The Stokes operator with Robin boundary conditions in solenoidal subspaces of $L^1(\mathbb{R}^n_+)$ and $L^\infty(\mathbb{R}^n_+)$

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Abstract

We prove that the Stokes operator with Robin boundary conditions is the generator of a bounded holomorphic semigroup on $L^\infty_\sigma(\mathbb{R}^n_+)$, which is even strongly continuous on the space $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$. Contrary to that result it is also proved that there exists no Stokes semigroup on $L^1_\sigma(\mathbb{R}^n_+)$, except if we assume the special case of Neumann boundary conditions. Nevertheless, we also obtain gradient estimates for the solution of the Stokes equations in $L^1_\sigma(\mathbb{R}^n_+)$ for all types of Robin boundary conditions.

Keywords Stokes system, Robin boundary conditions, resolvent estimates, Stokes operator, holomorphic semigroups, rotation invariant multipliers.

Mathematical Subject Classification Primary 35Q30, 76D07, Secondary 47D03.

1 Introduction

Here we consider the Stokes equations with Robin boundary conditions

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u + \nabla p &=& 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ \operatorname{div} u &=& 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ u(0) &=& u_0 & \text{in } \mathbb{R}^n_+, \\ T_\alpha u &=& 0 & \text{in } \partial\mathbb{R}^n_+ \times (0,\infty), \end{array} \right. \tag{1}$$

i.e., the trace operator T_α is given by

$$T_\alpha u := \left(\begin{array}{c} \alpha u' - \partial_n u' \\ u^n \end{array} \right)\Bigg|_{\partial\mathbb{R}^n_+}, \tag{2}$$

where u' denotes the tangential part of u and $\alpha \in [0,\infty]$[1]. Observe, that the cases $\alpha = 0$ or $\alpha = \infty$ correspond to classical Neumann or Dirichlet boundary conditions respectively.

The particular matter in our investigation of (1) are not only the Robin boundary conditions, but also the function spaces for the initial value u_0. The most available literature, which deals with the Stokes equations in the L^q-framework, only includes the case where $1 < q < \infty$. However, in this note we examine system (1) for initial values u_0 in $L^1(\mathbb{R}^n_+)$ or $L^\infty(\mathbb{R}^n_+)$. In particular

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[1]The case $\alpha = \infty$ is to understand in the following sense: divide the first line in (2) by α and let $\alpha \to \infty$.

we will prove resolvent estimates for the solution of the associated resolvent problem to (1) in solenoidal subspaces of $L^\infty(\mathbb{R}^n_+)$. This leads to a generation results for the Stokes operator with Robin boundary conditions in the spaces

$$L^\infty_\sigma(\mathbb{R}^n_+) := \{u \in L^\infty(\mathbb{R}^n_+) : \ (u, \nabla p) = 0, \ p \in \widehat{W}^{1,1}(\mathbb{R}^n_+)\}, \tag{3}$$

and

$$\mathrm{BUC}_\sigma(\mathbb{R}^n_+) = \{u \in \mathrm{BUC}(\mathbb{R}^n_+) : \operatorname{div} u = 0, u^n|_{\partial\mathbb{R}^n_+} = 0\},$$

where $\mathrm{BUC}(\mathbb{R}^n_+)$ denotes the space of bounded uniformly continuous functions in $\mathbb{R}^n_+$. One difficulty coming up in these spaces is to give a rigorous definition of the Stokes operator. Observe that it can not be defined in the usual way used in $L^q_\sigma(\mathbb{R}^n_+)$ for $1 < q < \infty$. This is due to the fact that the Helmholtz projection, associated to the Helmholtz decomposition $L^q(\mathbb{R}^n_+) = L^q_\sigma(\mathbb{R}^n_+) \oplus G_q(\mathbb{R}^n_+)$ for $1 < q < \infty$ (see [Sol77], [McC81], [BM88]), does not act as a bounded operator on $L^\infty(\mathbb{R}^n_+)$ or $L^1(\mathbb{R}^n_+)$. Therefore we will give a definition of the Stokes operator with Robin boundary conditions through its resolvent.

