NONLINEAR STABILITY OF EKMAN BOUNDARY LAYERS

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Abstract. Consider the initial value problem for the three dimensional Navier-Stokes equations with rotation in the half-space $\mathbb{R}^3_+$ subject to Dirichlet boundary conditions as well as the Ekman spiral which is a stationary solution to the above equations. It is proved that the Ekman spiral is nonlinearly stable with respect to L^2-perturbations provided the corresponding Reynolds number is small enough. Moreover, the decay rate can be computed in terms of the decay of the corresponding linear problem.

1. Introduction

Consider the the initial value problem for the three dimensional Navier-Stokes equations with rotation in the half-space $\mathbb{R}^3_+$ subject to Dirichlet boundary conditions, i.e. the set of equations

$$\left\{\begin{aligned} \partial_t u - \nu\Delta u + \Omega \mathbf{e}_3 \times u + (u\cdot\nabla)u + \nabla p = 0, &\quad t>0,\ x\in\mathbb{R}^3_+,\\ \operatorname{div} u = 0, &\quad t>0,\ x\in\mathbb{R}^3_+,\\ u(t,x_1,x_2,0) = 0, &\quad t>0,\ x_1,x_2\in\mathbb{R},\\ u(0,x) = u_0, &\quad x\in\mathbb{R}^3_+, \end{aligned}\right. \tag{1.1}$$

where $u=(u^1,u^2,u^3)$ denotes the velocity field and p the pressure of an incompressible, viscous fluid. Here, $\mathbf{e}_3$ denotes the unit vector in x_3-direction, $\nu>0$ the viscosity of the fluid, and the constant $\Omega\in\mathbb{R}$ is called the Coriolis parameter which is equal to twice the frequency of rotation around the x_3 axis.

It is well known that the above system has a stationary solution which can be expressed even explicitly as

$$u_E(x_3) = u_\infty(1-e^{-x_3/\delta}\cos(x_3/\delta), e^{-x_3/\delta}\sin(x_3/\delta), 0)^T, \tag{1.2}$$

$$p_E(x_2) = -\Omega u_\infty x_2, \tag{1.3}$$

where δ is defined by $\delta := (\frac{2\nu}{\Omega})^{1/2}$ and $u_\infty \geq 0$ is a constant. This stationary solution of equation (1.1) is called in honour of the swedish oceanograph V.W. Ekman, the *Ekman spiral*; see [8]. It describes mathematically rotating boundary layers in geophysical fluid dynamics (atmospheric and oceanic boundary layers) between a geostrophic flow and a solid boundary at which the no slip boundary condition applies. Moreover, δ denotes the thickness of the layer. In the geostrophic flow region corresponding to large x_3, there is a uniform flow with velocity u_∞ in the x_1 direction. Associated with u_∞, there is a pressure gradient in the x_2-direction. The Ekman spiral in $\mathbb{R}^3_+$ matches this uniform velocity for large x_3 with the no slip boundary condition at $x_3=0$, i.e. we have $u_E(0)=0$ and

$$u_E(x_3) \to (u_\infty,0,0) \quad \text{provided} \quad x_3\to\infty.$$

In this paper we are interested in stability questions for the Ekman spiral. More precisely, we consider perturbations of the Ekman spiral by functions u solving the above equation (1.1). To this end, set

$$w := u - u_E, \quad \text{and} \quad q := p - p_E.$$

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Since (u_E, p_E) is a stationary solution of (1.1), the pair (w, q) formally satisfies the equations

$$\tag{1.4} \left\{ \begin{array}{rcll} \partial_t w - \nu\Delta w + \Omega \mathbf{e}_3 \times w + (u_E \cdot \nabla)w + w_3\partial_3 u_E + (w \cdot \nabla)w + \nabla q &=& 0, & t > 0,\ x \in \mathbb{R}^3_+, \\ \text{div } w &=& 0, & t > 0,\ x \in \mathbb{R}^3_+, \\ w(x_1, x_2, 0) &=& 0, & t > 0,\ x_1, x_2 \in \mathbb{R}, \\ w(0, x) &=& w_0, & x \in \mathbb{R}^3_+, \end{array} \right.$$

where $w_0 = u_0 - u_E$.

It is natural to conjecture that there exists a critical Reynolds number Re_c with the property that if $Re < Re_c$, then the perturbed nonlinear flow is stable and that the flow is unstable provided $Re > Re_c$. Here $Re = u_\infty \delta \nu^{-1}$ denotes the Reynolds number of the given fluid. It seems that there is no mathematical proof of this statement so far.

Considering the linearized version of our problem, i.e.

$$\tag{1.5} \left\{ \begin{array}{rcll} \partial_t w - \nu\Delta w + \Omega \mathbf{e}_3 \times w + (u_E \cdot \nabla)w + w^3\partial_3 u_E + \nabla q &=& 0, & t > 0,\ x \in \mathbb{R}^3_+, \\ \text{div } w &=& 0, & t > 0,\ x \in \mathbb{R}^3_+, \\ w(x_1, x_2, 0) &=& 0, & t > 0,\ x_1, x_2 \in \mathbb{R}, \\ w(0, x) &=& w_0, & x \in \mathbb{R}^3_+, \end{array} \right.$$

we remark that linear stability results for the Ekman spiral can be obtained fairly easily by energy methods, again of course for small Reynolds number. Results on linear instability of w for large Reynolds number are more difficult to obtain. It was shown in [7] that in the case of flows between infinite layers, there exists a sequence of approximate solutions to (1.4) which is nonlinearly unstable for sufficiently large Reynolds numbers in the sense of [7].

Recently it was shown in [12] that the nonlinear equation (1.4) admits a unique, *local* mild solution for all non decaying initial data belonging to a certain Besov space. In this paper, we consider the problem of *global* weak solutions to (1.4) and study also their nonlinear stability behaviour for initial data belonging to $L^2_\sigma(\mathbb{R}^3_+)$.

