c_j^m\nabla\Phi^m\|_{L^{5/4}(Q_T)} \\ &\le C\sum_{j=1}^N \|\sqrt{c_j^m}\|_{L^{10/3}(Q_T)}\|\sqrt{c_j^m}\nabla\Phi\|_{L^2(Q_T)} \le C. \end{aligned} \tag{5.4}$$

independently of m. By maximal regularity of A_S, see Proposition 1, we observe that $w = w_2^m$ is a solution to

$$\partial_t w(t) + A_S w(t) = F_{\varepsilon^m} u^m(t) + G_{\varepsilon^m} u^m(t), \qquad w(0) = 0.$$

Moreover (5.2)-(5.4) and maximal regularity of A_S imply that

$$w_2^m \text{ is bounded in } W^{1,\frac{5}{4}}(0,T; L^{\frac{5}{4}}(\Omega)) \cap L^{\frac{5}{4}}(0,T; W^{2,\frac{5}{4}}(\Omega)).$$

With the facts that w_2^m is bounded in $L^2(0,T; \mathcal{D}(A_S^{1/2}))$ and that $\partial_t w_2^m$ is bounded in $L^{5/4}(Q_T)$ we deduce that w_2^m is relatively compact in $L^2(Q_T)$ by Aubin-Simon, see Proposition 2. Thus, we may assume $u^m \to u$ strongly in $L^2(Q_T)$. As a consequence, we also have $R_{\varepsilon^m}^{1/2} u^m \to u$ and $R_{\varepsilon^m} u^m \to u$ in $L^2(Q_T)$. From the bound in $L^2(0,T; \mathcal{D}(A_S^{1/2}))$ we infer that $\nabla u^m \to \nabla u$ weakly in $L^2(Q_T)$.

Next, we deduce relative compactness for c^m. Recall that $\sqrt{c_i^m}$ is bounded in $L^2(0,T; W^{1,2}(\Omega))$. In order to avoid singularities at zero, we consider $\sqrt{c_i^m + 1}$. For its time derivative we have

$$\begin{aligned} 2\partial_t \sqrt{c_i^m + 1} &= \frac{\partial_t c_i^m}{\sqrt{c_i^m + 1}} = \frac{-\operatorname{div} J_i^m}{\sqrt{c_i^m + 1}} \\ &= -\operatorname{div}\Big(\frac{j_i^m}{\sqrt{c_i^m + 1}}\Big) - \frac{j_i^m \cdot \nabla c_i^m}{2(c_i^m + 1)^{3/2}} - \frac{R_{\varepsilon^m}^{1/2} u^m \cdot \nabla c_i^m}{\sqrt{c_i^m + 1}} \\ &= -\operatorname{div}\Big(\frac{j_i^m}{\sqrt{c_i^m + 1}}\Big) - \Big(\frac{1}{2\sqrt{c_i^m + 1}} \frac{j_i^m}{\sqrt{c_i^m + 1}} + R_{\varepsilon^m}^{1/2} u\Big) \cdot \frac{\nabla c_i^m}{\sqrt{c_i^m + 1}}, \end{aligned}$$

which is bounded in $L^1(0,T; W^{-1,2}(\Omega) + L^1(\Omega))$ by Lemma 4.3 *(iii)* and *(iv)*. Thus, using Aubin-Simon, it follows that $\sqrt{c_i^m + 1}$ is relatively compact in $L^2(0,T; L^p(\Omega))$ for any $p \in [1,6)$. Since $\sqrt{c_i^m}$ is bounded in $L^2(0,T; L^6(\Omega))$ there exists $c_i \in L^1(0,T; L^3(\Omega))$ such that, up to a subsequence,

$$\sqrt{c_i^m} \to \sqrt{c_i} \text{ strongly in } L^2(0,T; L^p(\Omega)) \text{ for } p \in [1,6). \tag{5.5}$$

Since $\nabla\sqrt{c_i^m}$ is bounded in $L^2(Q_T)$, we may assume $\nabla\sqrt{c_i^m} \to \nabla\sqrt{c_i}$ weakly in $L^2(Q_T)$. So $\nabla c_i^m = 2\sqrt{c_i^m}\nabla\sqrt{c_i^m}$ converges weakly to ∇c_i in $L^1(0,T; L^q(\Omega))$, $q \in [1, \frac{3}{2})$. With

$$\|\nabla c_i^m\|_{L^1(0,T;L^{3/2}(\Omega))} \leq \|\sqrt{c_i^m}\|_{L^2(0,T;L^6(\Omega))} \|(c_i^m)^{-1/2} \nabla c_i^m\|_{L^2(Q_T)} \leq C,$$

we deduce that in fact $\nabla c_i \in L^1(0,T; L^{3/2}(\Omega))$, hence $c \in L^1(0,T; W^{1,\frac{3}{2}}(\Omega))$. Note that from mass conservation, we also have $c_i \in L^\infty(0,T; L^1(\Omega)^+)$.

