THE PERMANENCE OF $\mathcal{R}$-BOUNDEDNESS AND PROPERTY(α) UNDER INTERPOLATION AND APPLICATIONS TO PARABOLIC SYSTEMS

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ABSTRACT. This note consists of two parts. In the first part we consider the behavior of $\mathcal{R}$-boundedness, $\mathcal{R}$-sectoriality, and property(α) under the interpolation of Banach spaces. In a general setting we prove that for interpolation functors of type h the $\mathcal{R}$-boundedness, the $\mathcal{R}$-sectoriality, and the property(α) preserve under interpolation. In particular, this is true for the standard real and complex interpolation methods. (Partly, these results were indicated in [12], however, with just a very brief outline of their proofs.) The second part represents an application of the first part. We prove $\mathcal{R}$-sectoriality, or equivalently, maximal L^p-regularity for a general class of parabolic systems on interpolation spaces including scales of Besov- and Bessel-potential spaces over $\mathbb{R}^n$.

1. Introduction

The concept of $\mathcal{R}$-bounded operator families nowadays plays an important role in the treatment of linear and nonlinear problems. By the celebrated result of L. Weis [21], it is known that $\mathcal{R}$-boundedness of the resolvent family $\lambda(\lambda + A)^{-1}$ for λ in a complex sector with opening angle greater that $\pi/2$ implies maximal regularity for a linear operator A. The maximal regularity, in turn, is fundamental in the treatment of linear and nonlinear PDEs for various reasons: the construction of local-in-time strong solutions, of global weak solutions, of real analytic solutions, uniqueness proofs, and so on.

Also, in combination with a holomorphic functional calculus, a so-called H^∞-calculus, the concept of $\mathcal{R}$-boundedness turned out to be very valuable. It allows for the introduction of a joint H^∞-calculus of two closed linear operators A, B, cf. [14]. In particular, it gives an answer to the question under what circumstances $f(A, B)$ gives rise

2010 *Mathematics Subject Classification.* Primary 46B70, 47F05; Secondary 35K41, 35K46.

Key words and phrases. Interpolation, $\mathcal{R}$-boundedness, maximal regularity, parabolic systems.

to a bounded operator for bounded real analytic functions $f(\lambda, z)$. In the simplest case we might have $f(\lambda, z) = (\lambda + z)^{-1}$, $A = d/dt$, and $B = -\Delta$, for instance. The full strength of such a joint H^∞-calculus reveals in the treatment of free boundary value problems, see e.g. [7], [17]. This type problems often can be reduced to the boundary, on which, however, one is faced to a mixed order system. The associated matrix symbols (Lopatinskii matrix) not seldom have a complicated structure, but still are real analytic functions and therefore fit into the framework of the joint H^∞-calculus developed in [14] or [11].

In this context also a geometric property of a Banach space (besides the property '*of class $\mathcal{HT}$*') comes into play: the so-called '*property(α)*'. In many situations it represents the crucial ingredient for the step from uniform boundedness to $\mathcal{R}$-boundedness. For instance, if property(α) for a Banach space X is assumed, the standard multiplier results, if applicable, yield $\mathcal{R}$-boundedness of an operator family $(M_\lambda)_\lambda$ in a parameter λ, instead of uniform boundedness only. For $M_\lambda = \lambda(\lambda + A)^{-1}$ this leads directly to the $\mathcal{R}$-sectoriality of A or, equivalently, to the maximal L^p-regularity. We refer to [15, Theorem 5.2 b)] for a multiplier result of this type. If an operator has a bounded H^∞-calculus on X, it automatically admits the stronger property of an $\mathcal{R}$-bounded H^∞-calculus, provided X has property(α). This is another significant consequence of property(α). This fact particularly matters for the initialization and the application of a joint H^∞-calculus, cf. [14].

For all these reasons, it is important to know about the behavior of the notions of $\mathcal{R}$-boundedness and property(α) with respect to other functional analytic operations, such as the interpolation of Banach spaces, for instance. In the first part of this paper we clarify this behavior. In fact, we will show that both properties preserve under interpolation. These results are indicated in [12]. However, their proofs are just outlined and for readers not so experienced in this topic it might be hard to follow the very brief argumentation given in [12]. It also seems that a rigorous proof so far is not contained anywhere else in the available literature, although the results are not seldom used in other works. With the aim to apply them in order to prove $\mathcal{R}$-sectoriality for a class of parameter-elliptic systems on Besov and Bessel-potential spaces, here we give a rigorous proof of the results on the interpolation of $\mathcal{R}$-boundedness and property(α) indicated in [12]. Indeed, we prove the preservation of these two properties in a very general setting for exact interpolation functors of type θ. This covers the cases of real and complex interpolation and also generalizes the results indicated in [12].

The proof of the permanence results under discussion is based on the characterization of $\mathcal{R}$-boundedness in terms of boundedness in Rademacher spaces. The Rademacher spaces are complementary in $L^p([0,1],X)$ if X is K-convex. By these facts the interpolation of $\mathcal{R}$-boundedness is reduced to the interpolation of X-valued L^p-spaces and to general facts concerning the interpolation of complementary subspaces. The preservation of property(α) under interpolation then can be reduced to the obtained results on the interpolation of $\mathcal{R}$-boundedness. Indeed, property(α) can be regarded as a special form of $\mathcal{R}$-boundedness on Rademacher spaces. Therefore the results on $\mathcal{R}$-boundedness apply.

This is also the reason, why we give the proof of the preservation of property(α) under interpolation here, although this result is not directly applied in this note. In a forthcoming work it will be applied in order to prove an $\mathcal{R}$-bounded H^∞-calculus for a certain class of elliptic operators on scales of Besov and Bessel-potential spaces.

In the second part of this note we will apply the results obtained in the first part in order to prove $\mathcal{R}$-sectoriality for a class of parameter-elliptic systems realized on interpolation spaces. For classical works on parameter-elliptic systems we refer to [2] and [19]. Particularly, we will apply the obtained permanence results to real and complex interpolation functors. This yields the $\mathcal{R}$-sectoriality on scales of Besov and Bessel-potential spaces. For this purpose, we first establish the corresponding result for the model problem (i.e., constant coefficients) in Sobolev spaces $W^{k,p}(\mathbb{R}^n, \mathbb{C}^n)$ (see Proposition 5.12). This will be based on a multiplier theorem. Employing a localization procedure and perturbation arguments, the result generalizes to a class of variable coefficients (see Theorem 5.28). This result can be found in [15] for the special case $k = 0$. Interpolation and the outcome of the first part of this note then imply the $\mathcal{R}$-sectoriality on scales of interpolation spaces (see Theorem 5.29).

We remark that the results in the second part seem to be available also by combining deep results from known literature. For instance, employing results on $\mathcal{R}$-sectoriality on L^p for higher order operators with top order coefficients in BUC achieved in [6] and classical results on elliptic regularity, via interpolation one might be able to derive $\mathcal{R}$-sectoriality of elliptic higher order operators in $W^{k,p}$ without localizing. Moreover, classical results of [5] or results obtained in [11] show that a sectorial operator always admits maximal regularity or a bounded H^∞-calculus in real interpolation spaces (however, at first with no explicit information on its domain). Since we want to keep our approach as selfcontained as possible, however, we give a direct and elementary proof of $\mathcal{R}$-sectoriality for a class of parabolic problems, which even works for general L^p-compatible interpolation scales.

The paper is organized as follows. The first part includes Section 2 to 4. In Section 2 we clarify the notation. Section 3 includes the introduction of $\mathcal{R}$-bounded families, the characterization via Rademacher spaces, and the result on the permanence of $\mathcal{R}$-boundedness under interpolation (Theorem 3.19). The introduction, the characterization in terms of $\mathcal{R}$-boundedness, and the corresponding results on the interpolation of property(α) are the content of Section 4. In the second part, i.e. in Section 5, we prove the mentioned $\mathcal{R}$-sectoriality for a class of parameter-elliptic systems realized on scales of interpolation spaces. The main results here are Theorem 5.29 and Corollary 5.31.

2. Notation

Definition 2.1. *In the sequel we use the following notation:*

- *The set* $\{X_0, X_1\}$ *is said to be an interpolation couple, if* X_0 *and* X_1 *are Banach spaces, which are embedded in a Hausdorff topological vector space* $\mathcal{X}$. *On* $X_0 + X_1$ *we define the norm* $\|x\|_{X_0+X_1} := \inf_{x_k\in X_k: x_0+x_1=x}(\|x_0\|_{X_0} + \|x_1\|_{X_1})$ *for* $x \in X_0+X_1$.
- *Let* $\{X_0, X_1\}$, $\{Y_0, Y_1\}$ *be interpolation couples, then we define*

$$\begin{aligned} L(\{X_0, X_1\},\{Y_0, Y_1\}) := \{T : X_0 + X_1 \to Y_0 + Y_1 |\\ T \text{ linear and } T_{|X_k} \in L(X_k, Y_k), k = 0, 1\} \end{aligned}$$

  and $L(\{X_0, X_1\}) := L(\{X_0, X_1\}, \{X_0, X_1\})$.
- *For normed spaces* X *and* Y *we denote the existence of an injective continuous linear mapping from* Y *to* X *by* $Y \hookrightarrow X$. *By* $L(Y, X)$ *we denote the space of all linear and bounded operators from* Y *into* X.
- *By* $Y \hookrightarrow_d X$ *we denote the existence of an injective continuous linear mapping from* Y *to* X *with dense image in* X.
- *Let* X *and* Y *be normed spaces. The equality* $X = Y$ *is used with the meaning that there exists an isomorphism between* X *and* Y. *In particular we have equivalence of the norms in this case.*
- *By virtue of the appearance of various spaces we often want to make clear in which space an integral or a series converges. Hence we write "*$\int \ldots dx_{[X]}$*" and "*${}_{[X]}\sum \ldots$*" to indicate the convergence in* X.

For the definition of an interpolation functor $\mathcal{F}$ we follow [20, 1.2.2]. Particularly we make use of interpolation functors of *type* h, which means that we have an estimate as

$$\|T_{|\mathcal{F}(\{X_0,X_1\})}\|_{L(\mathcal{F}(\{X_0,X_1\}),\mathcal{F}(\{Y_0,Y_1\}))} \\ \leq C_h \cdot h(\|T_{|X_0}\|_{L(X_0,Y_0)}, \|T_{|X_1}\|_{L(X_1,Y_1)})$$

with a constant $C_h > 0$. An interpolation functor is called *exact of type* θ, if it is of type $h(t_0, t_1) := t_0^{1-\theta} t_1^{\theta}$ with $C_h = 1$. Important examples of exact interpolation functors of type θ are given by the *real* and *complex interpolation methods*, which are defined as in [1, 7.9/7.51]. The proof of the exactness for these two methods can be found e.g. in [20, 1.3.3 (a)/1.9.3 (a)]. As usually, we denote the real interpolation space by $(X_0, X_1)_{\theta,p}$ and the complex interpolation space by $[X_0, X_1]_\theta$. We want to mention that in this note we only use the *K-method* for real interpolation.

Let $(\Omega, \mathscr{B}, \mu)$ be a measure space with a σ-finite measure μ on $\mathscr{B}$. For a Banach space X we denote the Banach space valued L^p-space $L^p(\Omega, \mathscr{B}, \mu, X)$ by $L^p(X)$. For an domain $\Omega \subset \mathbb{R}^n$ we denote the Sobolev space of order $m \in \mathbb{N}_0$ by $W^{m,p}(\Omega, X)$ or $W^{m,p}(X)$. Here and in the following we always consider the case $p \in (1, \infty)$ except in Section 3.1. An interpolation functor $\mathcal{F}$ is called *L^p-compatible*, if we have $\mathcal{F}(\{L^p(X_0), L^p(X_1)\}) = L^p(\mathcal{F}(\{X_0, X_1\}))$ for all interpolation couples $\{X_0, X_1\}$. In [20] it is proved that real and complex interpolation methods are *L^p-compatible* interpolation functors such that

$$\begin{aligned} C_p^{(1)} \|\cdot\|_{(L^p(X_0),L^p(X_1))_{\theta,p}} &\leq \|\cdot\|_{L^p((X_0,X_1)_{\theta,p})} \\ &\leq C_p^{(2)} \|\cdot\|_{(L^p(X_0),L^p(X_1))_{\theta,p}}, \\ \|\cdot\|_{[L^p(X_0),L^p(X_1)]_\theta} &= \|\cdot\|_{L^p([X_0,X_1]_\theta)} \end{aligned} \tag{1}$$

where the constants $C_p^{(1)} > 0$ and $C_p^{(2)} > 0$ are independent of the spaces X_0 and X_1.

To avoid confusion with different definitions of the resolvent set, we give it here. Under the resolvent set $\rho(A)$ of a linear and densely defined operator $A : D(A) \subset X \to X$ we understand the set of all $\lambda \in \mathbb{C}$ such that $(\lambda - A) : D(A) \to X$ is bijective and $(\lambda - A)^{-1} \in L(X)$.

3. $\mathcal{R}$-boundedness and Rademacher spaces

3.1. **Basic properties and definitions.** The following definitions and basic consequences can be found in detail in [15, Section 2] or [9, Section 11].

Definition 3.1. *Let X,Y be Banach spaces, $\mathcal{T} \subset L(X,Y)$, and $p \in [1,\infty)$. Then $\mathcal{T}$ is said to be $\mathcal{R}$-bounded, if there exists a constant $C > 0$ such that for all $m \in \mathbb{N}$, $(T_k)_{k=1,\dots,m} \subset \mathcal{T}$, and all $(x_k)_{k=1,\dots,m} \subset X$ we have*

$$\left\| \sum_{k=1}^{m} r_k T_k x_k \right\|_{L^p([0,1],Y)} \leq C \left\| \sum_{k=1}^{m} r_k x_k \right\|_{L^p([0,1],X)} . \tag{2}$$

Then $\mathcal{R}_p(\mathcal{T}) := \min\{C > 0 : (2) \text{ is satisfied }\}$ is called $\mathcal{R}$-bound of $\mathcal{T}$. For $k \in \mathbb{N}$ the functions $r_k : [0,1] \to \{-1,1\}, t \mapsto \operatorname{sign}(\sin(2^k \pi t))$ are called Rademacher functions.

Next we introduce the Rademacher spaces. With their help the intricate definition of $\mathcal{R}$-boundedness can be characterized in a convenient way.

Definition 3.2. *For a Banach space X, $p \in [1,\infty)$, and $m \in \mathbb{N}$ the spaces*

$$\begin{aligned} \operatorname{Rad}_p(X) &:= \left\{ (x_k)_{k\in\mathbb{N}} \subset X : \sum_{k=1}^{\infty} r_k x_k \text{ convergent in } L^p([0,1],X) \right\} \\ \operatorname{Rad}_p^m(X) &:= \{(x_k)_{k=1,\dots,m} \subset X\}, \end{aligned}$$

equipped with the norms

$$\begin{aligned} \|(x_k)_k\|_{\operatorname{Rad}_p(X)} &:= \| \sum_{k=1}^{\infty} r_k x_k \|_{L^p([0,1],X)}, \\ \|(x_k)_{k=1,\dots,m}\|_{\operatorname{Rad}_p^m(X)} &:= \| \sum_{k=1}^{m} r_k x_k \|_{L^p([0,1],X)} \end{aligned}$$

respectively, are called Rademacher spaces.

Remark 3.3.
(i) *The spaces $\operatorname{Rad}_p(X)$ and $\operatorname{Rad}_p^m(X)$ are Banach spaces and $\bigcup_{m=1}^{\infty} \operatorname{Rad}_p^m(X)$ is dense in $\operatorname{Rad}_p(X)$.*
(ii) *Let X be a Banach space, $p \in [1,\infty)$, and $m \in \mathbb{N}$. Then we have the following norm preserving embeddings*
$\operatorname{Rad}_p(X) \hookrightarrow L^p([0,1],X), \quad (x_k)_k \mapsto \sum_{k=1}^{\infty} r_k x_k,$
$\operatorname{Rad}_p^m(X) \hookrightarrow L^p([0,1],X), \quad (x_k)_{k=1,\dots,m} \mapsto \sum_{k=1}^{m} r_k x_k.$

Theorem 3.4. *Let X be a Banach space and $p \in [1,\infty)$. Then there exists a constant $C_p^{(K)} > 0$, such that for all $(x_k)_{k\in\mathbb{N}} \subset X$ we have*

$$\frac{1}{C_p^{(K)}} \left\| \sum_{k=1}^{\infty} r_k x_k \right\|_{L^1([0,1],X)} \leq \left\| \sum_{k=1}^{\infty} r_k x_k \right\|_{L^p([0,1],X)}$$

$$\leq C_p^{(K)} \left\| \sum_{k=1}^{\infty} r_k x_k \right\|_{L^1([0,1],X)}.$$

Theorem 3.5. *Let X be a Banach space and $p \in [1,\infty)$. Then we have*

$$\left\| \sum_{j=1}^{n} r_j a_j x_j \right\|_{L^p([0,1],X)} \leq 2 \left\| \sum_{j=1}^{n} r_j b_j x_j \right\|_{L^p([0,1],X)}$$

for all $n \in \mathbb{N}$, all $(a_j)_{j=1,\dots,n}, (b_j)_{j=1,\dots,n} \subset \mathbb{C}$ with $|a_j| \leq |b_j|$, and all $(x_j)_{j=1,\dots,n} \subset X$.

Remark 3.6. *The proof of the following results can be found in* [6].