We show in our first main result that there exists a global weak solution to the above set (1.4) of nonlinear equations provided the Reynolds number $Re = u_\infty \delta \nu^{-1}$ is small enough. Secondly, assuming this condition, for every initial data $w_0 \in L^p_\sigma(\mathbb{R}^3_+)$, there exists at least one global weak solution w to (1.4) such that

$$\lim_{t\to\infty} \|w(t)\|_2 = 0,$$

which shows in particular that the Ekman spiral is nonlinearly stable with respect to L^2-perturbations. Moreover, it is even possible to estimate the decay rate. Indeed, roughly speaking, if Re is small enough, $0 < \alpha \le \frac{1}{4}$ and $w_0 \in L^2_\sigma(\mathbb{R}^3_+)$ satisfies

$$\|e^{-tA_{SCE}} w_0\|_2 = O(t^{-\alpha}),$$

then there exists at least one global weak solution w to the nonlinear problem (1.1) having the same decay rate. A similar result applies for arbitrary $\alpha > 0$. Here $e^{-tA_{SCE}}$ denotes the semigroup on $L^2_\sigma(\mathbb{R}^3_+)$ generated by the Stokes-Coriolis-Ekman operator in $L^2_\sigma(\mathbb{R}^3_+)$ defined in the following section.

Our approach is inspired by the methods developed by Miyakawa and Sohr [15] and Borchers and Miyakawa [3],[2] in order to construct weak solutions to the Navier-Stokes equations – without rotational effects – on exterior domains. For more information on the Navier-Stokes equations with rotational effect, we refer to [13] or [1] and [4]. Although the assertions of our two main results are stated completely within the L^2-framework, our proof needs so-called maximal L^p-regularity estimates for the Stokes-Coriolis-Ekman operator in the halfspace $\mathbb{R}^3_+$ for $p \neq 2$. We sketch the proof of these estimates in Section 3.

2. Preliminaries and Main Results

For $1 < p < \infty$ denote by P the Helmholtz projection from $L^p(\mathbb{R}^3_+)$ to $L^p_\sigma(\mathbb{R}^3_+)$. We then may rewrite equation (1.4) as an evolution equation in $L^p_\sigma(\mathbb{R}^3_+)$ of the form

$$\left\{ \begin{array}{rcll} w' + A_{SCE}w + P(w\cdot\nabla)w &=& 0, & t>0,\\ w(0) &=& w_0, & \end{array}\right. \tag{2.1}$$

where the Stokes-Coriolis-Ekman operator A_{SCE} in $L^p_\sigma(\mathbb{R}^3_+)$ is defined by

$$\left\{ \begin{array}{rcl} A_{SCE}w &:=& P(-\nu\Delta w + \Omega\mathbf{e}_3\times w + [(u_E\cdot\nabla)w + w_3\partial_3 u_E]) = (A_S + A_C + A_E)w\\ D(A_{SCE}) &:=& W^{2,p}(\mathbb{R}^3_+)\cap W^{1,p}_0(\mathbb{R}^3_+)\cap L^p_\sigma(\mathbb{R}^3_+). \end{array}\right. \tag{2.2}$$

It follows for example from the results in [6] that the usual Stokes operator $A_{S,p} = -P\Delta$ generates a bounded analytic semigroup $e^{-tA_{S,p}}$ on $L^p_\sigma(\mathbb{R}^3_+)$ for all $p\in(1,\infty)$. By standard perturbation theory, we see that the Stokes-Coriolis-Ekman operator generates also an analytic semigroup $e^{-tA_{SCE,p}}$ on $L^p_\sigma(\mathbb{R}^3_+)$. In particular, after possible rescaling, the square root $(A_{SCE,2})^{1/2}$ of $A_{SCE,2}$ is a well defined operator in $L^2(\mathbb{R}^3_+)$. This allows us to define a weak solution to (1.4) as follows. For simplicity of notation, we omit the index 2 and write in the following $A_{SCE} = A_{SCE,2}$.

Definition 2.1. Let $w_0 \in L^2_\sigma(\mathbb{R}^3)$ and $f\in L^2((0,T);L^2_\sigma(\mathbb{R}^3_+)$ for all $T>0$. We call $w:[0,\infty)\to L^2_\sigma(\mathbb{R}^3_+)$ a *weak solution* of equation (1.4) if for all $T>0$,

i) $w\in L^\infty((0,T);L^2_\sigma(\mathbb{R}^3_+))\cap L^2((0,T);D(A^{1/2}_{SCE}))$ and
ii)

$$\begin{aligned} &-\int_0^T \langle w,\phi\rangle h'(t)dt + \nu\int_0^T\langle\nabla w,\nabla\phi\rangle h(t)dt + \int_0^T\langle(u_E\cdot\nabla)w,\phi\rangle h(t)dt\\ &+\int_0^T\langle w_3\cdot\partial_3 u_E,\phi\rangle h(t)dt + \Omega\int_0^T\langle\mathbf{e}_3\times w,\phi\rangle h(t)dt + \langle w\cdot\nabla w,\phi\rangle h(t)dt \quad = \quad \langle w_0,\phi\rangle h(0), \end{aligned}$$

holds for all $\phi\in D(A^{1/2}_{SCE})$ and all $h\in C^1([0,T],\mathbb{R})$ with $h(T)=0$.

Note that the sixth term on the left hand side above is meaningful since the dimension of the underlying space is 3.
We are now in the position to state our main results concerning global weak solutions of (1.4) and nonlinear stability of the Ekman spiral.