The situation in $L^1(\mathbb{R}^n_+)$ is different. According to a result in [DHP01] it is known that in the case of Dirichlet boundary conditions there exists no Stokes semigroup in

$$L^1_\sigma(\mathbb{R}^n_+) = \overline{L^1(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)}^{\|\cdot\|_1}.$$

On the other hand, by a reflection argument it can be easily seen that the Stokes operator with Neumann boundary conditions is the generator of a bounded holomorphic C_0-semigroup on $L^1_\sigma(\mathbb{R}^n_+)$. Thus, the natural question arises, what happens in between, i.e., if one considers a mixture of these two special types of Robin boundary conditions. In this paper we give a complete answer to this question. It turns out, that the generation result holds if and only if we assume Neumann boundary conditions, i.e., in the case $\alpha = 0$. In other words, whenever we add an arbitrary small Dirichlet part to the Neumann boundary conditions, the generation result in $L^1_\sigma(\mathbb{R}^n_+)$ fails. Let us remark that the non generation result in $L^1(\mathbb{R}^n_+)$ for Dirichlet boundary conditions is also physically motivated. Indeed, in [Koz98] it is proved that the existence of a local strong solution of the Navier-Stokes equations with Dirichlet boundary conditions in $L^1(\Omega)$ for an exterior domain $\Omega \subseteq \mathbb{R}^n$ implies that no force could act on the boundary. This would mean that the Navier-Stokes equations are physically meaningless, hence one does not expect a generation result in L^1 to be valid. In spite of that, we can prove gradient estimates $\|\nabla u\|_1 \le C\|u_0\|_1$ for all types of Robin boundary conditions and initial values in $L^1_\sigma(\mathbb{R}^n_+)$. We summarize the main results presented in this paper. In spaces of bounded functions we have

Theorem 1.1 *Let* $\alpha \in [0, \infty]$ *and*

$$X_\sigma \in \left\{L^\infty_\sigma(\mathbb{R}^n_+), \mathrm{BUC}_\sigma(\mathbb{R}^n_+)\right\}.$$

The Stokes operator with Robin boundary conditions $-A_\alpha$ *is the generator of a bounded holomorphic semigroup on* X_σ. *The semigroup is even strongly continuous if* $X_\sigma = \mathrm{BUC}_\sigma(\mathbb{R}^n_+)$.

In $L^1_\sigma(\mathbb{R}^n_+)$ we will obtain the following somehow surprising result.

Theorem 1.2 *Let* $\alpha \in [0, \infty]$. *The Stokes operator with Robin boundary conditions* A_α *is the generator of a semigroup on* $L^1_\sigma(\mathbb{R}^n_+)$ *if and only if* $\alpha = 0$, *i.e., in the case of Neumann boundary conditions.*

In spite of Theorem 1.2 we will be able to prove gradient estimates to be valid in all spaces considered.

Theorem 1.3 *Let $\alpha \in [0,\infty]$ and let*

$$X_\sigma \in \left\{ L^\infty_\sigma(\mathbb{R}^n_+), \mathrm{BUC}_\sigma(\mathbb{R}^n_+), L^1_\sigma(\mathbb{R}^n_+) \right\}.$$

The Stokes semigroup $(\mathrm{e}^{-tA_\alpha})_{t\geq 0}$ satisfies the estimate

$$\|\nabla \mathrm{e}^{-tA_\alpha} f\|_{X_\sigma} \leq C t^{-1/2} \|f\|_{X_\sigma}, \quad t>0,\ f\in X_\sigma.$$

Note that for $X_\sigma = L^1_\sigma(\mathbb{R}^n_+)$ the above estimate is obtained by a density argument and the fact that the Stokes semigroup on $L^2_\sigma(\mathbb{R}^n_+)$ can be applied to functions $f \in L^1(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$.

In the previous literature problem (1) with initial values in $L^\infty_\sigma(\mathbb{R}^n_+)$ and $L^1_\sigma(\mathbb{R}^n_+)$ was only investigated for the special case of Dirichlet boundary conditions. For example the counterexample in $L^1_\sigma(\mathbb{R}^n_+)$ and the generation result for the Stokes operator with Dirichlet boundary conditions in $L^\infty_\sigma(\mathbb{R}^n_+)$ and $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$ are contained in [DHP01]. Based on the Green matrix for the Stokes system in $\mathbb{R}^n_+$, a similar result is proved in a space of bounded and continuous solenoidal fields in [Sol03]. There the author also applied his result for the linear problem to the nonlinear Navier-Stokes equations and proved a local existence result for nondecaying initial values. Another existence result of local-in-time solutions for the Navier-Stokes equations is proved in [IM] for initial values in $L^\infty_\sigma(\mathbb{R}^n_+)$ and $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$. Their proof relies on the results obtained in [DHP01]. Gradient estimates for Dirichlet boundary conditions in $L^1_\sigma(\mathbb{R}^n_+)$ are proved in [GMS99], [SS01], and in $L^\infty_\sigma(\mathbb{R}^n_+)$ in [Shi99], [SS01].