(i) *If $\mathcal{T} \subset L(X,Y)$ is $\mathcal{R}$-bounded for one $p \in [1,\infty)$, then we have an estimate as (2) for all $p \in [1,\infty)$. The $\mathcal{R}$-bounds can be estimated as $[C_p^{(K)}]^{-2}\mathcal{R}_1(\mathcal{T}) \leq \mathcal{R}_p(\mathcal{T}) \leq [C_p^{(K)}]^2\mathcal{R}_1(\mathcal{T})$.*

(ii) *If $\mathcal{T}, \mathcal{S} \subset L(X,Y)$ are $\mathcal{R}$-bounded, then $\mathcal{T}+\mathcal{S} := \{T+S : T \in \mathcal{T}, S \in \mathcal{S}\}$ is also $\mathcal{R}$-bounded with $\mathcal{R}_p(\mathcal{T}+\mathcal{S}) \leq \mathcal{R}_p(\mathcal{R})+\mathcal{R}_p(\mathcal{S})$.*

(iii) *For two given $\mathcal{R}$-bounded families $\mathcal{T}_1 \subset L(Z,Y)$ and $\mathcal{T}_2 \subset L(X,Z)$ we obtain the $\mathcal{R}$-boundedness of $\mathcal{T}_1\mathcal{T}_2 := \{T_1T_2 : T_k \in \mathcal{T}_k, k=1,2\} \subset L(X,Y)$ with $\mathcal{R}_p(\mathcal{T}_1\mathcal{T}_2) \leq \mathcal{R}_p(\mathcal{T}_1) \cdot \mathcal{R}_p(\mathcal{T}_2)$.*

(iv) *If $\mathcal{T} \subset L(X,Y)$ is $\mathcal{R}$-bounded, then $\mathcal{T}$ is also uniformly bounded. The converse, in general, is only true if X and Y are both Hilbert spaces.*

Remark 3.7. *For a given $C > 0$ we have the equivalence of the following three statements:*

(i) *For all $m \in \mathbb{N}$ and $(T_k)_{k=1,\dots,m} \subset \mathcal{T}$ we have that*

$$\|\mathbb{T}_m\|_{L(\mathrm{Rad}_p^m(X),\mathrm{Rad}_p^m(Y))} \leq C,$$

where $\mathbb{T}_m$ is defined by

$$\mathbb{T}_m : \mathrm{Rad}_p^m(X) \to \mathrm{Rad}_p^m(Y), \quad \sum_{k=1}^{m} r_k x_k \mapsto \sum_{k=1}^{m} r_k T_k x_k.$$

(ii) *For all $(T_k)_{k\in\mathbb{N}} \subset \mathcal{T}$ the operator*

$$\mathbb{T} : \mathrm{Rad}_p(X) \longrightarrow \mathrm{Rad}_p(Y), \quad \sum_{k=1}^{\infty} r_k x_k \mapsto \sum_{k=1}^{\infty} r_k T_k x_k$$

is well-defined and $\|\mathbb{T}\|_{L(\mathrm{Rad}_p(X),\mathrm{Rad}_p(Y))} \leq C$.

(iii) *$\mathcal{T} \subset L(X,Y)$ is $\mathcal{R}$-bounded with $\mathcal{R}_p(\mathcal{T}) \leq C$.*

Proof. This is obtained as an easy consequence of Remark 3.3. □

Thanks to Remark 3.7 the behavior of $\mathcal{R}$-boundedness under interpolation is completely reduced to the investigation of the interpolation of the Rademacher spaces. Having this in mind, we next analyze the corresponding properties of these spaces. To this end, the existence of projections onto $\mathrm{Rad}_p^m(X)$ and $\mathrm{Rad}_p(X)$ will turn out to be helpful. For the space $\mathrm{Rad}_p^m(X)$ we easily obtain the following result:

Theorem 3.8. *Let X be a Banach space. Then the operator*

$$R_m^X : L^p([0,1],X) \longrightarrow L^p([0,1],X),\, f \mapsto \sum_{k=1}^{m} r_k \int_0^1 r_k(u)f(u)du$$

is continuous and even a projection onto $\mathrm{Rad}_p^m(X)$.

The existence of a projection onto $\mathrm{Rad}_p(X)$ is a more involved issue. The study of this problem requires some knowledge on the geometry of Banach spaces.

Definition 3.9. *A Banach space X is called K-convex if*

$$R^X : L^p([0,1],X) \to L^p([0,1],X), \qquad f \mapsto \sum_{k=1}^{\infty} r_k \left(\int_0^1 r_k(u)f(u)du \right)$$

defines a bounded operator. In this case the operator R^X is a projection onto $\mathrm{Rad}_p(X)$.

Remark 3.10. (i) *One can show that K-convexity is equivalent to 'B-convexity' and 'non-trivial type', see for example in* [9].

(ii) *It can be shown directly that K-convexity preserves under interpolation by L^p-compatible interpolation functors. See for example in* [13, Proposition 5.1] *for the real and complex interpolation.*

(iii) *Under use of Fubini's Theorem it is easy to show that $L_p(\Omega, X)$ is K-convex if X is K-convex. In particular this yields the K-convexity of* $\mathrm{Rad}_p(X)$ *and* $\mathrm{Rad}_p^m(X)$ *since they are closed subspaces of $L^p([0,1],X)$.*

Another important property of a Banach space X is the continuity of the Hilbert transform $H = \mathscr{F}^{-1}[i\xi/|\xi|]\mathscr{F}$ on $L^p(\mathbb{R}, X)$. If this is satisfied, X is said to be of class $\mathcal{HT}$ (or equivalently UMD), cf. [3, Theorem 4.4.1]. Here we just cite the following result which is obtained as a corollary of [18, Remark 3.1.] and [9, Section 13].

Theorem 3.11. *Let X be a Banach space of class $\mathcal{HT}$. Then X is K-convex.*

3.2. **Interpolation of $\mathbf{Rad}_p(X)$ and $\mathbf{Rad}_p^m(X)$.** To obtain a suitable characterization of the interpolation spaces $(\mathrm{Rad}_p(X), \mathrm{Rad}_p(Y))_{\theta,p}$ and $[\mathrm{Rad}_p(X), \mathrm{Rad}_p(Y)]_\theta$ we will apply the abstract isomorphism result derived in [20, 1.2.4]. The idea to use this isomorphism result is taken from [12, Prop. 3.7].

Definition 3.12. *Let X and Y be Banach spaces. The operator $R \in L(X, Y)$ is said to be a retraction if there exists an $S \in L(Y, X)$ with $RS = \mathrm{id}_Y$. In this case S is said to be the coretraction belonging to R.*

Remark 3.13. *Let X be a Banach space and $U \subset X$ be a closed subspace with the standard subspace topology. If there exists a projection $P \in L(X)$ with $\mathrm{range}(P) = U$ then it is easy to see, that P is a retraction with coretraction $S : U \to X, x \mapsto x$.*

Theorem 3.14 (see [20], Theorem 1.2.4)**.** *Let $\{A_0, A_1\}, \{B_0, B_1\}$ be two interpolation couples and let*

$$R \in L(\{A_0, A_1\}, \{B_0, B_1\}), \quad S \in L(\{B_0, B_1\}, \{A_0, A_1\})$$

such that $R_{|A_k} \in L(A_k, B_k)$ and $S_{|B_k} \in L(B_k, A_k)$ are retraction and coretraction ($k = 0, 1$). Then for an arbitrary interpolation functor $\mathcal{F}$ we have that $(SR)_{|\mathcal{F}(\{A_0,A_1\})} \in L(\mathcal{F}(\{A_0, A_1\}))$ is a projection onto $W := \mathrm{range}((SR)_{|\mathcal{F}(\{A_0,A_1\})}) \subset \mathcal{F}(\{A_0, A_1\})$, where the topology on W is given by the subspace topology relative to $\mathcal{F}(\{A_0, A_1\})$. In particular, the mapping $S_{|\mathcal{F}(\{B_0,B_1\})}$ yields an isomorphism between $\mathcal{F}(\{B_0, B_1\})$ and W.

Proposition 3.15. *Assume $\{X_0, X_1\}$ to be an interpolation couple of K-convex Banach spaces. For $p \in (1, \infty)$ and an L^p-compatible interpolation functor $\mathcal{F}$ of type h with $C_h \geq 1$ we have*

$$\mathcal{F}(\{\mathrm{Rad}_p(X_0), \mathrm{Rad}_p(X_1)\}) = \mathrm{Rad}_p(\mathcal{F}(\{X_0, X_1\}),$$

where the equivalence of the norms is given by

$$\begin{aligned} \frac{1}{C_h C_2} \|f\|_{\mathrm{Rad}_p(\mathcal{F}(\{X_0,X_1\}))} &\leq \|f\|_{\mathcal{F}(\{\mathrm{Rad}_p(X_0),\mathrm{Rad}_p(X_1)\})} \\ &\leq \frac{C_h}{C_1} \cdot h\left(\|R^{X_0}\|, \|R^{X_1}\|\right) \|f\|_{\mathrm{Rad}_p(\mathcal{F}(\{X_0,X_1\}))}, \end{aligned}$$

the constants $C_1, C_2 > 0$ come from the assumed equivalence

$$\begin{aligned} C_1 \|\cdot\|_{\mathcal{F}(\{L^p([0,1],X_0),L^p([0,1],X_1)\})} &\leq \|\cdot\|_{L^p([0,1],\mathcal{F}(\{X_0,X_1\}))} \\ &\leq C_2 \|\cdot\|_{\mathcal{F}(\{L^p([0,1],X_0),L^p([0,1],X_1)\})}. \end{aligned}$$

Proof. The aim is, of course, to apply Theorem 3.14. Therefore we define the spaces $A_k := L^p([0,1], X_k)$ and $B_k := \mathrm{Rad}_p(X_k)$. Let $R_k := R^{X_k} \in L(L^p([0,1], X_k), \mathrm{Rad}_p(X_k))$ be the projection given through K-convexity and let $S_k \in L(\mathrm{Rad}_p(X_k), L^p([0,1], X_k))$ be the embedding that exists according to Remark 3.13 and Remark 3.3 (ii). On $\mathbb{E} := L^p([0,1], X_0 + X_1)$ we define the operators

$$R : (A_0 + A_1) \to (B_0 + B_1), \qquad f \mapsto {}_{[\mathbb{E}]}\sum_{k=1}^{\infty} r_k \int_0^1 r_k(u) f(u) du_{[X_0+X_1]}$$
$$S : (B_0 + B_1) \to (A_0 + A_1), \qquad g \mapsto g.$$

It can be easily seen that R and S are well-defined and that we have $R_{|L^p([0,1],X_k)} = R_k$ and $S_{|\mathrm{Rad}_p(X_k)} = S_k$. It is also clear that S is norm preserving. Thus we can apply Theorem 3.14. At first this implies that

$$W := \mathrm{range}\left((SR)_{|\mathcal{F}(\{A_0,A_1\})}\right) = \mathrm{range}\left(R_{|\mathcal{F}(\{L^p([0,1],X_0),L^p([0,1],X_1)\})}\right)$$

is well-defined. Due to Remark 3.10 (ii) we see that $\mathcal{F}(\{X_0, X_1\})$ is also K-convex. Therefore we obtain the existence of the projection $R^{\mathcal{F}(\{X_0,X_1\})}$ onto $\mathrm{Rad}_p(\mathcal{F}(\{X_0, X_1\}))$. Employing the embedding $L^p([0,1], \mathcal{F}(\{X_0, X_1\})) \hookrightarrow \mathbb{E}$ and the L^p-compatibility of $\mathcal{F}$ we obtain

$$\begin{aligned} W &= \mathrm{range}\left(R_{|L^p([0,1],\mathcal{F}(\{X_0,X_1\}))}\right) \\ &= \mathrm{range}\left(R^{\mathcal{F}(\{X_0,X_1\})}\right) = \mathrm{Rad}_p(\mathcal{F}(\{X_0, X_1\})). \end{aligned}$$

So far these equalities are only equalities of sets by the fact that on W we have the relative topology with respect to the interpolation space $\mathcal{F}(\{L^p([0,1], X_0), L^p([0,1], X_1)\})$. However, the L^p-compatibility of $\mathcal{F}$ yields the topological equality of W and $\mathrm{Rad}_p(\mathcal{F}(\{X_0, X_1\}))$. So we have $\mathcal{F}(\{\mathrm{Rad}_p(X_0), \mathrm{Rad}_p(X_1)\}) = W$ thanks to Theorem 3.14.

It remains to determine the constants which are involved in the equivalence of the norms. The L^p-compatibility of $\mathcal{F}$ and the definition of $\|\cdot\|_{\mathrm{Rad}_p(\mathcal{F}(\{X_0,X_1\}))}$ yield

$$\begin{aligned} \|f\|_{\mathrm{Rad}_p(\mathcal{F}(\{X_0,X_1\}))} &= \|f\|_{L^p([0,1],\mathcal{F}(\{X_0,X_1\}))} \\ &\leq C_2 \|f\|_{\mathcal{F}(\{L^p([0,1],X_0),L^p([0,1],X_1)\})} \\ &\leq C_2 C_h \|f\|_{\mathcal{F}(\{\mathrm{Rad}_p(X_0),\mathrm{Rad}_p(X_1)\})} \quad (f \in W). \end{aligned}$$

In the last estimate we used the fact that the S_k's are norm preserving. In view of $R_{|\mathcal{F}(\{A_0,A_1\})} f = f$ for all $f \in W$ and again by the L^p-compatibility we obtain

$$\begin{aligned} &\|f\|_{\mathcal{F}(\{\mathrm{Rad}_p(X_0),\mathrm{Rad}_p(X_1)\})} \\ &\leq C_h \cdot h\left(\|R_0\|, \|R_1\|\right) \cdot \|f\|_{\mathcal{F}(\{L^p([0,1],X_0),L^p([0,1],X_1)\})} \end{aligned}$$

$$\leq \frac{C_h}{C_1} h\left(\|R_0\|, \|R_1\|\right) \cdot \|f\|_{L^p([0,1],\mathcal{F}(\{X_0,X_1\}))} \quad (f \in W).$$

This implies the claimed equivalence of the norms and therefore the assertion is proved. □

Completely analogous we can obtain the following interpolation result for the space $\mathrm{Rad}_p^m(X)$.

Proposition 3.16. *Assume $\{X_0, X_1\}$ to be an interpolation couple, $p \in (1,\infty)$, and $m \in \mathbb{N}$. If $\mathcal{F}$ is an L^p-compatible interpolation functor of type h with $C_h \geq 1$, then we have*

$$\mathcal{F}(\{\mathrm{Rad}_p^m(X_0), \mathrm{Rad}_p^m(X_1)\}) = \mathrm{Rad}_p^m(\mathcal{F}(\{X_0, X_1\}),$$

where the equivalence of the norms is given by

$$\frac{1}{C_h C_2}\|f\|_{\mathrm{Rad}_p^m(\mathcal{F}(\{X_0,X_1\}))} \leq \|f\|_{\mathcal{F}(\{\mathrm{Rad}_p^m(X_0),\mathrm{Rad}_p^m(X_1)\})}$$
$$\leq \frac{C_h}{C_1} h\left(\|R_m^{X_0}\|, \|R_m^{X_1}\|\right)\|f\|_{\mathrm{Rad}_p^m(\mathcal{F}(\{X_0,X_1\}))}.$$

The constants $C_1, C_2 > 0$ are the same as in Proposition 3.15.

Corollary 3.17. *The results of Proposition 3.15 and Proposition 3.16 in particular hold for the real and the complex interpolation functors. Let $\{X_0, X_1\}$ be an interpolation couple of K-convex Banach spaces. Then we have*

$$\begin{aligned}
(\mathrm{Rad}_p(X_0), \mathrm{Rad}_p(X_1))_{\theta,p} &= \mathrm{Rad}_p((X_0, X_1)_{\theta,p}), \\
[\mathrm{Rad}_p(X_0), \mathrm{Rad}_p(X_1)]_\theta &= \mathrm{Rad}_p([X_0, X_1]_\theta)
\end{aligned}$$

for $p \in (1,\infty)$ and $0 < \theta < 1$.

3.3. $\mathcal{R}$-boundedness and interpolation.

Definition 3.18. *Let $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ be interpolation couples and $\mathcal{T} \subset L(\{X_0, X_1\}, \{Y_0, Y_1\})$. Then we define*

$$\mathcal{T}_{|X_k} := \left\{T_{|X_k} : T \in \mathcal{T}\right\} \subset L(X_k, Y_k)$$

for $k = 0, 1$.

Notation: In the following we set $L^p(X) := L^p([0,1], X)$.