Theorem 2.2. *Assume that* $u_\infty\delta\nu^{-1}\le\frac{3}{2\sqrt{2}}$. *Then the following assertions hold.*

a) *There exists a weak solution to (1.4).*
b) *For every* $w_0\in L^2_\sigma(\mathbb{R}^3_+)$ *there exists at least one global weak solution* w *of (1.4) such that*

$$\lim_{t\to\infty}\|w(t)\|_2 = 0.$$

c) *Assume that for* $w_0\in L^2_\sigma(\mathbb{R}^3_+)$ *and some* $\alpha>0$

$$\|e^{-tA_{SCE}}w_0\|_2 = O(t^{-\alpha}).$$

Then there exists at least one global weak solution w *of (1.4) such that*

$$\|w(t)\|_2 = \left\{\begin{array}{ll} O(t^{-\alpha}), & \alpha\le\frac14,\\ O(t^{-\frac14}), & \alpha>\frac14. \end{array}\right.$$

3. Tools for the proof

Let $1<r<\infty$ and for $f\in L^r((0,T),L^r_\sigma(\mathbb{R}^3_+))$ consider the inhomogeneous equation

$$\left\{\begin{array}{rcll} u' + Au &=& f, & t\in(0,T),\\ u(0) &=& 0, & \end{array}\right. \tag{3.1}$$

associated with a sectorial operator A in $L^r_\sigma(\mathbb{R}^3_+)$. We say that A admits maximal L^r-regularity if there exists a unique function $u \in W^{1,r}((0,T); L^r_\sigma(\mathbb{R}^3_+)) \cap L^r((0,T); D(A))$ satisfying (3.1). In this case, there exists a constant $C > 0$ such that

$$\|u'\|_{L^r(0,T;L^r_\sigma(\mathbb{R}^3_+))} + \|Au\|_{L^r(0,T;L^r_\sigma(\mathbb{R}^3_+))} \leq C\|f\|_{L^r(0,T;L^r_\sigma(\mathbb{R}^3_+))}$$

holds. It is well known that the Stokes operator A_S in $L^r_\sigma(\mathbb{R}^3_+)$ admits maximal L^r-regularity; see e.g. [17],[6], or [10]. In order to prove that this holds true also for the Stokes-Coriolis-Ekman operator, we show that $A_{SCE} - A_S$ is relatively bounded with respect to A_S and apply Proposition 4.3 of [5]. Observe that there exists a constant $\mu > 0$ such that

$$\mathcal{R}\{\lambda(\lambda + \mu + A_{SCE})^{-1} : \lambda \in \Sigma_\theta\} < \infty,$$

where $\Sigma_\theta := \{z \in \mathbb{C}\backslash\{0\} : \arg z < \theta\}$ for some $\theta > \frac{\pi}{2}$ and $\mathcal{R}$ denotes the $\mathcal{R}$-bound of a family of bounded operators on $L^r_\sigma(\mathbb{R}^3_+)$. Thus, Proposition 4.3 in [5] implies the following proposition.

Proposition 3.1. *Let $1 < r < \infty$, $f \in L^r((0,T); L^r_\sigma(\mathbb{R}^3_+))$ and A_{SCE} in $L^r_\sigma(\mathbb{R}^3_+)$ be given as in (2.2). Then there exists $\mu > 0$ such that $A_{SCE} + \mu$ admits maximal L^r-regularity on $L^r_\sigma(\mathbb{R}^3_+)$. In particular, there exists a constant $C > 0$ such that*

$$\|u'\|_{L^r(0,T;L^r_\sigma(\mathbb{R}^3_+))} + \|(A_{SCE} + \mu)u\|_{L^r(0,T;L^r_\sigma(\mathbb{R}^3_+))} \leq C\|f\|_{L^r(0,T;L^r_\sigma(\mathbb{R}^3_+))}$$

holds.

Besides the above maximal L^r-estimates for our evolution equation, we need the following simple lemma on interpolation of L^p-spaces.

Lemma 3.2. *If $1 < p < q < r < \infty$, then $L^p(\mathbb{R}^n_+) \cap L^r(\mathbb{R}^n_+) \subset L^q(\mathbb{R}^n_+)$ and*

$$\|u\|_q \leq \|u\|_p^\alpha \|u\|_r^{1-\alpha}, \text{ where } \alpha = \frac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}}.$$

In fact, taking Hölder's inequality with exponents $\frac{p}{\alpha q}$ and $\frac{r}{(1-\alpha)q}$ yields the claim.

4. The Stokes-Coriolis-Ekman semigroup on $L^2_\sigma(\mathbb{R}^3_+)$

In this section we show that the Stokes-Coriolis-Ekman operator generates a contraction semigroup on $L^2_\sigma(\mathbb{R}^3_+)$ provided the Reynolds number $Re = u_\infty \delta \nu^{-1}$ is small enough. To this end, consider the Stokes-Coriolis-Ekman operator A_{SCE} in $L^2_\sigma(\mathbb{R}^3_+)$ defined by

$$\left\{ \begin{array}{rcl} A_{SCE}w & = & -\nu P\Delta w + \Omega PJPw + (P(u_E \cdot \nabla)w + Pw_3\partial_3 u_E) = (A_S + A_C + A_E)w \\ D(A_{SCE}) & = & H^2(\mathbb{R}^3_+) \cap H^1_0(\mathbb{R}^3_+) \cap L^2_\sigma(\mathbb{R}^3_+). \end{array} \right. \tag{4.1}$$

Then the following result holds:

Theorem 4.1. *The Stokes-Coriolis-Ekman operator A_{SCE} generates an analytic C_0-semigroup T_{SCE} of contractions on $L^2_\sigma(\mathbb{R}^3_+)$ provided*

$$\frac{u_\infty \delta}{\nu} \leq \frac{3}{2\sqrt{2}}. \tag{4.2}$$

Note that by the results of the previous sections we already know that T^p_{SCE} is an analytic C_0-semigroup on $L^p_\sigma(\mathbb{R}^3_+)$ for $1 < p < \infty$. The uniform boundedness of T_{SCE} in $L^2_\sigma(\mathbb{R}^3_+)$ will be essential in the following.

Lemma 4.2. *Let $p \in [1,\infty]$, $q \in [1,\infty)$ and $\alpha > 0$. Then there exists a constant $C > 0$ such that*

$$\|e^{-(\cdot)/\alpha} w(\cdot)\|_{L^p(\mathbb{R}_+)} \leq C\alpha^{1-1/q+1/p} \|\frac{d}{dx} w\|_{L^q(\mathbb{R}_+)}$$

for all $w \in W^{1,q}_0(\mathbb{R}_+)$.