The content of this article can be regarded as a generalization to Robin boundary conditions of the above mentioned results in [DHP01], [GMS99], [Shi99], and [SS01] for Dirichlet boundary conditions. Our results are based on an explicit solution formula constructed in [Saa06]. The construction of this formula is similar to [DHP01]. However, we use a different method to derive estimates for the solution. In [DHP01] the authors provide pointwise kernel estimates, whereas our proofs rely on a multiplier result for rotation invariant multipliers. The mentioned results in [GMS99], [Shi99], and [SS01] are based on Ukai's formula for the solution of the Stokes equations with Dirichlet boundary conditions (see [Uka87]).

One motivation for the author to consider the Stokes equations with Robin boundary conditions of course was the just mentioned question of what happens in between the generation result for Neumann boundary conditions and the non generation result for Dirichlet boundary conditions in $L^1(\mathbb{R}^n_+)$. But we want to point out that the Robin boundary conditions in (1) are equivalent to

$$\left.\begin{pmatrix} \alpha u' + (\nu T(u,p))' \\ u^n \end{pmatrix}\right|_{\partial\mathbb{R}^n_+} = 0,$$

where $T(u,p) = (\nabla u + (\nabla u)^\tau - Ip)$ denotes the stress tensor and ν the outer normal at $\partial\mathbb{R}^n_+$. These boundary conditions are usually called partial slip boundary conditions. This means, from the physical point of view, Robin boundary conditions describe something in between no slip, i.e., Dirichlet boundary conditions, and full slip, i.e., Neumann boundary conditions. Although the most common boundary conditions used in the fluid mechanics literature are no slip boundary conditions, it is known that in some situations, e.g. for gas flows, non-Newtonian fluids, or moving contact lines, partial slip can occur (see e.g. [Mun89], [BG92], and [DV79] respectively). Moreover, physico-chemical parameters as wetting, shear rate, surface charge, and surface roughness can influence the behavior of a fluid at the solid-liquid interface. We refer to [LBS05] for a review on recent investigations on this subject and to the literature cited therein. This shows that in certain situations it might be more appropriate to assume partial slip boundary conditions, which is another motivation for the examination of Stokes and Navier-Stokes equations with Robin boundary conditions.

We want to remark that the content of this paper is included in [Saa03] and it extends the previous work [Saa06], which deals with (1) in $L^q(\mathbb{R}^n_+)$ for $1 < q < \infty$. In order to prove resolvent estimates in [Saa06], we made use of the rotation invariance in $n-1$ dimensions of large parts of the constructed solution formula. This enabled us to apply the bounded H^∞-calculus of the Poisson operator $(-\Delta_{\mathbb{R}^{n-1}})^{1/2}$ on $L^q(\mathbb{R}^{n-1})$. In other words we regarded the computed representation of the solution as a function g of the Poisson operator $(-\Delta_{\mathbb{R}^{n-1}})^{1/2}$, which can be estimated by

$$\|g((-\Delta_{\mathbb{R}^{n-1}})^{1/2})\|_{\mathcal{L}(L^q(\mathbb{R}^{n-1}))} \leq C\|g\|_{H^\infty(\Sigma)}, \tag{4}$$

where $H^\infty(\Sigma)$ denotes the space of all bounded holomorphic functions on a certain complex sector Σ, equipped with the infinity norm. Thus, besides the holomorphy of g, it was sufficient to verify pointwise estimates on a complex sector for the terms in the representation of the Stokes resolvent, regarded as functions of $(-\Delta_{\mathbb{R}^{n-1}})^{1/2}$. These estimates also provided a sufficient decay in the normal component x_n, such that the q-integration over x_n afterwards was feasible for $1 < q < \infty$. In [Saa06] this method leads to resolvent estimates and a bounded H^∞-calculus for the Stokes operator with Robin boundary conditions in $L^q_\sigma(\mathbb{R}^n_+)$ for $1 < q < \infty$.

To estimate the solution formula in $L^1(\mathbb{R}^n_+)$ and $L^\infty(\mathbb{R}^n_+)$ we will adapt the methods in [Saa06]. However, for this purpose we have to circumvent the following two main difficulties. Firstly, the bounded H^∞-calculus for the Poisson operator $|\nabla'| = (-\Delta_{\mathbb{R}^{n-1}})^{1/2}$ on $L^q(\mathbb{R}^{n-1})$, which is the main ingredient for the proof of the estimates in [Saa06], is neither valid in $L^1(\mathbb{R}^{n-1})$ nor in $L^\infty(\mathbb{R}^{n-1})$. Here we have to provide an appropriate substitute, which is based on the above mentioned classical result on rotation invariant multipliers. This result enables us to obtain estimates as (4), but now also valid in $L^1(\mathbb{R}^{n-1})$ and $L^\infty(\mathbb{R}^{n-1})$. Besides the holomorphy and the boundedness on a complex sector, here the functions also have to be holomorphic in 0 and have to satisfy a certain decay at infinity, and we will show that the terms in our solution formula still satisfy these two additional conditions. The second problem we have to deal with is the unboundedness of the Riesz operator $R' := \mathcal{F}^{-1}\frac{i\xi'}{|\xi'|}\mathcal{F}$ in $L^1(\mathbb{R}^{n-1})$ and $L^\infty(\mathbb{R}^{n-1})$. Roughly speaking, we will overcome this problem by rephrasing the solution formula in a way such that no more Riesz operator appears.