To prove relative compactness for the sequence Φ_2^m recall the bound in $L^\infty(0,T; W^{1,2}(\Omega))$ from Lemma 4.3. Regarding regularity in time, note that we have $-\Delta\Phi_2^m = S_{\varepsilon^m} \sum_k z_k c_k^m$. Taking the time derivative, we get

$-\Delta\partial_t\Phi_2^m = S_{\varepsilon^m}\sum_k z_k\partial_t c_k^m$. Recall that $\partial_t c_k^m = -\operatorname{div}\big(j_k^m + c_k^m R_{\varepsilon^m}^{1/2}u^m\big)$. With Lemma 4.3 *(iv)* we estimate the contributing flux terms by

$$\|j_k^m\|_{L^{5/4}(Q_T)} \le \|\sqrt{c_k^m}\|_{L^{10/3}(Q_T)}\|(c_k^m)^{-1/2}\nabla j_k^m\|_{L^2(Q_T)} \le C,$$
$$\|c_k^m R_{\varepsilon^m}^{1/2}u^m\|_{L^{10/9}(Q_T)} \le C\|c_k^m\|_{L^{5/3}(Q_T)}\|u^m\|_{L^{10/3}(Q_T)} \le C$$

uniformly in $m\in\mathbb{N}$. In the last step we used (5.1) and (5.3). As a consequence, $\partial_t c_i^m$ can be bounded in $L^{10/9}(0,T;W_0^{-1,\frac{10}{9}}(\Omega))$, so $\partial_t\Phi_2^n$ is bounded in $L^{10/9}(0,T;W^{1,\frac{10}{9}}(\Omega))$. Then, again using Aubin-Simon, we observe that Φ_2^m is relatively compact in $C([0,T];L^r(\Omega))$, $r\in[1,6)$. Note that the bound in $L^\infty(0,T;W^{1,2}(\Omega))$ implies also weak relative compactness in $L^q(0,T;W^{1,2}(\Omega))$ for any $q\in[1,\infty)$. Therefore there is $\Phi_2\in L^\infty(0,T;W^{1,2}(\Omega))\cap C([0,T];L^r(\Omega))$, $r\in[1,6)$, such that, up to a subsequence, we can assume

$$\Phi_2^m \to \Phi_2 \quad \begin{cases} \text{weakly in } L^q(0,T;W^{1,2}(\Omega)) \text{ for any } q<\infty, \\ \text{strongly in } C([0,T];L^r(\Omega)),\ r\in[1,6), \end{cases}$$

as $m\to\infty$. Remark that u^m and c^m are solutions to

$$\int_{Q_T} -u^m\partial_t\phi + \nabla u^m\cdot\nabla\phi + \Big((R_{\varepsilon^m}u^m\cdot\nabla)u^m\cdot\phi\Big)$$
$$= -\sum_{j=1}^N z_j\int_{Q_T} R_{\varepsilon^m}^{1/2}(P(c_j^m\nabla\Phi^m))\cdot\phi + \int_\Omega u^{0m}\phi(0), \tag{5.6}$$
$$\int_{Q_T} -c_i^m\partial_t\psi + \Big(d_i^m\nabla c_i^m + d_i^m z_i c_i^m\nabla\Phi^m - c_i^m R_{\varepsilon^m}^{1/2}u^m\Big)\nabla\psi = \int_\Omega c_i^{0m}\psi(0). \tag{5.7}$$

We analyze the limit behavior of the nonlinear terms. Since $R_{\varepsilon^m}u^m\to u$ strongly in $L^2(Q_T)$ and $\nabla u^m\to\nabla u$ weakly in $L^2(Q_T)$ we have $(R_{\varepsilon^m}u^m\cdot\nabla)u^m\to(u\cdot\nabla)u$ weakly in $L^1(Q_T)$. Lemma 4.3 *(iv)* implies that $\sqrt{c_i^m}\nabla\Phi^m$ is weakly relatively compact in $L^2(Q_T)$. Since $\nabla\Phi^m$ is weakly* convergent in $L^\infty(0,T;L^2(\Omega))$ and c_i^m by (5.5) is strongly convergent in $L^1(0,T;L^2(\Omega))$ we conclude that $c_i^m\nabla\Phi^m\to c_i\nabla\Phi$ weakly in $L^1(Q_T)$. Finally, by $c_i^m\to c_i$ strongly in $L^1(0,T;L^2(\Omega))$ and $R_{\varepsilon^m}^{1/2}u^m\to u$ weakly-* in $L^\infty(0,T;L^2(\Omega))$, it follows that $c_i^m R_{\varepsilon^m}^{1/2}u^m\to c_i u$ weakly in $L^1(Q_T)$.