Proof. Note first that it suffices to show the assertion for $w \in C_c^\infty(\mathbb{R}_+)$. In this case the result then follows from the representation

$$\mathrm{e}^{-s/\alpha} w(s) = \mathrm{e}^{-s/\alpha} \int_0^s w'(t) \mathrm{d}t, \quad s > 0,$$

and the fact that the function $s \mapsto \mathrm{e}^{-s/\alpha}$ belongs to $L^p(\mathbb{R}_+)$ for all $p \in [1, \infty]$. □

Proof. of Theorem 4.1. For $w_0 \in L^2_\sigma(\mathbb{R}^3_+)$ set $w(t) := T_{SCE}(t) w_0$. Then w satisfies

$$\left\{ \begin{array}{rcl} w' + A_{SCE} w &=& 0, \quad t > 0, \\ w(0) &=& w_0. \end{array} \right.$$

Multiplying the above equation with w and taking into account the skew symmetry of the second and third term of A_{SCE} we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3_+} |w(t)|^2 dx + \nu \int_{\mathbb{R}^3_+} |\nabla w(t)|^2 dx + \int_{\mathbb{R}^3_+} w(t) \cdot (w_3(t) \cdot \partial_3 u_E) dx = 0, \quad t > 0.$$

Since

$$\int_{\mathbb{R}^3_+} w \cdot (w_3 \cdot \partial_3 u_E) dx \le \sum_{j=1}^2 \|\mathrm{e}^{(\cdot)/2\delta} (\partial_3 u_E)^j w^3\|_2 \|\mathrm{e}^{-(\cdot)/2\delta} w^j\|_2.$$

and since

$$\partial_3 u_E(x_3) = \frac{u_\infty}{\delta} \mathrm{e}^{-x_3/\delta} \begin{pmatrix} \cos(x_3/\delta) + \sin(x_3/\delta) \\ \cos(x_3/\delta) - \sin(x_3/\delta) \\ 0 \end{pmatrix},$$

we see that

$$\|\mathrm{e}^{(\cdot)/2\delta} (\partial_3 u_E)^j w^3\|_2 \le \sqrt{2} \frac{u_\infty}{\delta} \|\mathrm{e}^{-(\cdot)/2\delta} w^3\|_2. \tag{4.3}$$

The above Lemma 4.2 implies

$$\|\mathrm{e}^{-(\cdot)/2\delta} w^j\|_2 \le \sqrt{\frac{2}{3}} \delta \|\partial_3 w^j\|_2, \quad j = 1, 2, 3.$$

Combining these estimates, we finally have

$$\int_{\mathbb{R}^3_+} w \cdot (w_3 \cdot \partial_3 u_E) dx \le \frac{2\sqrt{2}}{3} u_\infty \delta \|\nabla w\|_2^2.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \|w(t)\|_2^2 \le 0$$

for all $t > 0$, provided

$$u_\infty \delta \le \frac{3\nu}{2\sqrt{2}}.$$

Therefore

$$\|T_{SCE}(t) w_0\|_2 = \|w(t)\|_2 \le \|w_0\|_2, \quad t > 0,$$

and the assertion is proved.

□

5. Existence of global weak solutions

In this section we prove the existence of a global weak solution to problem (1.4) or to problem (2.1) in the case where the Reynolds number $Re = u_\infty \delta \nu^{-1}$ is small enough. We therefore assume throughout this section that

$$Re = \frac{u_\infty \delta}{\nu} \leq \frac{3}{2\sqrt{2}}.$$

We subdivide the proof into three steps.

Step 1: Approximate local solutions
We start by constructing first approximate solutions to our problem. To this end, we introduce smoothing operators J_k given by

$$J_k := (1 + k^{-1} A_{SCE})^{-1}, \quad k \in \mathbb{N}.$$

Since A_{SCE} is dissipative in $L^2_\sigma(\mathbb{R}^3_+)$, see Theorem 4.1, J_k is a bounded operator in $L^2_\sigma(\mathbb{R}^3_+)$ with $\|J_k\|_{L^2_\sigma(\mathbb{R}^3_+)} \leq 1$ for all $k \in \mathbb{N}$. By Sobolev's embedding theorem we have

$$\|J_k u\|_\infty \leq C(k)\|u\|_2. \tag{5.1}$$

Moreover, if $1 < p < \infty$, then there exist $k_0 \in \mathbb{N}$ and $C > 0$ such that

$$\|J_k u\|_{L^p_\sigma(\mathbb{R}^3_+)} \leq C\|u\|_{L^p_\sigma(\mathbb{R}^3_+)}, k \geq k_0. \tag{5.2}$$

Indeed, this follows from the fact that A_{SCE} generates analytic semigroup on $L^p_\sigma(\mathbb{R}^3_+)$ and general properties of sectorial operators. We now set

$$w_{0k} := J_k w_0 \text{ and } F_k w := -P(J_k w \cdot \nabla) w$$

and construct approximate solutions w_k to equation (2.1) by solving the integral equations

$$w_k(t) = e^{-tA_{SCE}} w_{0k} + \int_0^t e^{-(t-s)A_{SCE}} F_k w_k(s) ds. \tag{5.3}$$

To this end, consider for $T > 0$ the Banach space $X := C([0,T]; D(A^{1/2}_{SCE}))$ equipped with the norm

$$\|u\|_T := \sup_{0 \leq t \leq T} (\|u(t)\|_2 + \|A^{1/2}_{SCE} u(t)\|_2)$$

and for $M > 0$ and $k \in \mathbb{N}$ the closed set

$$S(k, M, T) := \{u \in X, u(0) = w_{0k}, \|u\|_T \leq M\}.$$

as well as the nonlinear operator Γ_k defined on $S(k, M, T)$ by

$$\Gamma_k u(t) := e^{-tA_{SCE}} w_{0k} + \int_0^t e^{-(t-s)A_{SCE}} F_k u(s) \, \mathrm{d}s.$$