We organized this article as follows. In Section 2 we introduce the notation and establish the mentioned substitute (Corollary 2.3) for the bounded H^∞-calculus of the Poisson operator which is only valid in $L^q(\mathbb{R}^{n-1})$ for $1 < q < \infty$. We also recall some known properties of the Laplacian in $\mathbb{R}^n$ and $\mathbb{R}^n_+$ and the explicit solution formula for the system (1) constructed in [Saa06]. With the aid of these preparations, in Section 3 we will first prove that the Stokes operator with Neumann boundary conditions is the generator of a bounded holomorphic C_0-semigroup on $L^1_\sigma(\mathbb{R}^n_+)$. Furthermore, we state the counterexample for the other types of mixed boundary conditions, as well as the mentioned gradient estimates for the Stokes semigroup on $L^1(\mathbb{R}^n_+)$, valid for all considered boundary conditions. In Section 4 then we will verify resolvent estimates in $L^\infty(\mathbb{R}^n_+)$. This leads to the result that the Stokes operator is the generator of a bounded holomorphic semigroup on the space $L^\infty_\sigma(\mathbb{R}^n_+)$, which is even strongly continuous on the spaces $C_{0,\sigma}(\mathbb{R}^n_+)$ and $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$.

2 Preliminaries

2.1 Notations

In most parts of this note we use standard notation. For $m \in \{0, 1, \dots, \infty\}$ and a domain $\Omega \subseteq \mathbb{R}^n$, by $C^m(\Omega)$ we denote the space of all m-times continuously differentiable functions and by $C_c^m(\Omega)$ its subspace consisting of all functions in $C^m(\Omega)$ which are compactly supported. Furthermore, let $C_c^m(\overline{\Omega}) := \{u \restriction_{\overline{\Omega}}: u \in C_c^m(\mathbb{R}^n)\}$ and $C_b^m(\Omega)$ be the Banach space of all m-times continuously differentiable functions whose derivatives up to order m are bounded. Moreover, we write $\mathrm{BUC}(\Omega)$ for the space of all bounded and uniformly continuous functions in Ω.

We denote the Fourier transform defined on $\mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing functions, by

$$\hat{u}(\xi) := \mathcal{F}u(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi} u(x) \mathrm{d}x, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

For $q \in [1, \infty]$, $L^q(\Omega)$ denotes the Lebesgue space, which consists of all q-integrable functions if $1 \leq q < \infty$ and $L^\infty(\Omega)$ is the space of all functions u that satisfy $\|u\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{x\in\Omega}|u(x)| < \infty$. We define by $L^q_\sigma(\Omega) := \overline{C^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_q}$ the space of solenoidal functions in $L^q(\Omega)$ for $1 < q < \infty$, where $C^\infty_{c,\sigma}(\Omega)$ denotes all $C_c^\infty(\Omega)$-functions with vanishing divergence, i.e., $\operatorname{div} u = \nabla \cdot u = 0$. If $\Omega = \mathbb{R}^n_+$ we will also make use of the well-known fact that

$$L^q_\sigma(\mathbb{R}^n_+) = \{u \in L^q(\mathbb{R}^n_+) : \ \operatorname{div} u = 0, \ u^n \restriction_{\partial\mathbb{R}^n_+} = 0\}.$$

Furthermore, we set $L^q_{\mathrm{loc}}(\Omega) := \{u \in \mathcal{S}'(\Omega) : u \in L^q(K) \text{ for each compact } K \subseteq \Omega\}$. $W^{m,q}(\Omega)$ denotes the Sobolev space of order $m \in \mathbb{N}_0$. Its norm is given by

$$\|u\|_{W^{m,q}(\Omega)} := \left(\sum_{j=0}^m \|\nabla^j u\|^q_{L^q(\Omega)} \right)^{1/q},$$

where ∇^j is the tensor of all possible j-th order differentials. Moreover, $W^{m,q}_0(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $W^{m,q}(\Omega)$. If not otherwise stated, we also write $\|\cdot\|_q := \|\cdot\|_{L^q(\Omega)}$ and $\|\cdot\|_{m,q} := \|\cdot\|_{W^{m,q}(\Omega)}$.