Theorem 3.19. *Let $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ be interpolation couples of K-convex Banach spaces. Assume that $\mathcal{T} \subset L(\{X_0, X_1\}, \{Y_0, Y_1\})$ and that $\mathcal{F}$ is an L^p-compatible interpolation functor of type h with $C_h \geq 1$ for $p \in (1,\infty)$. If $\mathcal{T}_{|X_k} \subset L(X_k, Y_k)$ is $\mathcal{R}$-bounded with $\mathcal{R}$-bound $\mathcal{R}_p(\mathcal{T}_{|X_k})$ for $k = 0, 1$ then*

$$\mathcal{T}_{|\mathcal{F}(\{X_0,X_1\})} \subset L(\mathcal{F}(\{X_0, X_1\}), \mathcal{F}(\{Y_0, Y_1\}))$$

is also $\mathcal{R}$-bounded with

$$\mathcal{R}_p\left(\mathcal{T}_{|\mathcal{F}(\{X_0,X_1\})}\right) \leq C_0 \cdot h\left(\|R^{X_0}\|, \|R^{X_1}\|\right) \cdot h\left(\mathcal{R}_p(\mathcal{T}_{|X_0}), \mathcal{R}_p(\mathcal{T}_{|X_1})\right),$$

where $C_0 := \frac{C_2 C_h^3}{C_1}$ *and where* $C_1, C_2 > 0$ *come from Proposition 3.15.*

Proof. We use the characterization of $\mathcal{R}$-boundedness given in Remark 3.7 *(ii)*. Let $(T_j)_{j\in\mathbb{N}} \subset \mathcal{T}$ be an arbitrary series of operators. We have to show that

$$\begin{aligned} \mathbb{T} : \mathrm{Rad}_p(\mathcal{F}(\{X_0,X_1\})) &\longrightarrow \mathrm{Rad}_p(\mathcal{F}(\{Y_0,Y_1\})), \\ {}_{[L^p(\mathcal{F}(\{X_0,X_1\}))]}\sum_{j=1}^{\infty} r_j x_j &\mapsto {}_{[L^p(\mathcal{F}(\{Y_0,Y_1\}))]}\sum_{j=1}^{\infty} r_j T_j x_j \end{aligned}$$

is a well-defined operator satisfying

$$\begin{aligned} &\|\mathbb{T}\|_{L(\mathrm{Rad}_p(\mathcal{F}(\{X_0,X_1\})),\mathrm{Rad}_p(\mathcal{F}\{(Y_0,Y_1)\}))} \\ &\leq C_0 \cdot h\left(\|R^{X_0}\|, \|R^{X_1}\|\right) \cdot h\left(\mathcal{R}_p(\mathcal{T}_{|X_0}), \mathcal{R}_p(\mathcal{T}_{|X_1})\right) \end{aligned}$$

with C_0 as given in the statement of the theorem. We define the operator

$$\begin{aligned} \mathbb{S} : \mathrm{Rad}_p(X_0) + \mathrm{Rad}_p(X_1) &\to \mathrm{Rad}_p(Y_0) + \mathrm{Rad}_p(Y_1) \\ f = f_0 + f_1 &\mapsto \mathbb{T}_0 f_0 + \mathbb{T}_1 f_1, \end{aligned}$$

with

$$\mathbb{T}_k : \mathrm{Rad}_p(X_k) \longrightarrow \mathrm{Rad}_p(Y_k), \quad {}_{[L^p(X_k)]}\sum_{j=1}^{\infty} r_j x_j \mapsto {}_{[L^p(Y_k)]}\sum_{j=1}^{\infty} r_j T_j x_j.$$

So we get

$$\begin{aligned} \tilde{\mathbb{S}} &:= \mathbb{S}_{|\mathcal{F}(\{\mathrm{Rad}_p(X_0),\mathrm{Rad}_p(X_1)\})} \\ &\in L(\mathcal{F}(\{\mathrm{Rad}_p(X_0), \mathrm{Rad}_p(X_1)\}), \mathcal{F}(\{\mathrm{Rad}_p(Y_0), \mathrm{Rad}_p(Y_1)\})) \end{aligned}$$

with

$$\begin{aligned} &\|\tilde{\mathbb{S}}\|_{L(\mathcal{F}(\{\mathrm{Rad}_p(X_0),\mathrm{Rad}_p(X_1)\}),\mathcal{F}(\{\mathrm{Rad}_p(Y_0),\mathrm{Rad}_p(Y_1)\}))} \\ &\leq C_h \cdot h\left(\|\mathbb{T}_0\|, \|\mathbb{T}_1\|\right) \\ &\leq C_h \cdot h\left(\mathcal{R}_p(\mathcal{T}_{|X_0}), \mathcal{R}_p(\mathcal{T}_{|X_1})\right). \end{aligned}$$

In the last estimate we already used that $\|\mathbb{T}_k\|_{L(\mathrm{Rad}_p(X_k),\mathrm{Rad}_p(Y_k))} \leq \mathcal{R}_p(\mathcal{T}_{|X_k})$ for $k = 0, 1$. Hence we have

$$\begin{aligned} &\|\tilde{\mathbb{S}}\|_{L(\mathrm{Rad}_p(\mathcal{F}(\{X_0,X_1\})),\mathrm{Rad}_p(\mathcal{F}(\{Y_0,Y_1\})))} \\ &\leq \left[\frac{C_2 C_h^3}{C_1} h\left(\|R^{X_0}\|, \|R^{X_1}\|\right)\right] \cdot h\left(\mathcal{R}_p(\mathcal{T}_{|X_0}), \mathcal{R}_p(\mathcal{T}_{|X_1})\right) \end{aligned}$$

by Proposition 3.15. Pick

$$f :=_{[L^p(\mathcal{F}(X_0,X_1))]} \sum_{j=1}^{\infty} r_j x_j \in \mathrm{Rad}_p(\mathcal{F}(\{X_0,X_1\})).$$

Then we have $f =_{[L^p(X_0+X_1)]} \sum_{j=1}^{\infty} r_j x_j \in \mathrm{Rad}_p(X_0+X_1)$. In view of

$$\begin{aligned} \mathrm{Rad}_p(\mathcal{F}(\{X_0,X_1\})) &= \mathcal{F}(\{\mathrm{Rad}_p(X_0), \mathrm{Rad}_p(X_1)\}) \\ &\subset \mathrm{Rad}_p(X_0) + \mathrm{Rad}_p(X_1) \end{aligned}$$

we also have

$$\begin{aligned} f = f_0 + f_1 &= {}_{[L^p(X_0)]} \sum_{j=1}^{\infty} r_j (x_j)_0 +_{[L^p(X_1)]} \sum_{j=1}^{\infty} r_j (x_j)_1 \\ &= {}_{[L^p(X_0+X_1)]} \sum_{j=1}^{\infty} r_j [(x_j)_0 + (x_j)_1] \in \mathrm{Rad}_p(X_0+X_1) \end{aligned}$$

with $x_j = (x_j)_0 + (x_j)_1$ and $(x_j)_k \in X_k$ $(k = 0,1)$. Furthermore, we obtain

$$\begin{aligned} \tilde{\mathbb{S}} f = \mathbb{T}_0 f_0 + \mathbb{T}_1 f_1 &= {}_{[L^p(Y_0)]} \sum_{j=1}^{\infty} r_j T_j (x_j)_0 +_{[L^p(Y_1)]} \sum_{j=1}^{\infty} r_j T_j (x_j)_1 \\ &= {}_{[L^p(\mathcal{F}(\{Y_0,Y_1\}))]} \sum_{j=1}^{\infty} r_j T_j [\underbrace{(x_j)_0 + (x_j)_1}_{=x_j}] \\ &= \mathbb{T} f. \end{aligned}$$

This yields $\mathbb{T} = \tilde{\mathbb{S}}$ which completes the proof. □

Corollary 3.20. *Let $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ be interpolation couples of K-convex Banach spaces. For given $\mathcal{T} \subset L(\{X_0,X_1\},\{Y_0,Y_1\})$, $p \in (1,\infty)$, and $0 < \theta < 1$ we have:*
If $\mathcal{T}_{|X_k} \subset L(X_k, Y_k)$ is $\mathcal{R}$-bounded with $\mathcal{R}$-bound $\mathcal{R}_p(\mathcal{T}_{|X_k})$, $k = 0,1$, then

$$\begin{aligned} \mathcal{T}_{|(X_0,X_1)_{\theta,p}} &\subset L((X_0,X_1)_{\theta,p}, (Y_0,Y_1)_{\theta,p}), \\ \mathcal{T}_{|[X_0,X_1]_\theta} &\subset L([X_0,X_1]_\theta, [Y_0,Y_1]_\theta) \end{aligned}$$

are also $\mathcal{R}$-bounded with

$$\begin{aligned} \mathcal{R}_p\left(\mathcal{T}_{|(X_0,X_1)_{\theta,p}}\right) &\le C \cdot [\mathcal{R}_p(\mathcal{T}_{|X_0})]^{1-\theta} [\mathcal{R}_p(\mathcal{T}_{|X_1})]^{\theta}, \\ \mathcal{R}_p(\mathcal{T}_{|[X_0,X_1]_\theta}) &\le C' \cdot [\mathcal{R}_p(\mathcal{T}_{|X_0})]^{1-\theta} [\mathcal{R}_p(\mathcal{T}_{|X_1})]^{\theta}, \end{aligned}$$

and with $C := \frac{C_p^{(2)}}{C_p^{(1)}} \|R^{X_0}\|^{1-\theta} \|R^{X_1}\|^{\theta}$ and $C' := \|R^{X_0}\|^{1-\theta} \|R^{X_1}\|^{\theta}$. The constants $C_p^{(k)}$ are the same as in (2).

3.4. Maximal L^p-regularity, $\mathcal{R}$-sectoriality, and interpolation.

Definition 3.21. *A linear densely defined operator $A\colon D(A) \subset X \to X$ is called sectorial, if there exists a $\theta \in (0, \pi]$ such that $\Sigma_\theta \subset \rho(A)$ and*

$$\sup_{\lambda\in\Sigma_\theta} \|\lambda(\lambda - A)^{-1}\|_{L(X)} < \infty.$$

Here we define $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |arg(z)| < \theta\}$ as an open sector. The number

$$\varphi(A) := \sup\left\{\theta \in (0, \pi] : \Sigma_\theta \subset \rho(A) \wedge \sup_{\lambda\in\Sigma_\theta} \|\lambda(\lambda - A)^{-1}\|_{L(X)} < \infty\right\}$$

is called spectral angle of A.

Definition 3.22. *A linear densely defined operator $A\colon D(A) \subset X \to X$ is called $\mathcal{R}$-sectorial, if there exists a $\theta \in (0, \pi]$ such that $\Sigma_\theta \subset \rho(A)$ and such that*

$$\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_\theta\} \subset L(X)$$

is $\mathcal{R}$-bounded. The number $\varphi_{\mathcal{R}}(A)$ is defined as the supremum of all angles $\theta \in (0, \pi]$ such that we have $\Sigma_\theta \subset \rho(A)$ and the $\mathcal{R}$-boundedness of $\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_\theta\}$.

Observe that in view of Remark 3.6 $\mathcal{R}$-sectorality implies sectoriality and we always have $\varphi_{\mathcal{R}}(A) \geq \varphi(A)$. We can now apply the results obtained in the previous sections to conclude that $\mathcal{R}$-sectoriality is preserved under interpolation.

Theorem 3.23. *Let $p \in (1, \infty)$ and $\mathcal{F}$ be an arbitrary L^p-compatible interpolation functor of type h. Let X_0, X_1 be K-convex Banach spaces such that $X_0 \cap X_1 \hookrightarrow_d \mathcal{F}(\{X_0, X_1\})$. Furthermore, let*

$$A_0 : D(A_1) \subset X_0 \to X_0,$$
$$A_1 : D(A_2) \subset X_1 \to X_1,$$

be linear operators with the compatibility conditions $A_0u = A_1u$ for all $u \in D(A_0) \cap D(A_1) \hookrightarrow_d X_0 \cap X_1$ and

$$(\lambda - A_0)^{-1}u = (\lambda - A_1)^{-1}u, \quad \lambda \in \rho(A_0) \cap \rho(A_1), u \in X_0 \cap X_1. \tag{3}$$

If A_0 and A_1 are $\mathcal{R}$-sectorial, then the operator

$$B : D(B) \subset \mathcal{F}(\{X_0, X_1\}) \to \mathcal{F}(\{X_0, X_1\}),$$
$$D(B) := \mathcal{F}(\{D(A_0), D(A_1)\})$$

with $Bu := A_0u_0 + A_1u_1$ for $u = u_0 + u_1 \in D(B) \hookrightarrow D(A_0) + D(A_1)$ is also $\mathcal{R}$-sectorial. Note, that $D(A_k)$ is equipped with the graph norm $\|\cdot\|_{A_k}$. Moreover, we have $\varphi_{\mathcal{R}}(B) \geq \min_{k=0,1} \varphi_{\mathcal{R}}(A_k)$.

Proof. The operator B is densely defined in view of

$$D(A_0) \cap D(A_1) \hookrightarrow_d X_0 \cap X_1 \hookrightarrow_d \mathcal{F}(\{X_0, X_1\})$$

and since

$$D(A_0) \cap D(A_1) \hookrightarrow \mathcal{F}(\{D(A_0), D(A_1)\}) \hookrightarrow \mathcal{F}(\{X_0, X_1\}).$$

First we consider the relation of the resolvents of A_0, A_1, and B. Let $\lambda \in \rho(A_0) \cap \rho(A_1)$ then we can define

$$\begin{aligned} &R_\lambda : X_0 + X_1 \to D(A_0) + D(A_1), \\ &x_0 + x_1 \mapsto (\lambda - A_0)^{-1} x_0 + (\lambda - A_1)^{-1} x_1 \end{aligned}$$

due to (3) and get

$$\begin{aligned} [R_\lambda]_{X_k} &= (\lambda - A_k)^{-1}, \\ [R_\lambda]_{|\mathcal{F}(\{X_0,X_1\})} &\in L(\mathcal{F}(\{X_0, X_1\}), \mathcal{F}(\{D(A_0), D(A_1)\})). \end{aligned}$$

With this we can prove $\lambda \in \rho(B)$ and $(\lambda - B)^{-1} = [R_\lambda]_{|\mathcal{F}(\{X_0,X_1\})}$. For $0 < \theta < \min\limits_{k=1,2} \varphi_{\mathcal{R}}(A_k)$ we have $\Sigma_\theta \subset \rho(A_1) \cap \rho(A_2)$ and therefore also $\Sigma_\theta \subset \rho(B)$.

By assumption the families

$$\begin{aligned} \mathcal{T}_0 &:= \{[R_\lambda]_{X_0} : \lambda \in \Sigma_\theta\} \subset L(X_0), \\ \mathcal{T}_1 &:= \{[R_\lambda]_{X_1} : \lambda \in \Sigma_\theta\} \subset L(X_1) \end{aligned}$$

are $\mathcal{R}$-bounded. Thus Theorem 3.19 yields the $\mathcal{R}$-boundedness of

$$\{\lambda(\lambda - B)^{-1} : \lambda \in \Sigma_\theta\} = \left\{[R_\lambda]_{|\mathcal{F}(\{X_0,X_1\})} : \lambda \in \Sigma_\theta\right\} \subset L(\mathcal{F}(\{X, Y\})).$$

This proves the $\mathcal{R}$-sectoriality of B with $\varphi_{\mathcal{R}}(B) \geq \min\limits_{k=1,2} \varphi_{\mathcal{R}}(A_k)$. □

Remark 3.24. *Let $\{X_0, X_1\}$ be a couple of Banach space of class $\mathcal{HT}$ (cf. Theorem 3.11). Then the characterization of maximal L^p-regularity by $\mathcal{R}$-sectoriality with $\mathcal{R}$-angle bigger than $\frac{\pi}{2}$ allows for corresponding results on maximal L^p-regularity. The characterization mentioned above can e.g. be found in* [15].

Remark 3.25. *The results of Theorem 3.23 hold for interpolation functors of the real and the complex method, by the fact that they are L^p-compatible and since we always have $X_0 \cap X_1 \hookrightarrow_d \mathcal{F}(\{X_0, X_1\})$. A proof of the density of the last embedding can be found in* [20, 1.6.2,1.9.3], *for example.*

4. Property (α)

Our first aim in this section is to interpret property(α) as a special form of $\mathcal{R}$-boundedness. With the help of this interpretation we will show that also property(α) carries over to interpolation spaces provided that the interpolated Banach spaces are K-convex.

4.1. Fundamental facts about property(α).

First we recall the definition of property(α) from [15, Section 4.9]. This property is important in the context of an operator valued Fourier-multiplier theorem proved by L. Weis, cf. [15, Section 5.2 or Theorem 4.13]. Another application can be found in [15, Theorem 12.8] and [14, Theorem 5.3], where the authors proved, that the bounded H^∞-calculus is equivalent to the a priori stronger property of an $\mathcal{R}$-bounded H^∞-calculus, if the underlying Banach space has property(α). The H^∞-calculus is a powerful tool in the treatment of parabolic and elliptic partial differential equations. For more information on this topic we refer to [6] and [10], for instance.

Definition 4.1. *A Banach space X has property (α) if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, $(\alpha_{ij})_{i,j=1,\dots,n} \subset \mathbb{C}$, $|\alpha_{ij}| \leq 1$, and all $(x_{ij})_{i,j=1,\dots,n} \subset X$ we have that*

$$\int_0^1 \int_0^1 \left\| \sum_{i,j=1}^n r_i(u) r_j(v) \alpha_{ij} x_{ij} \right\|_X dudv$$

$$\leq C \int_0^1 \int_0^1 \left\| \sum_{i,j=1}^n r_i(u) r_j(v) x_{ij} \right\|_X dudv. \tag{4}$$

In this case we set $C_\alpha := \min\{C > 0 : \text{ estimate (4) holds}\}$.

By virtue of the following Lemma 4.3 *(ii)* with $p = 2$ and the orthogonality of the Rademacher functions we see that Hilbert spaces have property(α). Let X be a Banach space with property(α) then every closed subspace $Y \subset X$ has property(α). The cartesian product of Banach spaces with property(α) has also property(α). These results can be use to show that the Sobolev space $W^{m,p}(\Omega, X)$ has property(α) for $1 \leq p < \infty$ and $m \in \mathbb{N}_0$ if X possess property(α). This follows easily from the theorem of Tonelli for $m = 0$ and the fact that $W^{m,p}(\Omega, X)$ $(m > 0)$ is isometric isomorphic to a closed subspace of $(L^p(\Omega, X))^N$.

Definition 4.2. *We set*

$$\begin{aligned} T_\alpha^m : \mathrm{Rad}_p^m(X) &\rightarrow \mathrm{Rad}_p^m(X) \\ (x_i)_{i=1,\dots,m} &\mapsto (\alpha_i x_i)_{i=1,\dots,m} \end{aligned}$$

for $m \in \mathbb{N}$ *and* $\alpha \in \mathbb{C}^m$. *Additionally, we define the family*

$$\mathcal{T}^m := \{T_\alpha^m : \alpha \in \mathbb{C}^m, |\alpha_i| \leq 1, i = 1, \dots, m\}.$$

Lemma 4.3. *Let X be a Banach space. We have the following equivalences:*

(i) *X has property (α).*

(ii) *(p-independence) For all $p \in [1, \infty)$ there exists a constant $\tilde{C} > 0$ such that for all $n \in \mathbb{N}$,$(\alpha_{ij})_{i,j=1,\dots,n} \subset \mathbb{C}$, $|\alpha_{ij}| \leq 1$ and all $(x_{ij})_{i,j=1,\dots,n} \subset X$ we have*

$$\left(\int_0^1 \int_0^1 \left\| \sum_{i,j=1}^n r_i(u) r_j(v) \alpha_{ij} x_{ij} \right\|_X^p du dv \right)^{1/p}$$

$$\leq \tilde{C} \left(\int_0^1 \int_0^1 \left\| \sum_{i,j=1}^n r_i(u) r_j(v) x_{ij} \right\|_X^p du dv \right)^{1/p}. \tag{5}$$

(iii) *For all $p \in [1, \infty)$ there exists a $C > 0$ such that we have $\mathcal{R}_p(\mathcal{T}^m) \leq C$ in $L(\mathrm{Rad}_p^m(X))$ for all $m \in \mathbb{N}$.*

Proof. The equivalence of *(i)* and *(ii)* is an immediate consequence of the inequality of Kahane (Theorem 3.4). In fact, it yields

$$\begin{aligned}\left[C_p^{(K)}\right]^{-2} \|(\xi_k)_k\|_{\mathrm{Rad}_1^m(\mathrm{Rad}_1^m(X))} &\leq \|(\xi_k)_k\|_{\mathrm{Rad}_p^m(\mathrm{Rad}_p^m(X))} \\ &\leq \left[C_p^{(K)}\right]^2 \|(\xi_k)_k\|_{\mathrm{Rad}_1^m(\mathrm{Rad}_1^m(X))}\end{aligned}$$

for $(\xi_k)_k \in \mathrm{Rad}_1^m(\mathrm{Rad}_1^m(X))$.