Note that $\|F_k u\|_2 \leq C(k)\|u\|_2 \|\nabla u\|_2$. Since $D(A^{1/2}_{SCE}) = H^1_0(\mathbb{R}^3_+) \cap L^2_\sigma(\mathbb{R}^3_+)$ and therefore

$$\|\nabla u\|_2 \leq \|u\|_2 + \|A^{1/2}_{SCE} u\|_2 \tag{5.4}$$

we see that

$$\|F_k u\|_2 \leq C(k)(\|u\|_2^2 + \|u\|_2 \|A^{1/2}_{SCE} u\|_2). \tag{5.5}$$

Since $e^{-tA_{SCE}}$ is an analytic semigroup of contractions we also have

$$\|\nabla e^{-tA_{SCE}} w\|_2 \quad \leq \quad C t^{-\frac{1}{2}} \|w\|_2, \quad t > 0.$$

We thus may estimate $\Gamma_k u(t)$ as follows

$$\begin{aligned}
\|\Gamma_k u(t)\|_T &\leq \|w_{0k}\|_2 + \|A_{SCE}^{1/2} w_{0k}\|_2 \\
&\quad + \sup_{0\leq t\leq T}\{\int_0^t C(k)\|u\|_2^2 \,\mathrm{d}s\} + \sup_{0\leq t\leq T}\{\int_0^t C(k)\|u\|_2 \|A_{SCE}^{1/2} u\|_2 \,\mathrm{d}s\} \\
&\quad + \sup_{0\leq t\leq T}\{\int_0^t t^{-\frac{1}{2}} C(k)\|u\|_2^2 \,\mathrm{d}s\} + \sup_{0\leq t\leq T}\{\int_0^t t^{-\frac{1}{2}} C(k)\|u\|_2 \|A_{SCE}^{1/2} u\|_2 \,\mathrm{d}s\} \\
&\leq \|w_{0k}\|_2 + \|A_{SCE}^{1/2} w_{0k}\|_2 + C_1(k) M^2 (T + T^{\frac{1}{2}}).
\end{aligned}$$

Furthermore, since

$$\begin{aligned}
||F_k u_1 - F_k u_2||_2 &\leq ||J_k u_2 \nabla (u_1 - u_2)||_2 + ||(J_k u_1 - J_k u_2)\nabla u_1||_2 \\
&\leq CM||u_1 - u_2||_T, \quad u_1, u_2 \in S(k, M, T).
\end{aligned}$$

we obtain

$$\|\Gamma_k u_1(t) - \Gamma_k u_2(t)\|_T \leq C_2(k) M (T + T^{\frac{1}{2}}) \|u_1(t) - u_2(t)\|_T.$$

Fix now M in such a way that $\|w_{0k}\|_2 + \|A_{SCE}^{1/2} w_{0k}\|_2 \leq \frac{M}{2}$ and then T such that $C_1(k)M^2(T+T^{\frac{1}{2}}) \leq \frac{M}{2}$ and $C_2(k)M(T + T^{\frac{1}{2}}) < 1$. Then Γ_k is a strict contraction in $S(k, M, T)$ and by Banach fixed point theorem, there exists a unique w_k in $S(k, M, T)$ satisfying (5.3) for $t \in (0, T)$.

Step 2: Approximate global solutions

In the following we prove a priori bounds for $w_k(T)$ and $A_{SCE}^{1/2} w_k(T)$ for all $T > 0$. To this end, recall that w_k is the solution of the equation

$$w_k'(t) + A_{SCE} w_k = F_k w_k, \quad t \in (0, T). \tag{5.6}$$

Multiplying (5.6) with w_k and integrating by parts yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_k\|_2^2 + \langle A_{SCE} w_k, w_k\rangle = \langle F_k w_k, w_k\rangle.$$

Since $\langle (u_E \cdot \nabla) w_k, w_k\rangle = \Omega\langle(\mathbf{e}_3 \times w_k), w_k\rangle = \langle F_k w_k, w_k\rangle = 0$ it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_k\|_2^2 + \langle A_S w_k, w_k\rangle + \langle w_{k3}\partial_3 u_E, w_k\rangle = 0.$$

Theorem 4.1 implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_k\|_2^2 + C\|\nabla w_k\|_2^2 \leq 0 \tag{5.7}$$

for some $C > 0$. Integrating with respect to t yields

$$\|w_k(T)\|_2^2 + \int_0^T \|\nabla w_k(s)\|_2^2 ds \leq C\|w_0\|_2. \tag{5.8}$$

Next, forming the dual pairing of (5.6) with $A_{SCE} w_k$ we obtain

$$\begin{aligned}
&\langle w_k', A_S w_k\rangle + \langle w_k', (u_E \cdot \nabla) w_k\rangle + \langle w_k', w_{k3}\partial_3 u_E\rangle \\
&\quad + \langle w_k', \Omega(\mathbf{e}_3 \times w_k)\rangle + \langle A_{SCE} w_k, A_{SCE} w_k\rangle = \langle F_k w_k, A_{SCE} w_k\rangle.
\end{aligned}$$

Substituting $w_k' = -A_{SCE} w_k + F_k w_k$ leads to

$$\begin{aligned}
\langle w_k', A_S w_k\rangle + \|A_{SCE} w_k\|_2^2 &= \langle A_{SCE}, (u_E \cdot \nabla) w_k + w_{k3}\partial_3 u_E + \Omega(\mathbf{e}_3 \times w_k)\rangle \\
&\quad - \langle F_k w_k, (u_E \cdot \nabla) w_k + w_{k3}\partial_3 u_E + \Omega(\mathbf{e}_3 \times w_k)\rangle + \langle F_k w_k, A_{SCE} w_k\rangle.
\end{aligned}$$