We also make use of the homogeneous Sobolev space $\widehat{W}^{1,q}(\Omega)$ consisting of all $L^1_{\mathrm{loc}}(\Omega)$-functions u having finite Dirichlet energy $\int_\Omega |\nabla u|^q \mathrm{d}x$, modulo constants. It becomes a Banach space when equipped with the norm

$$\|u\|_{\widehat{W}^{1,q}(\Omega)} := \left(\int_\Omega |\nabla u|^q \mathrm{d}x \right)^{1/q}.$$

We will use the same notation for the corresponding spaces of vector fields on Ω, i.e., $(L^q(\Omega))^n = L^q(\Omega)$, $(W^{k,q}(\Omega))^n = W^{k,q}(\Omega)$, etc. We denote by q' the Hölder conjugated exponent, i.e., $\frac{1}{q'} + \frac{1}{q} = 1$. If $u \in L^q(\Omega)$ and $v \in L^{q'}(\Omega)$ we use the notation $(u, v) := (u, v)_\Omega := \int_\Omega uv \mathrm{d}x$ for the dual pairing.

If X and Y are Banach spaces, the space of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$, and $\mathcal{L}(X)$ is the abbreviation for $\mathcal{L}(X, X)$. For any closed operator A in X, its domain and range are denoted by $D(A)$ and $R(A)$, respectively. Its resolvent set is denoted by $\rho(A)$ and its spectrum by $\sigma(A)$. Furthermore, we call A a generator, if $(\mathrm{e}^{tA})_{t\geq 0}$ satisfies the semigroup properties.

As usual C, M, ... denote constants that may change from line to line. Sometimes we would like to express a special dependence on some parameter s. Then we use either the subscript notation C_s, M_s, ... or we write it as an argument $C(s)$, $M(s)$, ...

2.2 Rotation Invariant Multipliers

In this subsection we establish the substitute for the H^∞-calculus of the Poisson operator $|\nabla'|$, which is only valid in $L^q(\mathbb{R}^n)$ for $1 < q < \infty$ (Proposition 2.2 and Corollary 2.3). The result is essentially a consequence of a multiplier result for rotation invariant multipliers and Cauchy's estimate formula for holomorphic functions. Compared to the H^∞-calculus of $|\nabla'|$, besides the boundedness and the holomorphy on a sector, here a function m also has to be holomorphic in 0 and has to satisfy a decay condition (see condition (5)). But then the boundedness of $m(|\nabla'|)$ is valid in $L^q(\mathbb{R}^n)$ for all $q \in [1, \infty]$. We start with the multiplier result.

For $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we denote by BV_{k+1} the normed space of all functions $m \in C_0([0,\infty), \mathbb{C}) := \{m \in C([0,\infty), \mathbb{C}) : \lim_{t\to\infty} m(t) = 0\}$ with $m, m', \ldots, m^{(k)}$ being locally absolutely continuous on $(0,\infty)$ and satisfying $\lim_{t\to\infty} m^{(j)}(t) = 0$ for $j = 0, \ldots, k$ and

$$\|m\|_{BV_{k+1}} = \frac{1}{\Gamma(k+1)} \int_0^\infty t^k |m^{(k+1)}(t)| \mathrm{d}t < \infty.$$

By a simple calculation we can see that $BV_{k+1} \hookrightarrow BV_k$. The introduction of this space allows us to formulate the next lemma about rotation invariant multipliers. A proof can be found in [Tre73].

Lemma 2.1 *Let $n, k \in \mathbb{N}$ satisfy $k > n/2$ and let $m \in BV_{k+1}$. Then the function $m(|\cdot|) : \mathbb{R}^n \to \mathbb{C}$, $\xi \mapsto m(|\xi|)$ belongs to the space $\mathcal{F}L^1(\mathbb{R}^n) := \{\mathcal{F}f : f \in L^1(\mathbb{R}^n)\}$ and there is a constant $C = C(n,k) > 0$ such that*

$$\|\mathcal{F}^{-1} m(|\cdot|)\|_{L^1(\mathbb{R}^n)} \le C \|m\|_{BV_{k+1}}.$$

As a consequence we obtain the following proposition for bounded holomorphic functions m, where we denote the space of bounded holomorphic functions on a domain $G \subseteq \mathbb{C}$ by $H^\infty(G)$.