"*(ii)⇒(iii)*": Here we can use the characterization of $\mathcal{R}$-boundedness by Remark 3.7 *(i)*. For this purpose, we set $Y_m := \mathrm{Rad}_p^m(X)$ $(m \in \mathbb{N})$, choose arbitrary $\left(T^m_{\alpha^{(j)}}\right)_{j\in\mathbb{N}} \subset \mathcal{T}^m$, and define the operator

$$\mathbb{T}_n : \mathrm{Rad}_p^n(Y_m) \to \mathrm{Rad}_p^n(Y_m), \quad (x_j)_{j=1,\dots,n} \mapsto \left(T^m_{\alpha^{(j)}} x_j\right)_{j=1,\dots,n}$$

with $\alpha^{(j)} = (\alpha_{ij})_{i=1,\dots,m} \in \mathbb{C}^m$, $x_j = (x_{ij})_{i=1,\dots,m} \in Y_m$. Under use of (ii) we get

$$\|\mathbb{T}_n (x_j)_j\|_{\mathrm{Rad}_p^n(Y_m)} \leq \tilde{C} \|(x_j)_j\|_{\mathrm{Rad}_p^n(Y_m)}$$

for all $n, m \in \mathbb{N}$ and $(x_j)_j \in \mathrm{Rad}_p^n(Y_m)$. So, we have $\|\mathbb{T}_n\|_{L(\mathrm{Rad}_p^n(Y_m))} \leq \tilde{C}$ for all $n, m \in \mathbb{N}$. Now Remark 3.7 implies (iii).

"*(iii)⇒(ii)*": Can be done in an analogous way. □

4.2. **Property (α) and interpolation.**

Lemma 4.4. *Let X be a K-convex Banach space and let $Y \subset X$ be a Banach space such that $\|.\|_Y = \|.\|_X$ on Y. Then we have*

(i) $\|R_m^X\|_{L(L^p([0,1],X))} \leq 2\|R^X\|_{L(L^p([0,1],X))}$, $m \in \mathbb{N}$,

(ii) $\|R^Y\|_{L(L^p([0,1],Y))} \leq \|R^X\|_{L(L^p([0,1],X))}$

for all $p \in (1,\infty)$.

Proof. This follows easily by the contraction principle of Kahane (Theorem 3.5). □

Theorem 4.5. *Let $\{X,Y\}$ be an interpolation couple of K-convex Banach spaces, $p \in (1,\infty)$, and let $\mathcal{F}$ be an L^p-compatible interpolation functor of type h. If X and Y have property (α) with constants $C_\alpha^X > 0$ and $C_\alpha^Y > 0$, then also the interpolation space $\mathcal{F}(\{X,Y\})$ has property(α) with*

$$C_\alpha^{\mathcal{F}(\{X,Y\})} \leq M_0 \cdot h\left(M_1 C_\alpha^X, M_1 C_\alpha^Y\right)$$

for some constants $M_0, M_1 > 0$.

Proof. The family $\mathcal{T}^m := \{T_\alpha^m : \alpha \in \mathbb{C}^m, |\alpha_i| \leq 1, i = 1,\dots,m\}$ can easily be interpreted as an subset of $L(\{\mathrm{Rad}_p^m(X), \mathrm{Rad}_p^m(Y)\})$. Then we define $\mathcal{T}_Z^m := \mathcal{T}^m_{|\mathrm{Rad}_p^m(Z)}$ for $Z \in \{X, Y, \mathcal{F}(\{X,Y\})\}$. Thanks to Lemma 4.3 we already know, that the families $\mathcal{T}_X^m \subset L(\mathrm{Rad}_p^m(X))$ and $\mathcal{T}_Y^m \subset L(\mathrm{Rad}_p^m(Y))$ are $\mathcal{R}$-bounded uniformly in $m \in \mathbb{N}$ with

$$\mathcal{R}_p(\mathcal{T}_X^m) \leq \left[C_p^{(K)}\right]^4 C_\alpha^X \quad \text{and} \quad \mathcal{R}_p(\mathcal{T}_Y^m) \leq \left[C_p^{(K)}\right]^4 C_\alpha^Y.$$

Therefore we obtain the $\mathcal{R}$-boundedness of

$$[\mathcal{T}^m]_{|\mathcal{F}(\{\mathrm{Rad}_p^m(X),\mathrm{Rad}_p^m(Y)\})} \subset L(\mathcal{F}(\{\mathrm{Rad}_p^m(X), \mathrm{Rad}_p^m(Y)\}))$$

by Theorem 3.19 and Remark 3.10 (iii). Additionally, this leads to an estimate of the $\mathcal{R}$-bound

$$\mathcal{R}_p\left([\mathcal{T}^m]_{|\mathcal{F}(\{\mathrm{Rad}_p^m(X),\mathrm{Rad}_p^m(Y)\})}\right) \leq C^{(m)} \cdot h\left(\mathcal{R}_p(\mathcal{T}_X^m), \mathcal{R}_p(\mathcal{T}_Y^m)\right)$$

with $C^{(m)} := \frac{C_2 C_h^3}{C_1} \cdot h\left(\|R^{\mathrm{Rad}_p^m(X)}\|, \|R^{\mathrm{Rad}_p^m(Y)}\|\right)$. Proposition 3.16 yields

$$\begin{aligned}[\mathcal{T}^m]_{|\mathcal{F}(\{\mathrm{Rad}_p^m(X),\mathrm{Rad}_p^m(Y)\})} &= [\mathcal{T}^m]_{|\mathrm{Rad}_p^m(\mathcal{F}(\{X,Y\}))} \\ &\subset L(\mathrm{Rad}_p^m(\mathcal{F}(\{X,Y\}))).\end{aligned}$$

Thus we can consider the $\mathcal{R}$-bound of $\mathcal{T}^m$ in $L(\mathrm{Rad}_p^m(\mathcal{F}(\{X,Y\})))$ and obtain

$$\mathcal{R}_p\left([\mathcal{T}^m]_{|\mathrm{Rad}_p^m(\mathcal{F}(\{X,Y\}))}\right)$$

$$\leq \left[\frac{C_2 C_h^2}{C_1} h\left(\|R_m^X\|, \|R_m^Y\|\right)\right] \cdot \mathcal{R}_p\left([\mathcal{T}^m]_{|\mathcal{F}(\{\mathrm{Rad}_p^m(X), \mathrm{Rad}_p^m(Y)\})}\right)$$
$$\leq \underbrace{\left[\frac{C_2 C_h^2}{C_1} h\left(2\|R^X\|, 2\|R^Y\|\right)\right]}_{=:C'} \cdot \mathcal{R}_p\left([\mathcal{T}^m]_{|\mathcal{F}(\{\mathrm{Rad}_p^m(X), \mathrm{Rad}_p^m(Y)\})}\right)$$

by Lemma 4.4 *(i)*. Due to $\mathrm{Rad}_p^m(X) \subset \mathrm{Rad}_p(X)$, $\mathrm{Rad}_p^m(Y) \subset \mathrm{Rad}_p(Y)$, Remark 3.3 and Lemma 4.4 *(ii)* we have

$$\begin{aligned}\|R^{\mathrm{Rad}_p^m(Z)}\|_{L(L^p([0,1],\mathrm{Rad}_p^m(Z)))} &\leq \|R^{\mathrm{Rad}_p(Z)}\|_{L(L^p([0,1],\mathrm{Rad}_p(Z)))} \\ &< \infty \quad (m \in \mathbb{N})\end{aligned}$$

for $Z \in \{X, Y\}$. Hence there exists an upper bound for $(C^{(m)})_{m\in\mathbb{N}}$:

$$\begin{aligned}C^{(m)} &= \frac{C_2 C_h^3}{C_1} h\left(\|R^{\mathrm{Rad}_p^m(X)}\|, \|R^{\mathrm{Rad}_p^m(Y)}\|\right) \\ &\leq \frac{C_2 C_h^3}{C_1} h\left(\|R^{\mathrm{Rad}_p(X)}\|, \|R^{\mathrm{Rad}_p(Y)}\|\right) =: C'' \quad (m \in \mathbb{N}).\end{aligned}$$

Summarizing results in

$$\mathcal{R}_p([\mathcal{T}^m]_{|\mathrm{Rad}_p^m(\mathcal{F}(\{X,Y\}))}) \leq M_0 \cdot h\left(M_1 C_\alpha^X, M_1 C_\alpha^Y\right)$$

with $M_0 := C'C''\left[C_p^{(K)}\right]^4$ and $M_1 := \left[C_p^{(K)}\right]^4$. Due to Lemma 4.3 the assertion follows by the proved $\mathcal{R}$-boundedness in $\mathrm{Rad}_p^m(\mathcal{F}(\{X, Y\}))$. □

Corollary 4.6. *Let $\{X, Y\}$ be an interpolation couple of K-convex Banach spaces. If X and Y have property (α) with $C_\alpha^X > 0$ and $C_\alpha^Y > 0$, then the real and complex interpolation spaces $(X, Y)_{\theta,p}$ and $[X, Y]_\theta$ also have property(α) for $p \in (1, \infty)$, $\theta \in (0, 1)$. In this case we have*

$$\begin{aligned}C_\alpha^{(X,Y)_{\theta,p}} &\leq M' \left[C_\alpha^X\right]^{1-\theta} \left[C_\alpha^Y\right]^\theta, \\ C_\alpha^{[X,Y]_\theta} &\leq M'' \left[C_\alpha^X\right]^{1-\theta} \left[C_\alpha^Y\right]^\theta\end{aligned}$$

for some constants $M', M'' > 0$.

5. Application to parabolic systems

In this chapter we consider realizations of parabolic differential equation systems in higher order spaces over $\mathbb{R}^n$. For example we define the Laplace operator on Sobolev spaces as

$$A_{k,p} : D(A_{k,p}) \subset W^{k,p}(\mathbb{R}^n) \to W^{k,p}(\mathbb{R}^n), \ f \mapsto \Delta f,$$

with $D(A_{k,p}) := W^{k+2,p}(\mathbb{R}^n)$ $(k \in \mathbb{N}_0)$. Similarly, we can define realizations on interpolation spaces such as Besov and Bessel-potential spaces. In the following we show that the realizations of parabolic systems on Sobolev spaces are $\mathcal{R}$-sectorial. The case $k = 0$ is proved in [15]. Here we generalize this result to the case $k \in \mathbb{N}$. Basically we follow the proof given in [15] for the case $k = 0$. However, due to the lack of differentiability of some cut-off functions used in [15], here we are forced to adapt the localization procedure at some places suitably. Applying the interpolation results of the previous chapters, we then will obtain $\mathcal{R}$-sectoriality for realizations on certain scales of interpolation spaces. To handle the parabolic problems under consideration we make use of Fourier multiplier methods. For this purpose and for the definition of Bessel-potential-spaces, here we recall the notion of a Fourier multiplier.

Definition 5.1. *Let X, Y be Banach spaces, $1 < p < \infty$, and $m \in L^\infty(\mathbb{R}^n, L(X,Y))$. Then we define*

$$\begin{aligned} T_m : \mathscr{S}(\mathbb{R}^n, X) &\to L^\infty(\mathbb{R}^n, Y) \\ f &\mapsto \mathscr{F}^{-1} m \mathscr{F} f. \end{aligned}$$

The symbol m is said to be an L^p-Fourier-multiplier, if there exists a $C_p > 0$ such that

(i) $T_m f \in L^p(\mathbb{R}^n, Y)$ *for all* $f \in \mathscr{S}(\mathbb{R}^n, X)$,
(ii) $\|T_m f\|_{L^p(\mathbb{R}^n,Y)} \le C_p \|f\|_{L^p(\mathbb{R}^n,X)}$ *for all* $f \in \mathscr{S}(\mathbb{R}^n, X)$.

In this case there exists a unique continuous extension of T_m from $L^p(\mathbb{R}^n, X)$ to $L^p(\mathbb{R}^n, Y)$ which, for simplicity, is also denoted by T_m.

Remark 5.2. *Let X be a Banach space of class $\mathcal{HT}$ and $k \in \mathbb{N}$. We define $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$ and*

$$\Lambda_{-k} : L^p(\mathbb{R}^n, X) \to W^{k,p}(\mathbb{R}^n, X); \qquad f \mapsto T_{\langle\xi\rangle^{-k} \mathrm{id}_X} f$$

$$\Lambda_k : W^{k,p}(\mathbb{R}^n, X) \to L^p(\mathbb{R}^n, X); \qquad f \mapsto \mathscr{F}^{-1} \left[\langle\xi\rangle^k \, \mathrm{id}_X \right] \mathscr{F} f.$$

By standard arguments we get

$$[\Lambda_{-k}]_{|W^{j,p}(\mathbb{R}^n,X)} \in L(W^{j,p}(\mathbb{R}^n, X), W^{j+k,p}(\mathbb{R}^n, X))$$

for $j \in \mathbb{N}_0$ and $(\Lambda_{-k})^{-1} = \Lambda_k$.

5.1. **Besov- and Bessel-potential-spaces.** Next we recall some basic facts on Besov- and Bessel-potential spaces, which can be found e.g. in [1, 7.30-7.34] and [20, 2.3-2.4]. The Banach space-valued case can be found in [4].

Definition and Remark 5.3. *Let X be a Banach space and $1 < p, q < \infty$. Then*

(i) $B^s_{p,q}(\mathbb{R}^n, X) := (L^p(\mathbb{R}^n, X), W^{k,p}(\mathbb{R}^n, X))_{\frac{s}{k},q}$ *with* $s \in (0,\infty) \cap [k-1, k)$ *and equipped with the interpolation norm is called Besov space,*

(ii) $H^{s,p}(\mathbb{R}^n, X) := \{u \in \mathscr{S}'(\mathbb{R}^n, X) : \mathscr{F}^{-1} \langle \xi \rangle^s \mathscr{F} u \in L^p(\mathbb{R}^n, X)\}$ *with* $s \in \mathbb{R}_{\geq 0}$ *is called Bessel-potential space, where the norm is given by* $\|u\|_{H^{s,p}(\mathbb{R}^n,X)} := \|\mathscr{F}^{-1} \langle \xi \rangle^s \mathscr{F} u\|_{L^p(\mathbb{R}^n,X)}$.

Note that we also have the following representations.

(iii) *Let* $m, s, j \in \mathbb{N}_0$ *with* $0 \leq m < s < j$ *and* $\lambda \in (0,1)$ *such that* $s = (1-\lambda)m + \lambda j$. *Then we have*

$$B^s_{p,q}(\mathbb{R}^n, X) = (W^{m,p}(\mathbb{R}^n, X), W^{j,p}(\mathbb{R}^n, X))_{\lambda,q}.$$

(iv) *Let* X *be of class* $\mathcal{HT}$ *and* $m, j \in \mathbb{N}_0$ *with* $0 \leq m < s < j$ *and* $s = (1-\theta)m + \theta j$ *for a* $\theta \in (0,1)$. *Then we have:*

$$\begin{aligned} H^{s,p}(\mathbb{R}^n, X) &= [W^{m,p}(\mathbb{R}^n, X), W^{j,p}(\mathbb{R}^n, X)]_\theta \\ W^{k,p}(\mathbb{R}^n, X) &= H^{k,p}(\mathbb{R}^n, X), \quad (k \in \mathbb{N}_0). \end{aligned}$$

Remark 5.4. *Let* $s \in \mathbb{R}_{>0}$, $1 < p, q < \infty$, *and* X *be a Banach space of class* $\mathcal{HT}$.

(i) *The spaces* $B^s_{p,q}(\mathbb{R}^n, X)$ *and* $H^{s,p}(\mathbb{R}^n, X)$ *are of class* $\mathcal{HT}$, *too.*

(ii) *If* X *has property*(α), *then* $B^s_{p,q}(\mathbb{R}^n, X)$ *and* $H^{s,p}(\mathbb{R}^n, X)$ *have property*(α), *too.*

The last statement can be seen easily by a retraction argument.