Let $(T_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers with $T_k\to\infty$ as $k\to\infty$ and let $(u^{k,m},c^{k,m},\Phi_2^{k,m})$ be the solution of (4.2)-(4.4) on Q_{T_k} with parameter ε^m. From the above compactness results we find $(u,c,\Phi_2)\colon\mathbb{R}_+\to L^2_\sigma(\Omega)\times L^1(\Omega)^+\times W^{1,2}(\Omega)$ such that, up to a diagonal extraction, for all $k\in\mathbb{N}$

$$u^{k,m}\longrightarrow u \quad \begin{cases} \text{weakly in } L^2(0,T_k;\mathcal{D}(A_S^{1/2})), \\ \text{strongly in } L^2(Q_{T_k}), \end{cases} \tag{5.8}$$

$$c^{k,m} \longrightarrow c \quad \begin{cases} \text{weakly in } L^1(0,T_k;W^{1,\frac{3}{2}}(\Omega)), \\ \text{strongly in } L^1(0,T_k;L^p(\Omega)), \quad p \in [1,3), \end{cases} \tag{5.9}$$

$$\Phi_2^{k,m} \longrightarrow \Phi_2 \quad \begin{cases} \text{weakly in } L^q(0,T_k;W^{1,2}(\Omega)), \quad q \in [1,\infty), \\ \text{strongly in } C([0,T_k];L^r(\Omega)), \quad r \in [1,6), \end{cases} \tag{5.10}$$

as $m \to \infty$. In this situation it holds true that

$$\begin{aligned} (R_{\varepsilon^m} u^{k,m} \cdot \nabla) u^{k,m} &\longrightarrow (u \cdot \nabla) u, && \text{weakly in } L^1(Q_{T_k}), \\ c^{k,m} \nabla \Phi^{k,m} &\longrightarrow c \nabla \Phi, && \text{weakly in } L^1(Q_{T_k}), \\ c^{k,m} R_{\varepsilon^m}^{1/2} u^{k,m} &\longrightarrow cu, && \text{weakly in } L^1(Q_{T_k}), \\ u^{0m} &\longrightarrow u^0, && \text{strongly in } L^2_\sigma(\Omega), \\ c^{0m} &\longrightarrow c^0, && \text{strongly in } L^2(\Omega). \end{aligned}$$

Thus, passing to the limit in (5.6)-(5.7) yields (2.4)-(2.5).

With (5.9) we see that for a.e. $t \in (0,T_k)$, $c^m(t) \longrightarrow c(t)$ in $L^2(\Omega)$. By well-known properties of the resolvent of Δ_R this yields $\Phi_2^m(t) \longrightarrow \Phi_2(t)$ in $W^{2,2}(\Omega)$ for a.e. $t \in (0,T_k)$, where $\Phi_2(t)$ is the solution to

$$-\Delta \Phi_2 = \sum_{j=1}^N z_j c_j \text{ in } \Omega, \qquad \partial_\nu \Phi_2 + \tau \Phi_2 = 0 \text{ on } \partial\Omega.$$

Since $T_k \to \infty$, this is true for almost all $t \in \mathbb{R}_+$. The limit c enjoys better regularity, so $\Phi_2 \in L^1(0,T;W^{3,\frac{3}{2}}(\Omega))$.

It remains to show the assertion on the pressure π. We employ results on the pressure from [33]. In this respect, note that $c_i \nabla \Phi \in L^1(0,T;L^2(\Omega))$ for any $T < \infty$ by Hölder's inequality and Sobolev's embedding theorem. Then [33, Theorem V.1.7.1.] implies that there is a function $\widehat{\pi} \in L^{4/3}(0,T;L^2_{loc}(\Omega))$ such that $\pi = \partial_t \widehat{\pi}$ is an associated pressure. This means that (u,π) satisfies (1.1) in the sense of distributions in $[0,T) \times \Omega$ for every $T < \infty$. □

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References

[1] R.A. Adams and J.J.F. Fournier. *Sobolev Spaces.* Academic press, Kidlington, Oxford, 2 edition, 2003.

[2] H. Amann. *Linear and Quasilinear Parabolic Problems - Abstract Linear Theory*, volume 1 of *Monographs in mathematics.* Birkhäuser Verlag Basel, 1994.

[3] H. Amann and M. Renardy. Reaction-diffusion problems in electrolysis. *NoDEA Nonlinear Differential Equations Appl.*, 1:91–117, 1994.

[4] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[5] P. Biler. Existence and asymptotics of solutions for a parabolic-elliptic system with nonlinear no-flux boundary conditions. *Nonlinear Anal.*, 19:1121–1136, December 1992.