Proposition 2.2 *Let $m : [0,\infty) \to \mathbb{C}$. Assume there exist $\phi \in (0,\pi)$, $\varepsilon \in (0,1)$, and $C_0 > 0$ such that $m \in H^\infty(\Sigma_\phi \cup \{0\})$, where $\Sigma_\phi = \{z \in \mathbb{C} \setminus \{0\} : \arg z < \phi\}$, and*

$$|z^\varepsilon m(z)| \le K, \quad z \in \Sigma_\phi. \tag{5}$$

Then $[\xi \mapsto m(|\xi|)] \in \mathcal{F}L^1(\mathbb{R}^n)$ and there is a constant $C > 0$ such that

$$\|\mathcal{F}^{-1} m(|\cdot|)\|_{L^1(\mathbb{R}^n)} \le CK.$$

Proof. In view of Lemma 2.1 we have to prove that $m \in BV_{k+1}$ for some $k > n/2$. Since $m \in \mathcal{H}^\infty(\Sigma_\phi \cup \{0\}) \subseteq C^\infty([0,\infty))$, by (5) it immediately follows that $m \in C_0([0,\infty))$ and all derivatives $m^{(j)}$ with $j \le k$ are locally absolutely continuous functions on $(0,\infty)$. By the holomorphy of m in 0 there is a $\delta_0 > 0$ such that m is holomorphic in $B_{\delta_0}(0) \cup \Sigma_\phi$, where $B_r(w) := \{z \in \mathbb{C} : |z - w| < r\}$. Thus we may choose $\delta_i = \delta_i(\delta_0) \in (0,1)$, $i = 1, 2$, in a way such that for

$$r : \mathbb{C} \to (0,\infty), \quad z \mapsto r(z) := \delta_1(\mathrm{Re}\, z + \delta_2),$$

the ball $B_{r(z)}(z)$ lies completely in the domain $B_{\delta_0}(0) \cup \Sigma_\phi$ for all $z \in [0,\infty)$ (see Figure 1). For

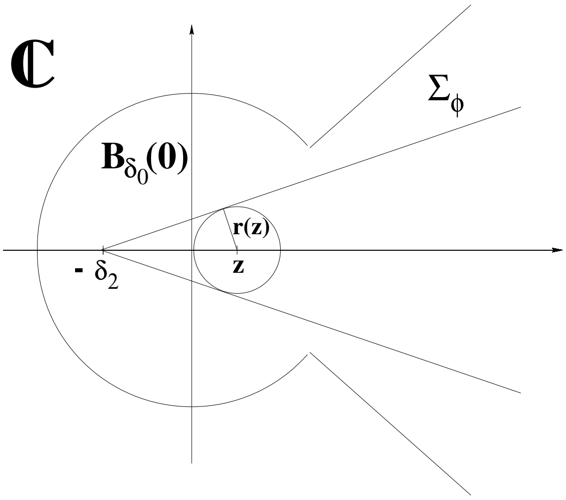

Figure 1: The ball with center $z \in [0,\infty)$ and radius $r(z)$ is contained in $B_{\delta_0}(0) \cup \Sigma_\phi$.

$t \in [0,\infty)$ we therefore get by Cauchy's formula and by our assumption (5) on m

$$\begin{aligned} |m^{(j)}(t)| &\leq \frac{j!}{r(t)^j} \max_{|t-z|=r(t)} |m(z)| \leq C(j,\delta_0) \frac{1}{(t+\delta_2)^j} \max_{|t-z|=r(t)} \frac{K}{|z|^\varepsilon} \\ &\leq \frac{C(j,\delta_0)}{(t+\delta_2)^j} \frac{K}{|t-r(t)|^\varepsilon} \leq \frac{C(j,\delta_0)K}{(t+\delta_2)^j |(1-\delta_1)t - \delta_1\delta_2|^\varepsilon} \end{aligned}$$

for $j \in \mathbb{N} \cup \{0\}$. This implies $\lim_{t\to\infty} m^{(j)}(t) = 0$ for $j \in \mathbb{N} \cup \{0\}$ and, if we set $j = k+1$, that

$$\begin{aligned} \|m\|_{BV_{k+1}} &= \frac{1}{\Gamma(k+1)} \int_0^\infty t^k |m^{(k+1)}(t)| \mathrm{d}t \\ &\leq C(k,\delta_0) K \int_0^\infty t^k \frac{1}{(t+\delta_2)^{k+1} |(1-\delta_1)t - \delta_1\delta_2|^\varepsilon} \mathrm{d}t \\ &\leq C(k,\delta_0,\varepsilon) K. \end{aligned} \tag{6}$$

This yields the assertion. □

In the subsequent sections we will also frequently make use of the following corollary on multipliers depending on parameters. It is an immediate consequence of estimate (6).