Hence,

$$\langle w_k', A_S w_k\rangle \leq C(\|w_k\|_2^2 + \|\nabla w_k\|_2^2) + C(k)\|w_k\|_2^2\|\nabla w_k\|_2^2 \leq C(\|w_0\|_2^2 + \|\nabla w_k\|_2^2) + C(k)\|w_0\|_2^2\|\nabla w_k\|_2^2,$$

where the second inequality is due to the fact that $\|w_k\|_2^2 \le \|w_{k0}\|_2^2 \le \|w_0\|_2^2$ which follows from (5.7) and the contractivity of J_k. Finally, since $\langle w_k', A_S w_k\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w_k\|_2^2$, integrating with respect to t yields together with (5.8)

$$\|\nabla w_k(T)\|_2^2 \le C(T\|w_0\|_2^2 + \|w_0\|_2^2) + C(k)\|w_0\|_2^4 + \|\nabla w_{0k}\|_2^2.$$

As $D(A_{SCE}^{1/2}) = H_0^1(\mathbb{R}^3_+) \cap L^2_\sigma(\mathbb{R}^3_+)$ the estimate

$$\|A_{SCE}^{1/2} u\|_2 \le \|u\|_2 + \|\nabla u\|_2$$

holds. Combining this with the estimate given in (5.8) we obtain an a-priori bound for $\|w_k\|_T$ for every $T > 0$.

Step 3: Weak convergence

In this final step we show that the approximate global solutions w_k constructed above converge in the weak sense to some function w satisfying the equation

$$w(t) = e^{-tA_{SCE}} w_0 - \int_0^t e^{-(t-s)A_{SCE}} P(w\cdot\nabla)w(s)ds. \tag{5.9}$$

We fix some time interval $[0, T]$. Since

$$\int_0^T \|A_{SCE}^{1/2} w_k(s)\|_2^2 ds \le \int_0^T \|w_k(s)\|_2^2 + \|\nabla w_k(s)\|_2^2 ds \le T\|w(0)\|_2^2 + \int_0^T \|\nabla w_k(s)\|_2^2 ds$$

the above inequality (5.8) implies

$$w_k \in L^2(0,T; D(A_{SCE}^{1/2})) \cap L^\infty(0,T; L^p_\sigma(\mathbb{R}^3_+)) =: Y =: Y_1 \cap Y_2, \quad k \in \mathbb{N}$$

and that (w_k) is even a bounded sequence in Y. Since Y_1 is reflexive, there exists a subsequence of (w_k) converging weakly in Y_1. Further, by Alaoglu's theorem, (w_k) possesses a weak-star convergent subsequence in Y_2 and thus there exists a function $w \in Y$ with (w_k) converging weakly to w in Y_1 and (w_k) converging in the weak-star topology to w in Y_2.

Next, we write $w_k(t) = w_k^{(1)}(t) + w_k^{(2)}(t)$ where

$$\begin{aligned} w_k^{(1)}(t) &:= e^{-tA_{SCE}} w_{0k}, \\ w_k^{(2)}(t) &:= \int_0^t e^{-(t-s)A_{SCE}} F_k w_k(s)\, \mathrm{d}s. \end{aligned}$$

Performing the same calculations which led to (5.8), we now obtain

$$\|w_k^{(1)}(t) - w_l^{(1)}(t)\|_2^2 + \int_0^t \|A_{SCE}^{1/2}(w_k^{(1)}(s) - w_l^{(1)}(t))\|_2^2 ds \le C\|w_{0k} - w_{0l}\|_2^2, \quad k,l \in \mathbb{N}.$$

Since $w_{0k} \to w_0$ in $L^p_\sigma(\mathbb{R}^3_+)$ as $k \to \infty$ we see that $(w_k^{(1)})$ and $(w_k^{(2)})$ are bounded sequences in Y.

Next, we set $r = \frac{n+2}{n+1}$ for $n = 3$. By Lemma 3.2, Hölder's and Sobolev's inequalities

$$\|F_k w_k\|_r \le C\|J_k w_k\|_{\frac{2(n+2)}{n}} \|\nabla w_k\|_2 \le \|w_k\|_2^{\frac{2}{n+2}} \|\nabla J_k w_k\|_2^{\frac{n}{n+2}} \|\nabla w_k\|_2.$$

Since $\|\nabla J_k v_k\|_2 \le \|A_{SCE}^{1/2} v_k\|_2 + \|v_k\|_2$ by (5.4), we see that

$$\|F_k w_k\|_r \le C\|w_0\|_2^{\frac{2}{n+2}} (\|A_{SCE}^{1/2} w_k\|_2 + \|w_0\|_2)^{\frac{2}{r}}.$$

Further, since (w_k) is bounded in Y_2, it follows from (5.8) that

$$\int_0^T \|F_k w_k\|_r^r dt \le C\|w_0\|_2^{\frac{2r}{n+2}} \int_0^T (\|A_{SCE}^{1/2} w_k\|_2 + \|w_0\|_2)^2 dt \le C(T+1)\|w_0\|_2^{\frac{2r}{n+2}+2}.$$

Hence, $(F_k w_k)$ is a bounded sequence in $L^r(0,T; L^r_\sigma(\mathbb{R}^3_+))$. By construction, $w_k^{(2)}$ is a solution of the Cauchy problem

$$\begin{aligned} w_k'(t) + A_{SCE} w_k(t) &= F_k w_k(t), \quad t \ge 0, \\ w(0) &= 0. \end{aligned}$$

Thus our result on maximal L^r-regularity for A_{SCE}, i.e. Proposition 3.1, implies that $(w_k^{(2)})$ is a bounded sequence in $L^r(0,T;D(A_{SCE,r})) \cap W^{1,r}(0,T;L^r_\sigma(\mathbb{R}^3_+))$. The operators J_k are uniformly bounded on $L^r_\sigma(\mathbb{R}^3_+)$ in $k \geq k_0$; see (5.2). Thus $(J_k w_k^{(2)})_{k\geq k_0}$ is a bounded sequence in $L^r(0,T;D(A_{SCE,r})$ $\cap W^{1,r}(0,T;L^r_\sigma(\mathbb{R}^3_+))$ as well.