5.2. **Parabolic systems of differential equations.** In the following we always assume that $1 < p < \infty$. Furthermore, we set

$$A(x, D) := \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha,$$

for $a_\alpha : \mathbb{R}^n \to \mathbb{C}^{N \times N}$, $m, N \in \mathbb{N}$, and for $D^\alpha := (-i)^{|\alpha|} \partial^\alpha$. We define the "$W^{k,p}$-realization" of the formal differential operator $A(x, D)$ by

$$A_{k,p} : D(A_{k,p}) \subset W^{k,p}(\mathbb{R}^n, \mathbb{C}^N) \to W^{k,p}(\mathbb{R}^n, \mathbb{C}^N)$$

with $D(A_{k,p}) := W^{k+m,p}(\mathbb{R}^n, \mathbb{C}^N)$ and $A_{k,p} f := A(x, D) f$ for all $f \in W^{k+m,p}(\mathbb{R}^n, \mathbb{C}^N)$. For an interpolation functor $\mathcal{F}$ the "$(\mathcal{F}, k, p)$-realization" of $A(x, D)$ is defined by

$$A_{\mathcal{F},k,p} : D(A_{\mathcal{F},k,p}) \subset \mathcal{F}(L^p, W^{k,p}) \to \mathcal{F}(L^p, W^{k,p}) \tag{6}$$

with

$$\begin{aligned} \mathcal{F}(L^p, W^{k,p}) &:= \mathcal{F}(\{L^p(\mathbb{R}^n, \mathbb{C}^N), W^{k,p}(\mathbb{R}^n, \mathbb{C}^N)\}), \\ D(A_{\mathcal{F},k,p}) &:= \mathcal{F}(\{W^{m,p}(\mathbb{R}^n, \mathbb{C}^N), W^{k+m,p}(\mathbb{R}^n, \mathbb{C}^N)\}) \end{aligned}$$

and $A_{\mathcal{F},k,p}g := A(x,D)g$ for all $g \in D(A_{\mathcal{F},k,p})$. In particular, we define the "L^p-realization" of $A(x,D)$ by $A_p := A_{0,p}$.

In order to obtain well-defined operators we need to assume some regularity for the coefficients. For the well-definedness of the $W^{k,p}$-realization it is sufficient to assume that

$$a_\alpha \in W^{k,p_\infty}(\mathbb{R}^n, \mathbb{C}^{N\times N}), \quad p_\infty := \begin{cases} p, & kp > n, \\ \infty, & \text{otherwise} \end{cases} \tag{7}$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$. Here we define $W^{k,\infty}(\mathbb{R}^n, \mathbb{C}^{N\times N}) := \{f \in L^\infty(\mathbb{R}^n, \mathbb{C}^{N\times N}) : D^\beta f \in L^\infty(\mathbb{R}^n, \mathbb{C}^{N\times N}), |\beta| \le k\}$. As in [1, Theorem 4.39] the Sobolev space $W^{k,p}$ has the algebra property in case of $kp > n$. Let $K^* > 0$ such that

$$\|uv\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} \le K^* \|u\|_{W^{k,p_\infty}(\mathbb{R}^n,\mathbb{C}^N)} \|v\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)}$$

for all $u \in W^{k,p_\infty}(\mathbb{R}^n, \mathbb{C}^N)$ and $v \in W^{k,p}(\mathbb{R}^n, \mathbb{C}^N)$.

Remark 5.5. *Let $s \in (0,\infty) \cap [k-1,k)$, $k \in \mathbb{N}$, and $1 < q < \infty$. If $\mathcal{F}$ is the real or the complex interpolation functor, $A_{\mathcal{F},k,p}$ represents the Besov or Bessel-potential realization of $A(x,D)$:*

(i) *For "$\mathcal{F} = (.,.)_{\frac{s}{k},q}$" we have*

$$A^{\mathcal{B}}_{s,p,q} : D(A^{\mathcal{B}}_{s,p,q}) \subset B^s_{p,q}(\mathbb{R}^n, \mathbb{C}^N) \to B^s_{p,q}(\mathbb{R}^n, \mathbb{C}^N),$$
$$D(A^{\mathcal{B}}_{s,p,q}) = B^{s+m}_{p,q}(\mathbb{R}^n, \mathbb{C}^N), \quad A^{\mathcal{B}}_{s,p,q} := A_{\mathcal{F},k,p}.$$

(ii) *For "$\mathcal{F} = [.,.]_{\frac{s}{k}}$" we have*

$$A^{\mathcal{H}}_{s,p} : D(A^{\mathcal{H}}_{s,p}) \subset H^{s,p}(\mathbb{R}^n, \mathbb{C}^N) \to H^{s,p}(\mathbb{R}^n, \mathbb{C}^N),$$
$$D(A^{\mathcal{H}}_{s,p}) = H^{s+m,p}(\mathbb{R}^n, \mathbb{C}^N), \quad A^{\mathcal{H}}_{s,p} := A_{\mathcal{F},k,p}.$$

Remark 5.6. *Our approach to obtain $\mathcal{R}$-sectoriality for Besov- and Bessel-potential space realizations is a sort of 'decent method'. The regularity assumption W^{k,p_∞} for the coefficients, of course, is not optimal for the interpolated operators. On the other hand, notice that by a standard perturbation argument the regularity for the coefficients of the interpolated operators can always be reduced to close to optimal. In* [8, Section 5]*, for instance, this argument is used for a reduction from smooth to Hölder continuous coefficients. However, optimal conditions on the coefficients is none of our purposes here. Therefore we will not carry out this argument in what follows.*

Definition 5.7. *The symbol of $A(x,D)$ is defined by*

$$a(x,\xi) := \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \quad (x,\xi \in \mathbb{R}^n).$$

The principal part of $A(x,D)$ is defined as

$$A_0(x,D) := \sum_{|\alpha|=m} a_\alpha(x) D^\alpha.$$

The principal symbol of $A(x,D)$ is then given by the symbol of the principal part, i.e. by $a_0(x,\xi) := \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Definition 5.8. *Let $A(x,D)$ be the formal differential operator given above.*

(i) *The operator $A(x,D)$ is said to be parameter-elliptic in $\overline{\Sigma}_\theta$, if there exists a constant $C_P > 0$ such that we have*

$$|\det(a_0(x,\xi) - \lambda)| \geq C_P(|\xi|^m + |\lambda|)^N$$

for all $x \in \mathbb{R}^n$ and $(\xi,\lambda) \in (\mathbb{R}^n \times \overline{\Sigma}_\theta) \setminus \{0\}$.

(ii) *The operator $A(x,D)$ is called parabolic, if $A(x,D)$ is parameter-elliptic in $\overline{\Sigma}_{\pi/2}$.*

Remark 5.9. *If we consider a parabolic $A(x,D)$ with bounded coefficients a_α ($|\alpha| = m$), then there exists a $\theta \in \left(\frac{\pi}{2}, \pi\right)$ such that $A(x,D)$ is even parameter-elliptic in $\overline{\Sigma}_\theta$.*

5.2.1. *The model-problem.* In this section we consider the model problem, i.e., we assume the matrix-valued coefficients of $A(x,D)$ to be constant and that $A(x,D)$ is just a principal part, that is $A(x,D) = A_0(x,D)$. Hence the formal differential operator is of the form

$$A(D) := \sum_{|\alpha|=m} a_\alpha D^\alpha, \text{ with } a_\alpha \in \mathbb{C}^{N\times N}. \tag{8}$$

Lemma 5.10. *Let $v : \mathbb{R}^n \to \mathbb{C}$ be a function such that $\frac{1}{v} \in C^{|\alpha|}(\mathbb{R}^n)$ and $\alpha \in \{0,1\}^n$. We set*

$$M(\alpha) := \left\{ (\beta_1, \dots, \beta_{|\alpha|}) \in (\mathbb{N}_0^n)^{|\alpha|} : \sum_{k=1}^{|\alpha|} \beta_k = \alpha \right\}.$$

Then it is easily seen, that there are constants $C(\beta_1,\dots,\beta_{|\alpha|}) \in \mathbb{Z}$ for $(\beta_1,\dots,\beta_{|\alpha|}) \in M(\alpha)$ satisfying $|C(\beta_1,\dots,\beta_{|\alpha|})| \leq |\alpha|!$ and such that we have

$$D^\alpha \frac{1}{v} = \frac{1}{v^{|\alpha|+1}} \sum_{(\beta_1,\dots,\beta_{|\alpha|})\in M(\alpha)} C(\beta_1,\dots,\beta_{|\alpha|}) \prod_{k=1}^{|\alpha|} (D^{\beta_k} v).$$

Proof. This result follows by induction over $|\alpha|$. □

The next result is proved in [15, Theorem 6.2] for the case $|\beta| = m$. We need the following extension in order to handle diagonal operators in a suitable way.

Lemma 5.11. *Let $A(D)$ be given as in (8) and assume that it is parameter-elliptic in $\overline{\Sigma_\theta}$ for a $\theta \in (0,\pi)$ and a constant $C_P > 0$. Furthermore, let $M > 0$ such that $\sum_{|\alpha|=m} \|a_\alpha\|_{\mathbb{C}^{N\times N}} \le M$, and for $|\beta| \le m$ let*

$$m_\beta : (\mathbb{R}^n \times \overline{\Sigma_\theta}) \setminus \{0\} \to \mathbb{C}^{N\times N},\ (\xi,\lambda) \mapsto \xi^\beta \lambda^{\frac{m-|\beta|}{m}} (\lambda - a_0(\xi))^{-1}.$$

Then we have

(i) *that $m_\beta(\cdot,\lambda)$ is a Fourier multiplier for all $\lambda \in \overline{\Sigma_\theta} \setminus \{0\}$,*

(ii) *that the families*

$$\begin{aligned} \mathcal{T}_\beta &:= \{T_{m_\beta(\cdot,\lambda)} : \lambda \in \overline{\Sigma_\theta} \setminus \{0\}\} \subset L(L^p(\mathbb{R}^n,\mathbb{C}^N)) \\ \mathcal{T}_{\beta,0} &:= \{T_{m_\beta(\cdot,\lambda)} : \lambda \in \overline{\Sigma_\theta}\} \subset L(L^p(\mathbb{R}^n,\mathbb{C}^N)) \end{aligned}$$

are $\mathcal{R}$-bounded. Moreover, the $\mathcal{R}_p$-bounds of $\mathcal{T}_\beta$ and $\mathcal{T}_{\beta,0}$ can be estimated from above by a constant only depending on p, n, m, N, M, and C_P.

(iii) *For every $k \in \mathbb{N}$ we also have the $\mathcal{R}$-boundedness of*

$$\mathcal{T}_\beta^k := \left\{ [T_{m_\beta(\cdot,\lambda)}]_{|W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} : \lambda \in \overline{\Sigma_\theta} \setminus \{0\} \right\} \subset L(W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)),$$

and again $\mathcal{R}_p(\mathcal{T}_\beta^k)$ is bounded from above by a constant only depending on k, p, n, m, N, M, and C_P.

Proof. Let $\beta \in \mathbb{N}_0^n$, $|\beta| \le m$, and define the function

$$\widetilde{m}_\beta : (\mathbb{R}^n \times \overline{\Sigma_{\theta/m}}) \setminus \{0\} \to \mathbb{C}^{N\times N}, \quad (\xi,q) \mapsto \xi^\beta q^{m-|\beta|} (q^m - a_0(\xi))^{-1}.$$

By the homogeneity of $\widetilde{m}_\beta$ and the version of Michlins multiplier theorem given by [15, Theorem 5.2 b)], we obtain the $\mathcal{R}$-boundedness of $\{T_{\widetilde{m}_\beta(\cdot,q)} : q \in \overline{\Sigma_{\theta/m}}\} \subset L(L^p(\mathbb{R}^n,\mathbb{C}^N))$. Thus, it remains to prove an estimate for the $\mathcal{R}$-bound as asserted. To this end, we derive explicit estimates for $\xi^\alpha D^\alpha m_{\beta,\lambda}(\xi)$, $\alpha \in \{0,1\}^n$, and where $m_{\beta,\lambda} := \widetilde{m}_\beta(\cdot,\lambda^{1/m})$. Here we use the representation of the inverse matrix $(\lambda - a_0(\xi))^{-1}$ by the adjugate $(\lambda - a_0(\xi))^{\#} := ((-1)^{i+j} \cdot \det(\lambda - a_0(\xi))^+_{j,i})_{i,j=1,\dots,N}$, where $(\lambda - a_0(\xi))^+_{j,i}$ is the $(N-1)\times(N-1)$-matrix that results from deleting row j and column i in $(\lambda - a_0(\xi))$. Then we obtain

$$\begin{aligned} &\xi^\alpha D^\alpha m_{\beta,\lambda}(\xi) \\ &= \lambda^{\frac{m-|\beta|}{m}} \sum_{\gamma\le\alpha} \left(\xi^{\alpha-\gamma} D^{\alpha-\gamma} \frac{\xi^\beta}{\det(\lambda - a_0(\xi))} \right) \left(\xi^\gamma D^\gamma (\lambda - a_0(\xi))^{\#} \right) \end{aligned} \tag{9}$$

by the Leibniz rule. The equivalence of norms in $\mathbb{C}^{N\times N}$ yields

$$\|\xi^\gamma D^\gamma(\lambda - a_0(\xi))^\#\|_{\mathbb{C}^{N\times N}} \leq C(N)\cdot \sum_{i,j=1}^{N} |\xi^\gamma D^\gamma \det(\lambda - a_0(\xi))^+_{j,i}|.$$

For every $1 \leq i, j \leq N$ the Leibniz formula for the determinant implies

$$|\xi^\gamma D^\gamma \det(\lambda - a_0(\xi))^+_{j,i}| \leq \sum_{\sigma\in S_{N-1}} \left|\xi^\gamma D^\gamma \prod_{k=1}^{N-1} [(\lambda - a_0(\xi))^+_{j,i}]_{k,\sigma(k)}\right|.$$

Since $[(\lambda - a_0(\xi))^+_{j,i}]_{k,\sigma(k)}$ is just a component of the matrix $\lambda - a_0(\xi)$ and by $|\xi^\beta D^\beta \xi^\alpha| \leq |\xi^\alpha|$ and $|\xi^\beta D^\beta \lambda| \leq |\lambda|$, we obtain

$$|\xi^\beta D^\beta [(\lambda - a_0(\xi))^+_{j,i}]_{k,\sigma(k)}| \leq C(N, M)(|\xi|^m + |\lambda|) \quad (\beta \leq \gamma).$$

Altogether we therefore have

$$\|\xi^\gamma D^\gamma(\lambda - a_0(\xi))^\#\|_{\mathbb{C}^{N\times N}} \leq C(N, M, n)\cdot (|\xi|^m + |\lambda|)^{N-1} \tag{10}$$

for $\gamma \in \{0, 1\}^n$. In order to estimate $\xi^{\alpha-\gamma}D^{\alpha-\gamma}\frac{\xi^\beta}{\det(\lambda-a_0(\xi))}$, we apply Lemma 5.10, the parabolicity condition, and again the Leibniz rule to obtain

$$\begin{aligned}
&\left|\xi^{\alpha-\gamma}D^{\alpha-\gamma}\frac{\xi^\beta}{\det(\lambda - a_0(\xi))}\right| \\
&\leq \sum_{\delta\leq\alpha-\gamma} \left|\xi^{(\alpha-\gamma)-\delta}D^{(\alpha-\gamma)-\delta}\xi^\beta\right| \left|\xi^\delta D^\delta \frac{1}{\det(\lambda - a_0(\xi))}\right| \\
&\leq C(m, n, N, M, C_P)\frac{|\xi|^{|\beta|}}{(|\xi|^m + |\lambda|)^N}.
\end{aligned} \tag{11}$$

Now (9), (10), and (11) imply

$$\begin{aligned}
&\|\xi^\alpha D^\alpha m_{\beta,\lambda}(\xi)\|_{\mathbb{C}^{N\times N}} \\
&\leq C(n, m, N, M, C_P)\frac{(|\lambda|^{1/m})^{m-|\beta|}|\xi|^{|\beta|}}{(|\xi|^m + |\lambda|)} \\
&\leq C(n, m, N, M, C_P)\frac{(|\xi| + |\lambda|^{1/m})^m}{(|\xi|^m + |\lambda|)} \leq C(n, m, N, M, C_P).
\end{aligned}$$

Note that the operator $T_{m_\beta(\cdot,\lambda)}$ commutes with Λ_k and therefore

$$[T_{m_\beta(\cdot,\lambda)}]_{|W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} = \Lambda_{-k}T_{m_\beta(\cdot,\lambda)}\Lambda_k.$$

Assertion *(iii)* now follows immediately from *(ii)*. □

Proposition 5.12. *Let $A(D)$ be given as in (8) and assume that it is parameter-elliptic in $\overline{\Sigma}_\theta$ for a $\theta \in (0,\pi)$ and a constant $C_P > 0$. Furthermore, assume that we have $\sum_{|\alpha|=m} \|a_\alpha\|_{\mathbb{C}^{N\times N}} \le M$ for an $M > 0$. Then, there hold the following assertions for the $W^{k,p}$- realization $A_{k,p}$ $(k \in \mathbb{N}_0)$ of $A(D)$:*

(i) $\overline{\Sigma}_\theta \setminus \{0\} \subset \rho(A_{k,p})$.

(ii) *For every $\beta \in \mathbb{N}_0^n$ with $0 \le |\beta| \le m$ we have, that $\{\lambda^{\frac{m-|\beta|}{m}} D^\beta(\lambda - A_{k,p})^{-1} : \lambda \in \overline{\Sigma}_\theta \setminus \{0\}\} \subset L(W^{k,p}(\mathbb{R}^n,\mathbb{C}^N))$ is $\mathcal{R}$-bounded. In addition, we have $\lambda^{\frac{m-|\beta|}{m}} D^\beta(\lambda - A_{k,p})^{-1} = [T_{m_{\beta,\lambda}}]_{|W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)}$ for $m_{\beta,\lambda}(\xi) := \lambda^{\frac{m-|\beta|}{m}} \xi^\beta (\lambda - a_0(\xi))^{-1}$ $(\lambda \in \overline{\Sigma}_\theta \setminus \{0\}, \xi \in \mathbb{R}^n)$. Furthermore, the $\mathcal{R}_p$-bound is bounded from above by a constant only depending on k,p,n,m,N,C_P and M. In particular, this yields the $\mathcal{R}$-sectoriality of $A_{k,p}$ with $\varphi_{\mathcal{R}}(A_{k,p}) \ge \theta$.*

(iii) *For all $\lambda \in \overline{\Sigma}_\theta \setminus \{0\}$ there exists a constant $C_\lambda > 0$ depending on k,p,n,m,N,C_P,M such that*

$$\|(\lambda - A_{k,p})^{-1}\|_{L(W^{k,p}(\mathbb{R}^n,\mathbb{C}^N),W^{k+m,p}(\mathbb{R}^n,\mathbb{C}^N))} \le C_\lambda.$$

Proof. For $\lambda \in \overline{\Sigma_\theta} \setminus \{0\}$ and $0 \le |\beta| \le m$ we define

$$T_{\beta,\lambda} := [T_{m_{\beta,\lambda}}]_{|W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} \in L(W^{k,p}(\mathbb{R}^n,\mathbb{C}^N), W^{k+m-|\beta|,p}(\mathbb{R}^n,\mathbb{C}^N)).$$

Then it follows easily $\lambda(\lambda - A_{k,p})^{-1} = T_{0,\lambda}$ or rather

$$\lambda^{\frac{m-|\beta|}{m}} D^\beta (\lambda - A_{k,p})^{-1} = T_{\beta,\lambda}.$$

Therefore we proved *(i)* and

$$\{\lambda^{\frac{m-|\beta|}{m}} D^\beta(\lambda - A_{k,p})^{-1} : \lambda \in \overline{\Sigma}_\theta \setminus \{0\}\} = \mathcal{T}_\beta^k.$$

By Lemma 5.11 *(iii)* assertion *(ii)* follows.