Corollary 2.3 *Let $n \in \mathbb{N}$, $I \subseteq \mathbb{R}^n$, and $b : I \to [0,\infty)$, $s \mapsto b(s)$. Assume there exist $\phi \in (0,\pi)$, $\varepsilon \in (0,1)$, and $\delta_0 > 0$, such that $m : [0,\infty) \times I \to \mathbb{C}$ satisfies $m(\cdot,s) \in H^\infty(\Sigma_\phi \cup B_{\delta_0}(0))$ for all $s \in I$ and*

$$|z^\varepsilon m(z,s)| \leq b(s), \quad z \in \Sigma_\phi, \quad s \in I.$$

Then $[\xi \mapsto m(|\xi|,s)] \in \mathcal{F}L^1(\mathbb{R}^n)$ and there is a constant $C > 0$ such that

$$\|\mathcal{F}^{-1} m(|\cdot|,s)\|_{L^1(\mathbb{R}^n)} \leq C b(s), \quad s \in I. \tag{7}$$

2.3 Some known results for the Laplacian

Here we recall some well-known results for the Laplace and Poisson operator in L^1- and L^∞-spaces, which we will use in the sequel. Let $n \in \mathbb{N}$ and $|\nabla| = (-\Delta_{\mathbb{R}^n})^{1/2}$ the Poisson operator in $L^q(\mathbb{R}^n)$ for $1 \leq q \leq \infty$. Then we have the estimates

$$\begin{aligned} \||\nabla|^s \mathrm{e}^{-|\nabla|^{2m}t}\|_{\mathcal{L}(L^q(\mathbb{R}^n))} &\leq \frac{C}{t^{s/2m}}, \quad t>0,\ m\in\mathbb{N},\ s\geq 0,\ 1\leq q\leq\infty, &(8)\\ \|\nabla^j \mathrm{e}^{-|\nabla|^{2m}t}\|_{\mathcal{L}(L^q(\mathbb{R}^n))} &\leq \frac{C}{t^{j/2m}}, \quad t>0,\ m\in\mathbb{N},\ j\in\mathbb{N}_0,\ 1\leq q\leq\infty. &(9)\end{aligned}$$

Let Δ_D and Δ_N be the Dirichlet Laplacian and the Neumann Laplacian on $L^q(\mathbb{R}^n_+)$ for $1 \leq q \leq \infty$ respectively, defined by their L^q-realizations

$$\begin{aligned} \Delta_D u := \Delta u, &\qquad u \in D(\Delta_D) := \{v \in L^q(\mathbb{R}^n_+) : \Delta v \in L^q(\mathbb{R}^n_+), v|_{\partial\mathbb{R}^n_+} = 0\},\\ \Delta_N u := \Delta u, &\qquad u \in D(\Delta_N) := \{v \in L^q(\mathbb{R}^n_+) : \Delta v \in L^q(\mathbb{R}^n_+), \partial_n v|_{\partial\mathbb{R}^n_+} = 0\}.\end{aligned}$$

Then for each $\varphi_0 \in (0,\pi)$ and $1 \leq q \leq \infty$ there is a $C = C(\varphi_0) > 0$ such that the resolvent estimates

$$\begin{aligned} |\lambda|\|(\lambda-\Delta_D)^{-1}\|_{\mathcal{L}(L^q(\mathbb{R}^n_+))} + \sqrt{|\lambda|}\|\nabla(\lambda-\Delta_D)^{-1}\|_{\mathcal{L}(L^q(\mathbb{R}^n_+))} &\leq C, &(10)\\ |\lambda|\|(\lambda-\Delta_N)^{-1}\|_{\mathcal{L}(L^q(\mathbb{R}^n_+))} + \sqrt{|\lambda|}\|\nabla(\lambda-\Delta_N)^{-1}\|_{\mathcal{L}(L^q(\mathbb{R}^n_+))} &\leq C &(11)\end{aligned}$$

are valid for all $\lambda \in \Sigma_{\pi-\varphi_0}$. It is well-known that the resolvents of Δ_D and Δ_N can be represented through the resolvent of the Laplacian in the whole space via reflection, namely as

$$(\lambda-\Delta_D)^{-1} f = \big((\lambda-\Delta_{\mathbb{R}^n})^{-1} E^+ f\big)\restriction_{\mathbb{R}^n_+}, \quad f \in L^q(\mathbb{R}^n_+), \tag{12}$$

and

$$(\lambda-\Delta_N)^{-1} f = \big((\lambda-\Delta_{\mathbb{R}^n})^{-1} E^- f\big)\restriction_{\mathbb{R}^n_+}, \quad f \in L^q(\mathbb{R}^n_+), \tag{13}$$

respectively, where

$$(E^\pm f)(x', x_n) = \begin{cases} f(x', x_n) & : \quad x_n > 0, \\ \pm f(x', -x_n) & : \quad x_n < 0. \end{cases} \tag{14}$$

Thus, estimates (10) and (11) immediately follow from estimates for the Laplacian in the whole space $\Delta_{\mathbb{R}^n}$, which can be obtained, as well as (8) and (9), for instance by an application of the Mikhlin multiplier result.