Since (w_k) is a bounded sequence in Y by (5.8) and since (w_k) and $(J_k w_k)$ are bounded in the space of maximal regularity, if follows from theorem III.2.1 in [18] and lemma 1.4.6 in [14] that $(w_k^{(2)})$ and $(J_k w_k^{(2)})_{k\geq k_0}$ are relatively compact in $L^2(K\times(0,T))$ for any fixed compact set $K\subset\mathbb{R}^3_+$. It follows that $(w_k^{(2)})$ and $(J_k w_k^{(2)})$ converge in $L^2(\mathbb{R}^3_+\times(0,T))$. Therefore, $w_k(s)\to w(s)$ and $J_k w_k(s)\to w(s)$ for a.a. $s\in(0,T)$ for some function $w\in Y$.

Finally, we need to verify that function w constructed above is in fact a weak solution of our problem (1.4). To this end, note that by the weak convergence of (w_k) in Y_1 we have

$$\begin{aligned}
\lim_{k\to\infty}\int_0^T -\langle w_k,\phi\rangle h' \,\mathrm{d}t &= \int_0^T -\langle w,\phi\rangle h' \,\mathrm{d}t,\\
\lim_{k\to\infty}\int_0^T \langle \nabla w_k,\nabla\phi\rangle h \,\mathrm{d}t &= \int_0^T \langle \nabla w,\nabla\phi\rangle h \,\mathrm{d}t,\\
\lim_{k\to\infty}\int_0^T \langle (u_E\cdot\nabla) w_k,\phi\rangle h \,\mathrm{d}t &= \int_0^T \langle (u_E\cdot\nabla) w,\phi\rangle h \,\mathrm{d}t,\\
\lim_{k\to\infty}\int_0^T \langle w_{k3}\cdot\partial_3 u_E,\phi\rangle h \,\mathrm{d}t &= \int_0^T \langle w_3\cdot\partial_3 u_E,\phi\rangle h \,\mathrm{d}t,\\
\lim_{k\to\infty}\int_0^T \omega\langle \mathbf{e}_3\times w_k,\phi\rangle h \,\mathrm{d}t &= \int_0^T \omega\langle \mathbf{e}_3\times w,\phi\rangle h \,\mathrm{d}t,\\
\lim_{k\to\infty}\langle w_{0k},\phi\rangle h(0) &= \langle w_0,\phi\rangle h(0)
\end{aligned}$$

since all these terms are linear. It remains to show that

$$\lim_{k\to\infty}\int_0^T \langle J_k w_k\cdot\nabla w_k,\phi\rangle h \,\mathrm{d}t = \int_0^T \langle w\cdot\nabla w,\phi\rangle h \,\mathrm{d}t.$$

Let $\chi_N:\mathbb{R}^3_+\to\{0,1\}$ be given by

$$\chi_N(x) = \begin{cases} 1, & x\in\mathbb{R}^3_+\cap B(0,N),\\ 0, & \text{otherwise}\end{cases}$$

where $B(x,n)$ denotes the ball with center x and radius n and consider

$$\int_0^T \langle J_k w_k\cdot\nabla w_k,\phi\rangle h \,\mathrm{d}t = \int_0^T \langle J_k w_k\cdot\nabla w_k,\chi_N\phi\rangle h \,\mathrm{d}t + \int_0^T \langle J_k w_k\cdot\nabla w_k,(1-\chi_N)\phi\rangle h \,\mathrm{d}t.$$

Assume first that $w\in D(A_{SCE}^{1/2})\cap L^\infty$. Then

$$\lim_{N\to\infty}\int_0^T \langle J_k w_k\cdot\nabla w_k,\chi_N\phi\rangle h \,\mathrm{d}t = \int_0^T \langle w\cdot\nabla w,\chi_N\phi\rangle h \,\mathrm{d}t$$

since $\chi_N v$ is bounded and has bounded support. If $v\notin L^\infty$, take $J_n w\in D(A_{SCE}^{1/2})\cap L^\infty$ and pass then to the limit $N\to\infty$.

Since $n=3$ we obtain by Hölder's and by Sobolev's inequality with $r=(\frac{1}{2}+\frac{1}{n})^{-1}=\frac{6}{5}$ as well as by the Gagliardo-Nirenberg theorem that

$$\begin{aligned}|\int_0^T \langle J_k w_k\cdot\nabla w_k,(1-\chi_N)\phi\rangle h\ \mathrm{d}t| &\le C\int_0^T \|J_k w_k\cdot\nabla w_k\|_r\|(1-\chi_n)\phi\|_{2^*}\ \mathrm{d}t\\ &\le C\int_0^T \|J_k w_k\|_n\|\nabla w_k\|_2\|(1-\chi_n)\phi\|_{2^*}\ \mathrm{d}t\\ &\le C\int_0^T \|w_k\|_2^{\frac{1}{2}}\|\nabla J_k w_k\|_2^{\frac{1}{2}}\|\nabla w_k\|_2\|(1-\chi_N)\phi\|_{2^*}\ \mathrm{d}t.\end{aligned}$$

Since $\phi\in D(A_{SCE}^{1/2})$ and $D(A_{SCE}^{1/2})\hookrightarrow L^{2^*}(\mathbb{R}^3_+)$, we obtain

$$\lim_{N\to\infty}|\int_0^T\langle J_k w_k\cdot\nabla w_k,(1-\chi_N)\phi\rangle h\ \mathrm{d}t|=0.$$

Since our choice of T was arbitrary, there exists for every $T>0$ a weak solution $w\in L^2(0,T;D(A_{SCE}^{1/2}))\cap L^\infty(0,T;L^2_\sigma(\mathbb{R}^3_+))$ of (2.1).