To prove *(iii)*, we use $\lambda(\lambda - A_{k,p})^{-1} = T_{0,\lambda}$ and *(ii)* to obtain the estimate

$$\begin{aligned}
&\|(\lambda - A_{k,p})^{-1}\|_{L(W^{k,p}(\mathbb{R}^n,\mathbb{C}^N),W^{k+m,p}(\mathbb{R}^n,\mathbb{C}^N))} \\
&\le \frac{C}{|\lambda|} \max_{|\beta|\le m} \|T_{\xi^\beta m_{0,\lambda}}\|_{L(L^p(\mathbb{R}^n,\mathbb{C}^N))} \\
&\le C(\lambda) C(p,n,m,N,C_P,M) =: C_\lambda.
\end{aligned}$$

Hence the assertion is proved. □

By means of interpolation this results extends to the model-problem of the $(\mathcal{F},k,p)$-realization of $A(D)$.

Corollary 5.13. *Let $p, q \in (1,\infty)$ and let $\mathcal{F}$ be an L^q-compatible interpolation functor of type h. If $A(D)$ is parameter-elliptic in $\overline{\Sigma}_\theta$ ($\theta \in (0,\pi)$) with constant $C_P > 0$, then the $(\mathcal{F}, k, p)$-realization of $A(D)$ is $\mathcal{R}$-sectorial with $\varphi_{\mathcal{R}}(A_{\mathcal{F},k,p}) \geq \theta$, provided that*

$$W^{k,p}(\mathbb{R}^n, \mathbb{C}^N) \hookrightarrow_d \mathcal{F}(\{L^p(\mathbb{R}^n, \mathbb{C}^N), W^{k,p}(\mathbb{R}^n, \mathbb{C}^N)\}).$$

Proof. For simplicity we set

$$X_0 := L^p(\mathbb{R}^n, \mathbb{C}^N), \qquad D(A_0) := W^{m,p}(\mathbb{R}^n, \mathbb{C}^N),$$
$$X_1 := W^{k,p}(\mathbb{R}^n, \mathbb{C}^N), \qquad D(A_1) := W^{k+m,p}(\mathbb{R}^n, \mathbb{C}^N).$$

According to Remark 5.4 the spaces X_0 and X_1 are of class $\mathcal{HT}$. Setting $A_0 := A_p$ and $A_1 := A_{k,p}$, Proposition 5.12 yields the $\mathcal{R}$-sectoriality of A_0 and A_1 with $\min_{j=0,1} \varphi_{\mathcal{R}}(A_j) \geq \theta$. Hence we obtain the $\mathcal{R}$-sectoriality of the $(\mathcal{F}, k, p)$-realization by Theorem 3.23. Note that we use graph norms on $D(A_j)$ in Theorem 3.23. On the other hand, we have the equivalence of the graph norms and the Sobolev norms by the fact that $A_0 \in L(W^{m,p}(\mathbb{R}^n, \mathbb{C}^N), L^p(\mathbb{R}^n, \mathbb{C}^N))$ and $A_1 \in L(W^{k+m,p}(\mathbb{R}^n, \mathbb{C}^N), W^{k,p}(\mathbb{R}^n, \mathbb{C}^N))$. Theorem 3.23 also yields the relation $\varphi_{\mathcal{R}}(A_{\mathcal{F},k,p}) \geq \theta$. □

Remark 5.14. *In particular, Corollary 5.13 holds for the real and the complex interpolation method. This follows directly from Remark 3.25.*

Corollary 5.15. *The result of Proposition 5.12 and Corollary 5.13 is also true for a $\theta > \frac{\pi}{2}$, if we regard parabolic model problems. This follows immediately from Remark 5.9.*

Example 5.16. *Let $X_s \in \{B^s_{p,q}(\mathbb{R}^n), H^{s,p}(\mathbb{R}^n)\}$. For all $s \in (0,\infty)$, $1 < p, q < \infty$ the Laplace operator $\Delta : D(\Delta) \subset X_s \longrightarrow X_s$ with $D(\Delta) := X_{s+2}$ is $\mathcal{R}$-sectorial on X_s, and we have $\varphi_{\mathcal{R}}(\Delta) = \pi$.*

5.2.2. *Perturbation results.* To handle the case of slightly varying coefficients we provide suitable perturbation results for $\mathcal{R}$-sectorial operators. The following notation as well as Theorem 5.20 are taken from [15].

Definition 5.17. *Let $A : D(A) \subset X \to X$ be an operator on a Banach space X. For $1 < p < \infty$ we define*

$$\Theta_{\mathcal{R}}(A) :=$$
$$\left\{\theta \in (0,\pi) : \Sigma_\theta \subset \rho(A) \wedge \mathcal{R}_p(\{\lambda(\lambda - A)^{-1} \subset L(X) : \lambda \in \Sigma_\theta\}) < \infty\right\}.$$

It is obvious that A is $\mathcal{R}$-sectorial with $\varphi_{\mathcal{R}}(A) = \sup \Theta_{\mathcal{R}}(A)$ provided that $\Theta_{\mathcal{R}}(A) \neq \emptyset$. For $\theta \in \Theta_{\mathcal{R}}(A)$ we define

$$N_\theta(A) := \sup\{\|\lambda(\lambda - A)^{-1}\|_{L(X)} : \lambda \in \Sigma_\theta\},$$

$$\begin{aligned} R_{\theta,p}(A) &:= \mathcal{R}_p(\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_\theta\}), \\ \widetilde{N}_\theta(A) &:= \sup\{\|A(\lambda - A)^{-1}\|_{L(X)} : \lambda \in \Sigma_\theta\}, \\ \widetilde{R}_{\theta,p}(A) &:= \mathcal{R}_p(\{A(\lambda - A)^{-1} : \lambda \in \Sigma_\theta\}). \end{aligned}$$

Of course $(0, \varphi_{\mathcal{R}}(A)) \subset \Theta_{\mathcal{R}}(A)$*, but* $\varphi_{\mathcal{R}}(A) \notin \Theta_{\mathcal{R}}(A)$ *in general.*

Remark 5.18. *It can be easily seen that* $\widetilde{R}_{\theta,p}(A) \leq 1 + R_{\theta,p}(A)$*,* $R_{\theta,p}(A) \leq 1 + \widetilde{R}_{\theta,p}(A)$*,* $N_\theta(A) \leq R_{\theta,p}(A)$ *and* $\widetilde{N}_\theta(A) \leq \widetilde{R}_{\theta,p}(A)$*.*

The next lemma is an obvious consequence of the definition of $\mathcal{R}$-sectoriality.

Lemma 5.19. *Let* $A : D(A) \subset X \to X$ *be an* $\mathcal{R}$*-sectorial operator and* $\theta \in \Theta_{\mathcal{R}}(A)$*,* $\mu > 0$ *be arbitrary. Then we have* $\widetilde{R}_{\theta,p}(A - \mu) \leq C'_\theta + C_\theta \widetilde{R}_{\theta,p}(A)$ *and therefore the* $\mathcal{R}$*-sectoriality of* $A - \mu : D(A) \subset X \to X$ *with constants* $C_\theta, C'_\theta > 0$ *only depending on* θ*. Furthermore,* $\theta \in \Theta_{\mathcal{R}}(A - \mu)$ *and* $\varphi_{\mathcal{R}}(A - \mu) \geq \theta$*.*

Theorem 5.20. *Let* X *be a Banach space,* $A : D(A) \subset X \to X$ *be an* $\mathcal{R}$*-sectorial operator, and suppose that* $\theta \in \Theta_{\mathcal{R}}(A)$*. Assume that* $B : D(B) \subset X \to X$ *is an operator satisfying* $D(A) \subset D(B)$ *and*

$$\|Bx\|_X \leq a\|Ax\|_X + b\|x\|_X \quad (x \in D(A))$$

for some $a, b \geq 0$ *such that* $a < \left(\widetilde{N}_\theta(A)(C'_\theta + C_\theta \widetilde{R}_{\theta,p}(A))\right)^{-1}$ *(*C'_θ, C_θ *from Lemma 5.19). Then there is a constant*

$$C(a, b, \theta, A) := \frac{bN_\theta(A_{k,p})(C'_\theta + C_\theta \widetilde{R}_{\theta,p}(A_{k,p}))}{1 - a\widetilde{N}_\theta(A_{k,p})(C'_\theta + C_\theta \widetilde{R}_{\theta,p}(A_{k,p}))} > 0$$

such that for all $\mu > C(a, b, \theta, A)$ *the operator* $A + B - \mu : D(A) \subset X \to X$ *is* $\mathcal{R}$*-sectorial with* $\theta \in \Theta_{\mathcal{R}}(A + B - \mu)$ *and* $\varphi_{\mathcal{R}}(A + B - \mu) \geq \theta$*. Moreover, for all* $\lambda \in \Sigma_\theta$ *the resolvent is represented through*

$$(\lambda - (A - \mu + B))^{-1} = (\lambda + \mu - A)^{-1}(1 - B(\lambda + \mu - A)^{-1})^{-1} \tag{12}$$

and we have

$$\begin{aligned} &\|(1 - B(\lambda + \mu - A)^{-1})^{-1}\|_{L(X)} \\ &\leq \left(1 - \left[a\widetilde{N}_\theta(A) + b\frac{1}{\mu}N_\theta(A)\right] \widetilde{R}_{\theta,p}(A - \mu)\right)^{-1} \end{aligned}$$

.

Lemma 5.21. *Let* $k \in \mathbb{N}$ *(*$k \neq 0$*) and* $A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ *be parameter-elliptic in* $\overline{\Sigma}_\theta$ *with constant* $C_P > 0$ *and constant coefficients*

$a_\alpha \in \mathbb{C}^{N\times N}$ such that $\sum_{|\alpha|=m} \|a_\alpha\|_{\mathbb{C}^{N\times N}} \le M$. Then there exists a constant $C = C(k,m,n,p,N,M,C_P) > 0$ such that

$$\begin{aligned}&\|D^\alpha f\|_{W^{k-1,p}(\mathbb{R}^n,\mathbb{C}^N)}\\ &\le C|\lambda|^{-\frac{1}{m}}\|A_{k,p}f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} + C|\lambda|^{1-\frac{1}{m}}\|f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)}\end{aligned}$$

for all $f \in W^{k+m,p}(\mathbb{R}^n,\mathbb{C}^N)$, $|\alpha| = m$, and $\lambda \in \overline{\Sigma_\theta} \setminus \{0\}$.

Proof. Let $f \in W^{k+m,p}(\mathbb{R}^n,\mathbb{C}^N)$, $|\alpha| = m$, and $\lambda \in \overline{\Sigma_\theta}\setminus\{0\}$ be arbitrary. Choose any $\beta \in \mathbb{N}_0^n$ with $|\beta| = m-1$ and $\beta_i \le \alpha_i$ for all $i = 1,\dots,n$. Proposition 5.12 yields $\lambda \in \rho(A_{k,p})$ and therefore

$$D^\alpha f = \lambda^{\frac{|\beta|-m}{m}} D^{\alpha-\beta}\left[\lambda^{\frac{m-|\beta|}{m}} D^\beta(\lambda - A_{k,p})^{-1}\right](\lambda - A_{k,p})f.$$

According to Proposition 5.12 *(ii)* there exists a constant $C > 0$ only depending on p,n,m,N,M,k and C_P such that $\max_{|\beta|=m-1}\mathcal{R}_p(\mathcal{T}_\beta^k) \le C$. In view of $|\beta| = m-1$ and the triangle inequality we then deduce

$$\begin{aligned}&\|D^\alpha f\|_{W^{k-1,p}(\mathbb{R}^n,\mathbb{C}^N)}\\ &\le C|\lambda|^{\frac{|\beta|-m}{m}}\big(|\lambda|\|f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} + \|A_{k,p}f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)}\big).\end{aligned}$$

□

We are now in position to handle perturbations of the principal part of a parameter-elliptic differential operator with constant coefficients.

Notation: In the following context the constants k,m,n,p,N are always fixed. So, we do not mention the dependence of them explicitly.

Proposition 5.22. *Let $M, C_P, \tau > 0$, $k \in \mathbb{N}_0$, $1 < p < \infty$, and $\theta \in (0,\pi)$. There exist constants $\varepsilon(M,C_P,\theta) > 0$, $K(M,C_P,\theta) > 0$, and $\mu(M,C_P,\tau,\theta) > 0$ such that for all $A(D) := \sum_{|\alpha|=m} a_\alpha D^\alpha$ and $S(x,D) := \sum_{|\alpha|=m} s_\alpha(x)D^\alpha$ with*

(i) *$A(D)$ parameter-elliptic in $\overline{\Sigma}_\theta$ with constant C_P and $a_\alpha \in \mathbb{C}^{N\times N}$ such that $\sum_{|\alpha|=m}\|a_\alpha\|_{\mathbb{C}^{N\times N}} \le M$,*

(ii) *$s_\alpha \in W^{k,\infty}(\mathbb{R}^n,\mathbb{C}^{N\times N})$ satisfying $\sum_{|\alpha|=m}\|s_\alpha\|_\infty < \varepsilon$ and in the case of $k \ne 0$ additionally that*

$$\max_{0<|\gamma|\le k, |\alpha|=m}\|D^\gamma s_\alpha\|_\infty \le \tau$$

we have that

$$R_{\theta,p}(A_{k,p} + S_{k,p} - \mu) \le K.$$

Thus the $W^{k,p}$-realization $A_{k,p} + S_{k,p} - \mu$ is $\mathcal{R}$-sectorial and we have $\theta \in \Theta_{\mathcal{R}}(A_{k,p} + S_{k,p} - \mu)$, i.e., in particular that $\varphi_{\mathcal{R}}(A_{p,k} + S_{p,k} - \mu) \ge \theta$.

Furthermore, for all $\lambda \in \Sigma_\theta$ there exists a constant $C_\lambda(M, C_P, \mu) > 0$ such that

$$\|(\lambda - (A_{k,p} - \mu + S_{k,p}))^{-1}\|_{L(W^{k,p}(\mathbb{R}^n,\mathbb{C}^N),W^{k+m,p}(\mathbb{R}^N,\mathbb{C}^N))} \leq C_\lambda. \tag{13}$$

Proof. We define $m_\alpha(\xi) := \xi^\alpha a(\xi)^{-1}$ with $|\alpha| = m$. Due to the homogeneity of m_α this symbol is a Fourier multiplier. Note that

$$T_{m_\alpha} A(D)g = T_{m_\alpha} \mathscr{F}^{-1} a \mathscr{F} g = \mathscr{F}^{-1} \xi^\alpha \mathscr{F} g = D^\alpha g. \tag{14}$$

Thanks to $T_{m_\alpha} \in \mathcal{T}_{\alpha,0}$ Lemma 5.11 *(ii)* yields

$$\max_{|\alpha|=m} \|T_{m_\alpha}\|_{L(L^p(\mathbb{R}^n,\mathbb{C}^N))} \leq \eta \tag{15}$$

for a constant $\eta(M, C_p) > 0$ that does not depend explicitly on the coefficients a_α.

According to Proposition 5.12 we have the $\mathcal{R}$-sectoriality of $A_{k,p}$ with $\theta \in \Theta_\mathcal{R}(A_{k,p})$ and a constant $\mathcal{K}(M, C_P) > 0$ with

$$\widetilde{N}_\theta(A_{k,p}), N_\theta(A_{k,p}), \widetilde{R}_{\theta,p}(A_{k,p}), R_{\theta,p}(A_{k,p}) < \mathcal{K}/2.$$

In the following we aim for an application of Theorem 5.20. Thus we have to show that

$$\|S_{k,p} f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} \leq a\|A_{k,p} f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} + b\|f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} \tag{16}$$

for

$$\begin{aligned} a := a(M, C_P, \theta) &:= (\mathcal{K}(C'_\theta + C_\theta \mathcal{K}))^{-1} \\ &< (\widetilde{N}_\theta(A_{k,p})(C'_\theta + C_\theta \widetilde{R}_{\theta,p}(A_{k,p})))^{-1} \end{aligned}$$

and a $b(M, C_P, \tau) \geq 0$.

Step 1: Proof of estimate (16).