[6] P. Biler and J. Dolbeault. Long time behavior of solutions to Nernst-Planck and Debye-Hckel drift-diffusion systems. *Ann. Henri Poincaré*, 1:461–472, 2000.
[7] P. Biler, W. Hebisch, and T. Nadzieja. The Debye system: existence and large time behavior of solutions. *Nonlinear Anal.*, 23:1189–1209, November 1994.
[8] D. Bothe, A. Fischer, M. Pierre, and G. Rolland. Global wellposedness for a class of reaction-advection-anisotropic-diffusion systems. *In preparation.*
[9] D. Bothe, A. Fischer, M. Pierre, and G. Rolland. Global existence for diffusion-electromigration systems in space dimension three and higher. *Nonlinear Analysis*, 99:152–166, 2014.
[10] D. Bothe, A. Fischer, and J. Saal. Global well-posedness and stability of electrokinetic flows. *SIAM Journal of Mathematical Analysis*, 46:1263–1316, 2014.
[11] D. Bothe and J. Prüss. Mass transport through charged membranes. *Proc. 4th European Conf. on Elliptic and Parabolic Problems*, pages 332–342, 2002.
[12] S.T. Chang. *New Electrokinetic Techniques for Material Manipulation on the Microscale.* BiblioBazaar, 2011.
[13] Y.S. Choi and R. Lui. Analysis of an electrochemistry model with zero-flux boundary conditions. *Appl. Anal.*, 49:277–288, 1993.
[14] Y.S. Choi and R. Lui. Multi-dimensional electrochemistry model. *Arch. Rational Mech. Anal.*, 130:315–342, 1995.
[15] E.L. Cussler. *Diffusion: Mass Transfer in Fluid Systems.* Cambridge University Press, 2 edition, 1997.
[16] W. Dreyer, C. Guhlke, and R. Müller. Overcoming the shortcomings of the Nernst-Planck model. *Phys Chem Chem Phys.*, 15(19):7075–7086, 2013.
[17] H. Gajewski. On existence, uniqueness and asymptotic behavior of solutions of the basic equations for carrier transport in semiconductors. *Z. Angew. Math. Mech.*, 65:101–108, 1985.
[18] H. Gajewski and K. Gröger. Reaction-diffusion processes of electrically charged species. *Math. Nachr.*, 177:109–130, 1996.
[19] G.P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems.* Springer-Verlag, 2 edition, 2011.
[20] Y. Giga. Domains of fractional powers of the Stokes operator in L_r spaces. *Arch. Ration. Mech. Anal.*, 89:251–265, 1985.
[21] A. Glitzky, K. Gröger, and R. Hünlich. Free energy and dissipation rate for reaction diffusion processes of electrically charged species. *Appl. Anal.*, 60:201–217, 1995.
[22] A. Glitzky and R. Hünlich. Energetic estimates and asymptotics for electro-reaction-diffusion systems. *Z. Angew. Math. Mech.*, 77:823–832, 1997.
[23] A. Glitzky and R. Hünlich. Global estimates and asymptotics for electro-reaction-diffusion systems in heterostructures. *Appl. Anal.*, 66:205–226, 1997.
[24] J.W. Jerome and R. Sacco. Global weak solutions for an incompressible charged fluid with multi-scale couplings: Initialboundary-value problem. *Nonlinear Anal.*, 71:e2487 – e2497, 2009.
[25] B. Kirby. *Micro- and Nanoscale Fluid Mechanics: Transport in Microfluidic Devices.* Cambridge University Press, 2010.
[26] O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Ural'ceva. *Linear and quasilinear equations of parabolic type*, volume 23 of *Translation of Mathematical Monographs.* Amer. Math. Soc, 1968.
[27] J.S. Newman. *Electrochemical systems.* Prentice Hall, 2 edition, 1991.
[28] A. Noll and J. Saal. $\mathcal{H}^\infty$-calculus for the Stokes operator on L_q-spaces. *Math. Z.*, 244:651–688, 2003.
[29] R.F. Probstein. *Physicochemical Hydrodynamics.* Butterworths, 1989.
[30] R. Ryham. Existence, uniqueness, regularity and long-term behavior for dissipative systems modeling electrohydrodynamics. *arXiv:0910.4973v1*, 2009.

[31] M. Schmuck. Analysis of the Navier-Stokes-Nernst-Planck-Poisson system. *Math. Models Methods Appl.*, 19:993–1014, 2009.

[32] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl.*, 146:65–96, 1987.

[33] H. Sohr. *The Navier-Stokes Equations: An Elementary Functional Analytic Approach*. Birkhäuser, 2001.

[34] V.A. Solonnikov. Estimates for solutions of nonstationary Navier-Stokes equations. *J. Soviet Math*, 8:213–317, 1977.

[35] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. North Holland, 1978.

[36] M. Trojanowicz. *Advances in Flow Analysis*. Wiley-VCH, 2008.

[37] J. Wiedmann. *An electrolysis model and its solutions*. Phd-thesis, University of Zurich, 1997.

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