□

6. Proof of the stability estimates

In this section we prove assertions a) and b) of Theorem 2.2. Note first that the solution of problem (2.1) is given by

$$w(t)=e^{-tA_{SCE}}w_0-\int_0^t e^{-(t-s)A_{SCE}}P(w\cdot\nabla)w(s)ds.$$

In order to estimate the nonlinear term, recall that

$$\|\nabla e^{-tA_{SCE}}u\|_2\le Ct^{-\frac{1}{2}}\|u\|_2,\quad u\in L^p_\sigma(\mathbb{R}^3_+).$$

The same estimate is of course also valid for the adjoint operator ${A_{SCE}}^*$. This implies

$$\begin{aligned}\|e^{-tA_{SCE}}P(w\cdot\nabla)w\|_2=\sup_{\|\phi\|_2=1}|\langle e^{-tA_{SCE}}P(w\cdot\nabla)w,\phi\rangle| &= \sup_{\|\phi\|_2=1}|\langle w\otimes w,\nabla e^{-tA_{SCE}{}^*}\phi\rangle|\\ &\le \sup_{\|\phi\|_2=1}\|w\otimes w\|_2\|\nabla e^{-tA_{SCE}{}^*}\phi\|_2\\ &\le Ct^{-\frac{1}{2}}\|w\|_4^2.\end{aligned}$$

Since $w\in D(A_{SCE}^{1/2})=D(A_S^{1/2})$, it follows that

$$\|w\|_4^2\le C\|A_S^{3/8}w\|_2^2\le C\|w\|_2^{\frac{1}{2}}\|\nabla w\|_2^{\frac{3}{2}}$$

and thus we may estimate the nonlinear term as

$$\|e^{-tA_{SCE}}P(w\cdot\nabla)w\|_2\le Ct^{-\frac{1}{2}}\|w\|_2^{\frac{1}{2}}\|\nabla w\|_2^{\frac{3}{2}}. \tag{6.1}$$

Hence,

$$\|w(t)\|_2\le\|e^{-tA_{SCE}}w_0\|_2+C\int_0^t(t-s)^{-\frac{1}{2}}\|w\|_2^{\frac{1}{2}}\|\nabla w\|_2^{\frac{3}{2}}\ \mathrm{d}s. \tag{6.2}$$

By (5.7), $\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_2\le 0$ for all $t>0$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_2^2+\frac{m}{t}\|w\|_2\le\frac{m}{t}\|w\|_2$$

for every integer $m>0$. Multiplying this inequality by t^m and applying estimate (6.1), yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^m\|w\|_2)\le mt^{m-1}\|e^{-tA_{SCE}}w_0\|_2+Cmt^{m-1}(\int_0^t(t-s)^{-\frac{1}{2}}\|w\|_2^2\ \mathrm{d}s)^{\frac{1}{4}}(\int_0^t(t-s)^{-\frac{1}{2}}\|\nabla w\|_2^2\ \mathrm{d}s)^{\frac{3}{4}}.$$

Denoting the second term on the right hand side above by $F(t)$, we obtain after integrating in t and dividing by t^m

$$\|w(t)\|_2 \leq t^{-m} \int_0^t m\tau^{m-1} \|e^{-tA_{SCE}} w_0\|_2 \, \mathrm{d}\tau + t^{-m} \int_0^t F(\tau) \, \mathrm{d}\tau.$$

Furthermore, we set

$$F_1(t) := \int_0^t (t-s)^{-\frac{1}{2}} \|w\|_2^2 \, \mathrm{d}s \text{ and } F_2(t) := \int_0^t (t-s)^{-\frac{1}{2}} \|\nabla w\|_2^2 \, \mathrm{d}s$$

and estimate

$$\begin{aligned} t^{-m} \int_0^t F(\tau) \, \mathrm{d}\tau &\leq Ct^{-m} \int_0^t m\tau^{m-1} (F_1(\tau))^{\frac{1}{4}} (F_2(\tau))^{\frac{3}{4}} \, \mathrm{d}\tau \\ &\leq Cmt^{-1} \Big(\int_0^t F_1(\tau) \, \mathrm{d}\tau\Big)^{\frac{1}{4}} \Big(\int_0^t F_2(\tau) \, \mathrm{d}\tau\Big)^{\frac{3}{4}} \\ &\leq Cm\Big(t^{-1} \int_0^t F_1(\tau) \, \mathrm{d}\tau\Big)^{\frac{1}{4}} \Big(t^{-1} \int_0^t F_2(\tau) \, \mathrm{d}\tau\Big)^{\frac{3}{4}}. \end{aligned}$$

Recall that $\frac{\mathrm{d}}{\mathrm{d}t} \|w(t)\|_2^2 + C\|\nabla w\|_2^2 \leq 0$ for all $t > 0$. In particular, $\|w\|_2 \in L^\infty(\mathbb{R}_+)$ and

$$t^{-1} \int_0^t F_1(\tau) \, \mathrm{d}\tau \leq Ct^{\frac{1}{2}}$$

and also $\|\nabla w\|_2 \in L^1(\mathbb{R}_+)$ with

$$t^{-1} \int_0^t F_2(\tau) \, \mathrm{d}\tau \leq Ct^{-\frac{1}{2}}.$$

Summing up, we proved that

$$\|w(t)\|_2 \leq t^{-m} \int_0^t m\tau^{m-1} \|e^{-tA_{SCE}} w_0\|_2 \, \mathrm{d}\tau + Ct^{-\frac{1}{4}}. \tag{6.3}$$

In order to prove assertion i), we only have to observe that $\lim_{t\to\infty} \|e^{-tA_{SCE}} w_0\|_2 = 0$. The assertion then follows from the estimate (6.3).

Finally, assume in addition that $\|e^{-tA_{SCE}} w_0\|_2 = O(t^{-\alpha})$ for some $\alpha > 0$. The estimate (6.3) implies that $t^{-m} \int_0^t m\tau^{m-1} \|e^{-tA_{SCE}} w_0\|_2 \, \mathrm{d}\tau = O(t^{-\alpha})$, too, which proves assertion ii) of our theorem.

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