Here we will only give the proof for the case $k \geq 1$, since the case $k = 0$ is given in [15]:

Let $g \in W^{k,p}(\mathbb{R}^n, \mathbb{C}^N)$ with $|\alpha| = m$. Then we have thanks to assumption *(ii)* that

$$\begin{aligned} \|s_\alpha g\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} &\leq C\Bigg[\sum_{|\beta|\leq k} \|s_\alpha\|_\infty \|D^\beta g\|_{L^p(\mathbb{R}^n,\mathbb{C}^N)} \\ &\quad + \sum_{|\beta|\leq k} \sum_{\gamma<\beta} \|D^{\beta-\gamma} s_\alpha\|_\infty \|D^\gamma g\|_{L^p(\mathbb{R}^n,\mathbb{C}^N)}\Bigg] \qquad (17) \\ &\leq C\|s_\alpha\|_\infty \|g\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} + C(\tau)\|g\|_{W^{k-1,p}(\mathbb{R}^n,\mathbb{C}^N)}. \end{aligned}$$

Summing up, we obtain for $f \in W^{k+m,p}(\mathbb{R}^n, \mathbb{C}^N)$ that

$$\|S_{k,p} f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} \leq C \max_{|\alpha|=m} \|s_\alpha\|_\infty \sum_{|\alpha|=m} \|D^\alpha f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)}$$

$$+ C(\tau) \sum_{|\alpha|=m} \|D^\alpha f\|_{W^{k-1,p}(\mathbb{R}^n,\mathbb{C}^N)}.$$

Now we have to find estimates for the expressions $\|D^\alpha f\|_{W^{k-1,p}(\mathbb{R}^n,\mathbb{C}^N)}$ and $\|D^\alpha f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)}$. By applying (14) and (15) we can conclude

$$\|D^\alpha f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} \leq \eta \|A_{k,p} f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)}. \tag{18}$$

For $\|D^\alpha f\|_{W^{k-1,p}(\mathbb{R}^n,\mathbb{C}^N)}$ we have by Lemma 5.21 for $\lambda_0 > 0$ and a constant $C(M, C_P) > 0$ that

$$\begin{aligned} \|D^\alpha f\|_{W^{k-1,p}(\mathbb{R}^n,\mathbb{C}^N)} \leq\ & C(M, C_P)\lambda_0^{-\frac{1}{m}} \|A_{k,p} f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} \\ & + C(M, C_P, \lambda_0)\|f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)}. \end{aligned} \tag{19}$$

This results in

$$\begin{aligned} & \|S_{k,p} f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} \\ \leq & \left[C(\eta) \max_{|\alpha|=m} \|s_\alpha\|_\infty + C(M, C_P, \tau,)\lambda_0^{-\frac{1}{m}} \right] \|A_{k,p} f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} \\ & + C(M, C_P, \tau, \lambda_0) \cdot \|f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)}. \end{aligned} \tag{20}$$

Step 2: Application of Theorem 5.20
We set $\varepsilon(M, C_P, \theta) := \frac{1}{2}\frac{a}{C(\eta)} > 0$ and fix $\lambda_0 > 0$ such that

$$C(M, C_P, \tau)\lambda_0^{-\frac{1}{m}} < \frac{1}{2} a$$

with a as be given before. Since the choice of λ_0 only depends on the variables M, C_P, τ and θ we obtain that $b := C(M, C_P, \tau, \lambda_0)$ does not depend explicitly on the coefficients a_α and s_α. Since $\sum_{|\alpha|=m} \|s_\alpha\|_\infty < \varepsilon$ we obtain

$$\|S_{k,p} f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} \leq a\|A_{k,p} f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} + b\|f\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)}.$$

By the fact that $\mathcal{K}/2 > \widetilde{N}_\theta(A_{k,p})$, $\mathcal{K} > N_\theta(A_{k,p}), \widetilde{R}_\theta(A_{k,p})$ and since

$$\begin{aligned} \mu(M, C_P, \tau, \theta) :=\ & \frac{b\mathcal{K}(C'_\theta + C_\theta \mathcal{K})}{1 - a\frac{\mathcal{K}}{2}(C'_\theta + C_\theta \mathcal{K})} \\ & > \frac{bN_\theta(A_{k,p})(C'_\theta + C_\theta \widetilde{R}_{\theta,p}(A_{k,p}))}{1 - a\widetilde{N}_\theta(A_{k,p})(C'_\theta + C_\theta \widetilde{R}_{\theta,p}(A_{k,p}))} > 0, \end{aligned}$$

we obtain the $\mathcal{R}$-sectoriality of $A_{k,p} + S_{k,p} - \mu$ with $\theta \in \Theta_{\mathcal{R}}(A_{k,p} + S_{k,p} - \mu)$. Observe that we have $\mu = 2b\mathcal{K}(C'_\theta + C_\theta \mathcal{K})$. Moreover, Theorem 5.20 yields the estimate

$$R_{\theta,p}(A_{k,p} + S_{k,p} - \mu) \leq \frac{R_{\theta,p}(A_{k,p} - \mu)}{1 - [a\tilde{N}_\theta(A_{k,p}) + b\frac{1}{\mu}N_\theta(A_{k,p})]\tilde{R}_{\theta,p}(A_{k,p} - \mu)}$$

$$\begin{aligned} &\le \frac{R_{\theta,p}(A_{k,p}-\mu)}{1-\frac{\mathcal{K}}{2}[a+b\frac{1}{\mu}](C'_\theta+C_\theta\mathcal{K})} \\ &\le 4(1+C'_\theta+C_\theta\mathcal{K}) =: K(M,C_P,\theta), \end{aligned}$$

where we used that $\widetilde{R}_{\theta,p}(A_{k,p}-\mu) \le C'_\theta + C_\theta \widetilde{R}_{\theta,p}(A_{k,p}) \le C'_\theta + C_\theta\mathcal{K}$ and $R_{\theta,p}(A_{k,p}-\mu) \le 1 + \widetilde{R}_{\theta,p}(A_{k,p}-\mu)$.

The proof of (13) with $\lambda \in \Sigma_\theta$ follows directly from

$$\begin{aligned} &\|(1-S_{k,p}(\lambda+\mu-A_{k,p})^{-1})^{-1}\|_{L(W^{k,p}(\mathbb{R}^n,\mathbb{C}^N))} \\ &\le \left(1-\left[a\widetilde{N}_\theta(A_{k,p})+b\frac{1}{\mu}N_\theta(A_{k,p})\right]\widetilde{R}_{\theta,p}(A_{k,p}-\mu)\right)^{-1} \le 4 \end{aligned}$$

and the representation of the resolvent in (12) and Proposition 5.12 *(iii)*. □

5.2.3. *Some helpful facts on diagonal-operators.* To prove our main theorem for parabolic systems we next establish some facts on diagonal operators. Diagonal operators appear in a natural way during the process of localization. The proof of the following results is rather elementary and therefore omitted.

Lemma 5.23. *Let $(T_l)_{l\in\mathbb{N}}$ be a sequence of operators on a Banach space X with $D(T_l) := Y$ for another Banach space $Y \hookrightarrow X$. If there additionally hold the conditions*

(i) $T_l \in L(Y,X)$ *for all* $l \in \mathbb{N}$,
(ii) $\sup_{l\in\mathbb{N}} \|T_l\|_{L(Y,X)} < \infty$,

then the diagonal-operator

$$\mathbb{A} : D(\mathbb{A}) \subset \mathbb{X} \to \mathbb{X}, \quad (u_l)_{l\in\mathbb{N}} \mapsto (T_l u_l)_{l\in\mathbb{N}}$$

with $\mathbb{X} := \ell^p(\mathbb{N},X)$ and $D(\mathbb{A}) := \mathbb{Y} := \ell^p(\mathbb{N},Y)$ is well-defined and we have $\mathbb{A} \in L(\mathbb{Y},\mathbb{X})$.

If the T_l's are densely defined, then $\mathbb{A}$ is densely defined as well and we have

$$\rho(\mathbb{A}) = \left\{\lambda \in \bigcap_{l=1}^\infty \rho(T_l) : \exists C_\lambda > 0 : \sup_{l\in\mathbb{N}} \|(\lambda - T_l)^{-1}\|_{L(X,Y)} \le C_\lambda\right\}.$$

Furthermore, we obtain for all $\lambda \in \rho(\mathbb{A})$ and $(u_l)_{l\in\mathbb{N}} \in \mathbb{X}$ the representation

$$(\lambda - \mathbb{A})^{-1}(u_l)_{l\in\mathbb{N}} = ((\lambda - T_l)^{-1}u_l)_{l\in\mathbb{N}}.$$

Lemma 5.24. *Assume that $(T_l)_l$ is a sequence satisfying the conditions of Lemma 5.23. Then the diagonal-operator $\mathbb{A}$ is $\mathcal{R}$-sectorial with $\theta \in \Theta_\mathcal{R}(\mathbb{A})$, if there exists a $\theta \in (0,\pi)$ such that we have:*

(i) *T_l is $\mathcal{R}$-sectorial with $\theta \in \Theta_{\mathcal{R}}(T_l)$ for all $l \in \mathbb{N}$.*
(ii) *There is a $1 < p < \infty$ such that there exists a $K_p > 0$ with $\sup_{l\in\mathbb{N}} R_{\theta,p}(T_l) \leq K_p$.*
(iii) *For all $\lambda \in \Sigma_\theta$ there exists a $C_\lambda > 0$ with*

$$\sup_{l\in\mathbb{N}} \|(\lambda - T_l)^{-1}\|_{L(X,Y)} \leq C_\lambda.$$

In particular, we have $\varphi_{\mathcal{R}}(\mathbb{A}) \geq \theta$ and $R_{\theta,p}(\mathbb{A}) \leq K_p$ in this case.

Proof. Thanks to Lemma 5.23 and condition *(iii)*, we have $\Sigma_\theta \subset \rho(\mathbb{A})$ and

$$(\lambda - \mathbb{A})^{-1}(u_l)_{l\in\Gamma} = ((\lambda - T_l)^{-1}u_l)_{l\in\Gamma}$$

for all $(u_l)_{l\in\Gamma} \in \mathbb{X}$ and $\lambda \in \Sigma_\theta$.

It remains to prove the $\mathcal{R}$-boundedness of $\{\lambda(\lambda-\mathbb{A})^{-1} : \lambda \in \Sigma_\theta\}$. Let $M \in \mathbb{N}$, $(\lambda_k)_{k=1,\dots,M} \subset \Sigma_\theta$, and let $(x_k)_{k=1,\dots,M} \subset \mathbb{X}$ with $x_k =: (u_l^{(k)})_{l\in\mathbb{N}}$. Then we obtain

$$\begin{aligned}
&\left\|\sum_{k=1}^{M} r_k\lambda_k(\lambda_k - \mathbb{A})^{-1}x_k\right\|_{L^p([0,1],\mathbb{X})} \\
&= \left(\sum_{l\in\mathbb{N}}^{\infty} \int_0^1 \left\|\sum_{k=1}^{M} r_k(t)\lambda_k(\lambda_k - T_l)^{-1}u_l^{(k)}\right\|_X^p dt\right)^{1/p} \\
&\leq \left(\sum_{l\in\mathbb{N}}^{\infty} K_p^p \int_0^1 \left\|\sum_{k=1}^{M} r_k(t)u_l^{(k)}\right\|_X^p dt\right)^{1/p} = K_p \left\|\sum_{k=1}^{M} r_k x_k\right\|_{L^p([0,1],\mathbb{X})}.
\end{aligned}$$

This implies $\mathcal{R}_p(\{\lambda(\lambda - \mathbb{A})^{-1} : \lambda \in \Sigma_\theta\}) \leq K_p$ and $\varphi_{\mathcal{R}}(\mathbb{A}) \geq \theta$. □

5.2.4. *Main result on parabolic systems of differential equations.* First we make a preliminary remark on the approach we use in this section: our first aim is to prove $\mathcal{R}$-sectoriality of the $W^{k,p}$-realization of a parabolic system. To this end, we essentially follow the approach given in [15, Chapter 6]. However, since we deal with Sobolev spaces of arbitrary order we need differentiability of the localized coefficients, which is not required for the localization in L^p as performed in [15, Chapter 6]. Therefore, we have to slightly modify the method used in [15, Chapter 6] by introducing a smoother localization that assures the well-definedness of the localized operators on the Sobolev space $W^{k,p}$.

Our second aim is the transference of $\mathcal{R}$-sectoriality to the realization of a parabolic system on certain interpolation spaces. This, in turn, is then obtained as an easy consequence of Theorem 3.19. As a well-known fact we first have

Lemma 5.25. *For all $r > 0$ there exists a $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $0 \leq \varphi \leq 1$,* $\mathrm{supp}\varphi \subset (-r,r)^n$ *and*

$$\sum_{l\in\Gamma} \varphi_l^2(x) = 1 \text{ for all } x \in \mathbb{R}^n.$$

Here we set $\Gamma := r\mathbb{Z}^n$ and $\varphi_l := \varphi(.-l)$ for all $l \in \Gamma$. Additionally, we have $\mathrm{supp}\varphi_l \subset l + [-\frac{3}{4}r, \frac{3}{4}r]^n$.

Definition and Lemma 5.26. *Let*

$$\mathbb{X}_k \quad := \quad \ell^p(\Gamma, W^{k,p}(\mathbb{R}^n, \mathbb{C}^N))$$

for $1 < p < \infty$, $N \in \mathbb{N}$, and $k \in \mathbb{N}_0$ with $\Gamma = r\mathbb{Z}$, $r > 0$. Let the sequence $(\varphi_l)_{l\in\Gamma}$ be as given in Lemma 5.25 and define the 'localization-operator'

$$J : L^p(\mathbb{R}^n, \mathbb{C}^N) \to \mathbb{X}_0, \quad f \mapsto (\varphi_l f)_{l\in\Gamma}$$

and the 'patching-together-operator'

$$P : \mathbb{X}_0 \to L^p(\mathbb{R}^n, \mathbb{C}^N), \quad (f_l)_{l\in\Gamma} \mapsto \left(x \mapsto \sum_{l\in\Gamma} \varphi_l(x) f_l(x) \right).$$

For $k \in \mathbb{N}_0$ we have

$$J_{|W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} \in L(W^{k,p}(\mathbb{R}^n, \mathbb{C}^N), \mathbb{X}_k), \qquad P_{|\mathbb{X}_k} \in L(\mathbb{X}_k, W^{k,p}(\mathbb{R}^n, \mathbb{C}^N)),$$

and $PJ = \mathrm{id}_{L^p(\mathbb{R}^n,\mathbb{C}^N)}$.

Proof. Follows easily by the smoothness of φ. □

Next we recall an interpolation inequality for $\mathbb{C}^N$-valued functions. The scalar case is proved in [1, Theorem 5.2], for instance. The $\mathbb{C}^N$-valued case follows directly from the scalar case. We will apply this inequality later to verify the conditions of the perturbation result.

Lemma 5.27. *For every $m \in \mathbb{N}_0$ and each $\varepsilon_0 > 0$ there exists a constant $\mathcal{K}(p, n, k, N, \varepsilon_0) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and $j \in \mathbb{N}_0$ with $0 \leq j < k$ and $u \in W^{k,p}(\mathbb{R}^n, \mathbb{C}^N)$ we have*

$$\|u\|_{W^{j,p}(\mathbb{R}^n,\mathbb{C}^N)} \leq \mathcal{K}(\varepsilon \|u\|_{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)} + \varepsilon^{-j/(k-j)} \|u\|_{L^p(\mathbb{R}^n,\mathbb{C}^N)}).$$

Theorem 5.28. *Let $1 < p < \infty$, $k \in \mathbb{N}_0$, and the differential operator $A(x, D) := \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ be given. Furthermore, assume that the coefficients satisfy the following regularities:*

$$a_\alpha \in BUC(\mathbb{R}^n, \mathbb{C}^{N\times N}) \cap C_b^k(\mathbb{R}^n, \mathbb{C}^{N\times N}) \text{ for } |\alpha| = m,$$
$$a_\alpha \in W^{k,p_\infty}(\mathbb{R}^n, \mathbb{C}^{N\times N}) \text{ for } |\alpha| < m$$

(see (7) for the definition of p_∞). If $A(x,D)$ is parabolic with constant $C_P > 0$, then there exists a $\nu > 0$ such that the $W^{k,p}$-realization $A_{k,p} - \nu$ is $\mathcal{R}$-sectorial with $\varphi_{\mathcal{R}}(A_{k,p} - \nu) > \frac{\pi}{2}$.

Proof. **1. Localization ("Freezing the coefficients"):**
First, Remark 5.9 yields a $\theta \in (\frac{\pi}{2}, \pi)$ such that we even have the parameter-ellipticity of $A(x,D)$ in $\overline{\Sigma}_\theta$. For $M := \sum_{|\alpha|=m} \|a_\alpha\|_\infty$ let $\varepsilon = \varepsilon(M, C_P, \theta), K = K(M, C_P, \theta) > 0$ be the constants as given in the statement of Proposition 5.22. By the uniform continuity of all a_α with $|\alpha| = m$ there exists a $\delta > 0$ such that

$$\sum_{|\alpha|=m} \|a_\alpha(x) - a_\alpha(y)\|_{\mathbb{C}^{N\times N}} < \varepsilon \quad (x, y \in \mathbb{R}^n,\ |x-y| < \delta).$$

Next, we choose $r > 0$ such that $\operatorname{diam}{(-r,r)}^n < \delta$. For this r we choose a $\varphi \in C_0^\infty(\mathbb{R}^n)$ as in Lemma 5.25. Furthermore, we choose $\chi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\chi(x) = 1$ for $\|x\|_1 := \sum_{k=1}^n |x_k| \le \frac{7}{8}$, $\chi(x) = 0$ for $\|x\|_1 \ge 1$, and $0 \le \chi(x) \le 1$ for all $x \in \mathbb{R}^n$.

Then we define for $l \in \Gamma := r\mathbb{Z}$ the localized differential operator

$$A^l(x,D) := \sum_{|\alpha|=m} a_\alpha^l(x) D^\alpha$$

with coefficients

$$\begin{aligned} a_\alpha^l(x) &:= a_\alpha\left(l + \chi\left(\frac{x-l}{r}\right)(x-l)\right) \\ &= \begin{cases} a_\alpha(x) & , \|x-l\|_1 \le \frac{7}{8}r \\ a_\alpha(l) & , \|x-l\|_1 \ge r \\ a_\alpha(\tilde{x}) \text{ for some } \tilde{x} \in Q_l & , \text{ otherwise} \end{cases}, \end{aligned}$$

for $|\alpha| = m, l \in \Gamma$, and $Q_l := l + (-r,r)^n$. Next, we analyze the structure of these coefficients. For $\Phi_l : x \mapsto l + \chi\left(\frac{x-l}{r}\right)(x-l) \in Q_l$ we have $\Phi_l \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ and there exists a constant $C_\chi(r) > 0$ such that for all $l \in \Gamma$, $\gamma \in \mathbb{N}_0^n$ with $1 \le |\gamma| \le k$ we have $\|D^\gamma \Phi_l\|_\infty < C_\chi$. Thanks to $a_\alpha \in C_b^k(\mathbb{R}^n, \mathbb{C}^{n\times n})$ $(|\alpha| = m)$ the classical chain rule applies and we obtain that $a_\alpha^l \in C_b^k(\mathbb{R}^n, \mathbb{C}^{N\times N})$. Moreover, there exists a constant $\tau > 0$ only depending on χ, r, and $\max_{\substack{0\le|\gamma|\le k \\ |\alpha|=m}} \|D^\gamma a_\alpha\|_\infty$ such that

$$\max_{0\le|\gamma|\le k} \|D^\gamma a_\alpha^l\|_\infty \le \tau \text{ for all } l \in \Gamma, |\alpha| = m.$$

As before, let $A_{k,p}^l$ be the $W^{k,p}$-realization of A^l. Due to Lemma 5.23 we can form the diagonal-operator

$$\mathbb{A}_{k,p} : D(\mathbb{A}_{k,p}) \subset \mathbb{X}_k \to \mathbb{X}_k, \quad (u_l)_{l\in\Gamma} \mapsto (A_{k,p}^l u_l)_{l\in\Gamma}$$

with $D(\mathbb{A}_{k,p}) := \mathbb{X}_{k+m}$.

2. $\mathcal{R}$-sectoriality of $\mathbb{A}_{k,p} - \mu$:

For the formal differential operator $A^l(x, D)$ we have the decomposition

$$A^l(x,D) = \underbrace{\sum_{|\alpha|=m} \left[a^l_\alpha(x) - a_\alpha(l)\right] D^\alpha}_{=:A'(x,D)} + \underbrace{\sum_{|\alpha|=m} a_\alpha(l) D^\alpha}_{=:A''(D)} .$$

Note that the second part $A''(D)$ is parameter-elliptic in $\overline{\Sigma}_\theta$ with the same constant C_P. For all $l \in \Gamma$, we have

$$\sum_{|\alpha|=m} \|a_\alpha(l)\|_{\mathbb{C}^{N\times N}} \le M, \qquad \sum_{|\alpha|=m} \|a^l_\alpha(\cdot) - a_\alpha(l)\|_\infty < \varepsilon,$$

$$\max_{\substack{0<|\gamma|\le k\\ |\alpha|=m}} \|D^\gamma(a^l_\alpha(\cdot) - a_\alpha(l))\|_\infty \le \tau \quad \text{ for } k \ne 0.$$

Applying Proposition 5.22 yields a constant $\mu(M, C_P, \tau, \theta) > 0$ independent of $l \in \Gamma$ such that $A''_{k,p} + A'_{k,p} - \mu = A^l_{k,p} - \mu$ is $\mathcal{R}$-sectorial with $\theta \in \Theta_\mathcal{R}(A^l_{k,p} - \mu)$ and $R_{\theta,p}(A^l_{k,p} - \mu) \le K$. Since K according to Proposition 5.22 only depends on C_P, M, and θ and since these constants are uniformly in ℓ, we obtain

$$R_{\theta,p}(A^l_{k,p} - \mu) \le K \quad (\ell \in \Gamma).$$

Additionally, Proposition 5.22 yields

$$\sup_{l\in\Gamma} \|(\lambda + \mu - A^l_{k,p})^{-1}\|_{L(W^{k,p}(\mathbb{R}^n,\mathbb{C}^N), W^{k+m,p}(\mathbb{R}^n,\mathbb{C}^N))} \le C_\lambda,$$

where the constant C_λ only depends on λ, M, C_P und μ. Thus, Lemma 5.24 implies the $\mathcal{R}$-sectoriality of $\mathbb{A}_{k,p} - \mu$ with

$$\theta \in \Theta_\mathcal{R}(\mathbb{A}_{k,p} - \mu) \quad \text{and} \quad R_{\theta,p}(\mathbb{A}_{k,p} - \mu) \le K.$$

3. Determination of $JA_{k,p} - \mathbb{A}_{k,p}J$ and $A_{k,p}P - P\mathbb{A}_{k,p}$:

In essentially the same way as in [15], we obtain

$$(JA_{k,p} - \mathbb{A}_{k,p}J)u = \mathbb{B}_{k,p}Ju, \tag{21}$$

$$(A_{k,p}P - P\mathbb{A}_{k,p})(u_l)_{l\in\Gamma} = P\mathbb{D}_{k,p}(u_l)_{l\in\Gamma}, \tag{22}$$

where we set $D(\mathbb{B}_{k,p}) := \mathbb{X}_{k+m-1}$, $D(\mathbb{D}_{k,p}) := \mathbb{X}_{k+m-1}$, $A_{\text{low}}(x, D) := A(x, D) - A_0(x, D)$, and where the operators $\mathbb{B}_{k,p}$ and $\mathbb{D}_{k,p}$ are defined as

$$\mathbb{B}_{k,p} : D(\mathbb{B}_{k,p}) \subset \mathbb{X}_k \to \mathbb{X}_k,$$

$$(u_l)_{l\in\Gamma} \mapsto \left(A_{\text{low}}(x,D)u_l + \sum_{\substack{j\in\Gamma:\\ Q_j\cap Q_l\neq\emptyset}} (\varphi_l A(x,D) - A(x,D)\varphi_l)(\varphi_j u_j) \right)_{l\in\Gamma},$$

$$\mathbb{D}_{k,p} : D(\mathbb{D}_{k,p}) \subset \mathbb{X}_k \to \mathbb{X}_k,$$

$$(u_l)_{l\in\Gamma} \mapsto \left(A_{\text{low}}(x,D)u_l + \sum_{\substack{j\in\Gamma:\\ Q_j\cap Q_l\neq\emptyset}} \varphi_l (A(x,D)\varphi_j - \varphi_j A(x,D))u_j \right)_{l\in\Gamma}.$$

Observe that in $\mathbb{B}_{k,p}$ and $\mathbb{D}_{k,p}$ there appear only derivatives of order less or equal to $m-1$.

4. Perturbation of $\mathbb{A}_{k,p} - \mu$:

Our aim is to apply Theorem 5.20 to the operator $\mathbb{A}_{k,p}^{(\mu)} := \mathbb{A}_{k,p} - \mu$ combined with the perturbation $\mathbb{B}_{k,p}$ and to $\mathbb{A}_{k,p}^{(\mu)}$ combined with $\mathbb{D}_{k,p}$. At first we have $1 \in \rho\left(\mathbb{A}_{k,p}^{(\mu)}\right)$ and $\left(1 - \mathbb{A}_{k,p}^{(\mu)}\right)^{-1} \in L(\mathbb{X}_k, \mathbb{X}_{k+m})$ due to Lemma 5.23. Next we set $\varepsilon_0 := 1$ and choose $0 < \varepsilon' < 1$ such that

$$\begin{aligned} a &:= \varepsilon' \mathcal{K} M_{\mathbb{B},\mathbb{D}} \cdot \left\| \left(1 - \mathbb{A}_{k,p}^{(\mu)}\right)^{-1} \right\|_{L(\mathbb{X}_k,\mathbb{X}_{k+m})} \\ &< \left(\widetilde{N}_\theta \left(\mathbb{A}_{k,p}^{(\mu)}\right) \left(C'_\theta + C_\theta \widetilde{R}_{\theta,p}\left(\mathbb{A}_{k,p}^{(\mu)}\right)\right)\right)^{-1}, \end{aligned}$$

with $M_{\mathbb{B},\mathbb{D}} := \max\{\|\mathbb{B}_{k,p}\|_{L(\mathbb{X}_{k+m-1},\mathbb{X}_k)}, \|\mathbb{D}_{k,p}\|_{L(\mathbb{X}_{k+m-1},\mathbb{X}_k)}\}$ and with $\mathcal{K} > 0$ from Lemma 5.27. For $(u_l)_{l\in\Gamma} \in \mathbb{X}_{k+m-1}$, Lemma 5.27 and the boundedness of $\mathbb{B}_{k,p}$ and $\mathbb{D}_{k,p}$ yield

$$\left.\begin{array}{l} \|\mathbb{B}_{k,p}(u_l)_{l\in\Gamma}\|_{\mathbb{X}_k} \\ \|\mathbb{D}_{k,p}(u_l)_{l\in\Gamma}\|_{\mathbb{X}_k} \end{array}\right\} \begin{aligned} &\leq a' \|(u_l)_{l\in\Gamma}\|_{\mathbb{X}_{k+m}} + b'\|(u_l)_{l\in\Gamma}\|_{\mathbb{X}_0} \\ &\leq a' \|(u_l)_{l\in\Gamma}\|_{\mathbb{X}_{k+m}} + b'\|(u_l)_{l\in\Gamma}\|_{\mathbb{X}_k} \end{aligned}$$

for $a' := \varepsilon' \mathcal{K} M_{\mathbb{B},\mathbb{D}}$ and $b' := \mathcal{K} M_{\mathbb{B},\mathbb{D}} \cdot (\varepsilon')^{-(k+m-1)}$. By virtue of

$$\begin{aligned} \|(u_l)_{l\in\Gamma}\|_{\mathbb{X}_{k+m}} \leq{}& \left\| \left(1 - \mathbb{A}_{k,p}^{(\mu)}\right)^{-1} \right\|_{L(\mathbb{X}_k,\mathbb{X}_{k+m})} \\ &\cdot \left(\left\| \mathbb{A}_{k,p}^{(\mu)}(u_l)_{l\in\Gamma} \right\|_{\mathbb{X}_k} + \|(u_l)_{l\in\Gamma}\|_{\mathbb{X}_k} \right) \end{aligned}$$

we conclude

$$\left.\begin{array}{l} \|\mathbb{B}_{k,p}(u_l)_{l\in\Gamma}\|_{\mathbb{X}_k} \\ \|\mathbb{D}_{k,p}(u_l)_{l\in\Gamma}\|_{\mathbb{X}_k} \end{array}\right\} \leq a \left\| \left(\mathbb{A}_{k,p}^{(\mu)} u_l\right)_{l\in\Gamma} \right\|_{\mathbb{X}_k}$$

$$+\left(b'+a'\left\|\left(1-\mathbb{A}_{k,p}^{(\mu)}\right)^{-1}\right\|_{L(\mathbb{X}_k,\mathbb{X}_{k+m})}\right)\cdot\|(u_l)_{l\in\Gamma}\|_{\mathbb{X}_k}.$$

Hence, from Theorem 5.20 we infer that there exists an $\eta>0$ such that $\mathbb{A}_{k,p}^{(\mu)}+\mathbb{B}_{k,p}-\eta$ and $\mathbb{A}_{k,p}^{(\mu)}+\mathbb{D}_{k,p}-\eta$ are $\mathcal{R}$-sectorial with $\theta\in\Theta_{\mathcal{R}}\left(\mathbb{A}_{k,p}^{(\mu)}+\mathbb{B}_{k,p}-\eta\right)\cap\Theta_{\mathcal{R}}\left(\mathbb{A}_{k,p}^{(\mu)}+\mathbb{D}_{k,p}-\eta\right)$.

5. Determination of the resolvent $(A_{k,p}-(\mu+\eta))^{-1}$**:**
Let $\lambda\in\Sigma_\theta$ and $\nu:=\mu+\eta>0$; Due to (21) and (22) we obtain a left inverse of $\lambda-(A_{k,p}-\nu))$ in form of

$$P\left(\lambda+\eta-\left(\mathbb{A}_{k,p}^{(\mu)}+\mathbb{B}_{k,p}\right)\right)^{-1}J_{|W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)}\tag{23}$$

and a right inverse given as

$$P\left(\lambda+\eta-\left(\mathbb{A}_{k,p}^{(\mu)}+\mathbb{D}_{k,p}\right)\right)^{-1}J_{|W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)}.\tag{24}$$

Hence (23) and (24) coincide and we have $\lambda\in\rho(A_{k,p}-\nu)$. The $\mathcal{R}$-boundedness follows directly from

$$\lambda\left(\lambda-(A_{k,p}-\nu)\right)^{-1}=P\left[\lambda\left(\lambda-\left(\mathbb{A}_{k,p}^{(\mu)}+\mathbb{D}_{k,p}-\eta\right)\right)^{-1}\right]J\quad(\lambda\in\Sigma_\theta)$$

and the fact that $P_{|\mathbb{X}_k}\in L(\mathbb{X}_k,W^{k,p}(\mathbb{R}^n,\mathbb{C}^N))$ and $J_{|W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)}\in L(W^{k,p}(\mathbb{R}^n,\mathbb{C}^N),\mathbb{X}_k)$. Consequently,

$$R_{\theta,p}\left(A_{k,p}-\nu\right)\le C(P,J)R_{\theta,p}\left(\mathbb{A}_{k,p}^{(\mu)}+\mathbb{D}_{k,p}-\eta\right),$$

which implies $A_{k,p}-\nu$ to be $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\varphi_{\mathcal{R}}(A_{k,p}-\nu)\ge\theta>\frac{\pi}{2}$. □

With the help of Theorem 3.23 the above result generalizes to parabolic systems realized on interpolation spaces.

Theorem 5.29. *Let $k\in\mathbb{N}_0$ and $A(x,D):=\sum_{|\alpha|\le m}a_\alpha(x)D^\alpha$ be a formal differential operator, where we assume the coefficients to have the following regularities:*

$$a_\alpha\in BUC(\mathbb{R}^n,\mathbb{C}^{N\times N})\cap C_b^k(\mathbb{R}^n,\mathbb{C}^{N\times N})\ \textit{for}\ |\alpha|=m$$
$$a_\alpha\in W^{k,p_\infty}(\mathbb{R}^n,\mathbb{C}^{N\times N})\ \textit{for}\ |\alpha|<m.$$

Let $1<p,q<\infty$ and $\mathcal{F}$ be an L^q-compatible interpolation functor of type h such that

$$W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)\hookrightarrow_d\mathcal{F}(\{L^p(\mathbb{R}^n,\mathbb{C}^N),W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)\}),$$

and let $A_{\mathcal{F},k,p}$ be the $(\mathcal{F},k,p)$-realization of $A(x,D)$ as defined in (6). If $A(x,D)$ is parabolic, then there exists $\nu>0$ such that $A_{\mathcal{F},k,p}-\nu$ is $\mathcal{R}$-sectorial with $\varphi_{\mathcal{R}}(A_{\mathcal{F},k,p}-\nu)>\frac{\pi}{2}$.

Proof. By the assumed regularity of the coefficients the L^p- and $W^{k,p}$-realizations A_p and $A_{k,p}$ are well-defined. Theorem 5.28 yields the $\mathcal{R}$-sectoriality of $A_0:=A_p-\nu$ and $A_1:=A_{k,p}-\nu$ for a $\nu>0$ with $\varphi_{\mathcal{R}}(A_p-\nu)>\frac{\pi}{2}$ and $\varphi_{\mathcal{R}}(A_{k,p}-\nu)>\frac{\pi}{2}$. We set

$$X_0:=L^p(\mathbb{R}^n,\mathbb{C}^N),\qquad D(A_0):=W^{m,p}(\mathbb{R}^n,\mathbb{C}^N),$$
$$X_1:=W^{k,p}(\mathbb{R}^n,\mathbb{C}^N),\qquad D(A_1):=W^{k+m,p}(\mathbb{R}^n,\mathbb{C}^N).$$

By the same arguments as in Corollary 5.13 we obtain in combination with Theorem 3.23 the $\mathcal{R}$-sectoriality of the operator

$$A_{\mathcal{F},k,p}-\nu:\ D(A_{\mathcal{F},k,p})\to\mathcal{F}(\{L^p(\mathbb{R}^n,\mathbb{C}^N),W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)\}),$$
$$D(A_{\mathcal{F},k,p}):=\mathcal{F}(\{W^{k,p}(\mathbb{R}^n,\mathbb{C}^N),W^{k+m,p}(\mathbb{R}^n,\mathbb{C}^N)\}),$$
$$(A_{\mathcal{F},k,p}-\nu)f:=(A(x,D)-\nu)f,\quad f\in D(A_{\mathcal{F},k,p}).$$

Moreover, Theorem 3.23 yields $\varphi_{\mathcal{R}}(A_{\mathcal{F},k,p}-\nu)>\frac{\pi}{2}$, hence the assertion is proved. □

Corollary 5.30. *The parabolic system described in Theorem 5.29 has maximal L^p-regularity on the space $\mathcal{F}(\{L^p(\mathbb{R}^n,\mathbb{C}^N),W^{k,p}(\mathbb{R}^n,\mathbb{C}^N)\})$.*

Proof. The characterization of maximal L^p-regularity by $\mathcal{R}$-sectoriality with $\mathcal{R}$-angle bigger then $\pi/2$ yields the assertion. This characterization can be found in [21], [15], or [6]. □

Corollary 5.31. *Assume the situation of Theorem 5.29 to be given. Then the Besov- and Bessel-potential-space realizations $A^{\mathcal{B}}_{s,p,q}$ and $A^{\mathcal{H}}_{s,p}$ as defined in Remark 5.5 are $\mathcal{R}$-sectorial on the spaces $B^s_{p,q}(\mathbb{R}^n,\mathbb{C}^N)$ and $H^{s,p}(\mathbb{R}^n,\mathbb{C}^N)$ with $\varphi_{\mathcal{R}}(A^{\mathcal{B}}_{s,p,q}-\nu)>\frac{\pi}{2}$ and $\varphi_{\mathcal{R}}(A^{\mathcal{H}}_{s,p,q}-\nu)>\frac{\pi}{2}$, respectively.*

Proof. This follows directly from Theorem 5.29 and Remark 3.25. □

Acknowledgement: We want to thank Lutz Weis for helpful hints concerning the first part of this paper. Furthermore, we also want to express our gratitude to Robert Denk for his cooperativeness and a plenty of fruitful discussions.

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