2.4 A solution formula for the Stokes resolvent problem

Here we recall the explicit solution formula for the Stokes resolvent problem with Robin boundary conditions obtained in [Saa06]. We also recall from [Saa06] some complex estimates for specific terms occuring in the formula. In Sections 3 and 4 they will turn out to be the key estimates in order to obtain estimates for the solution in $L^1(\mathbb{R}^n_+)$ and $L^\infty(\mathbb{R}^n_+)$.

In the sequel we will adopt the notation in [Saa06]. In particular $x' := (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ always denotes the first $n-1$ components of the variable $x \in \mathbb{R}^n_+$. Also for vector fields u and operators R we write $u = (u', u^n) = (u^1, \ldots, u^{n-1}, u^n)$ and $R = (R', R_n) = (R_1, \ldots, R_{n-1}, R_n)$ respectivly.

Recall that the Stokes resolvent problem is given by

$$(SRP)_{f,\lambda,\alpha}\left\{\begin{array}{rcll}(\lambda-\Delta)u+\nabla p &=& f & \text{in } \mathbb{R}^n_+,\\ \operatorname{div} u &=& 0 & \text{in } \mathbb{R}^n_+,\\ T_\alpha u &=& 0 & \text{in } \mathbb{R}^{n-1}.\end{array}\right.$$

Here we assume $\lambda\in\Sigma_{\pi-\varphi_0}$ for some $\varphi_0\in(0,\pi)$ and that f satisfies the compatibility conditions $\operatorname{div} f=0$ and $f^n|_{\partial\mathbb{R}^n_+}=0$. By applying partial Fourier transform, which we denoted by $\mathcal{F}$, that is Fourier transform with respect to x', in [Saa06, section 6] we established the following representation for u:

$$u' = (\lambda-\Delta_D)^{-1}f' - R'v^n + \mathrm{e}^{-\omega(|\nabla'|)\cdot}\phi', \tag{15}$$

$$u^n = (\lambda-\Delta_D)^{-1}f^n + v^n. \tag{16}$$

Here $R'=\mathcal{F}^{-1}\frac{i\xi'}{|\xi'|}\mathcal{F}$ denotes the Riesz operator,

$$\hat v^n(\xi',x_n) := \mathcal{F}v^n(\xi',x_n) := M_{x_n,\lambda}(|\xi'|)\,[1-(\omega(|\xi'|)+|\xi'|)m_\lambda(|\xi'|)]\,\hat h^n(\xi') \tag{17}$$

for $(\xi',x_n)\in\mathbb{R}^n_+$, and

$$\hat\phi'(\xi') = \frac{1}{\omega(|\xi'|)+\alpha}\left(\hat h'(\xi')+\alpha\frac{i\xi'}{|\xi'|}m_\lambda(|\xi'|)\hat h^n(\xi')\right),\quad \xi'\in\mathbb{R}^{n-1}. \tag{18}$$

The functions ω, $M_{x_n,\lambda}$, and m_λ are given by

$$\omega(z) := \sqrt{\lambda+z^2}, \tag{19}$$

$$M_{x_n,\lambda}(z) := \frac{e^{-\omega(z)x_n}-e^{-zx_n}}{\omega(z)-z}, \tag{20}$$

$$m_\lambda(z) := \frac{1}{\omega(z)+z+\alpha}, \tag{21}$$

whereas $h=(h',h^n)$ is defined as

$$\hat h^n(\xi') := \int_0^\infty e^{-\omega(|\xi'|)s}\hat f^n(\xi',s)\mathrm{d}s,\qquad \xi'\in\mathbb{R}^{n-1}, \tag{22}$$

$$\hat h'(\xi') := \int_0^\infty e^{-\omega(|\xi'|)s}\hat f'(\xi',s)\mathrm{d}s,\qquad \xi'\in\mathbb{R}^{n-1}. \tag{23}$$

Applying once integration by parts and using the compatibility conditions $\operatorname{div} f=0$ and $f^n\restriction_{\partial\mathbb{R}^n_+}=0$ for f we easily observe that

$$\omega(|\xi'|)\hat h^n(\xi') = -i\xi'\cdot\hat h'(\xi'),\quad \xi'\in\mathbb{R}^{n-1}. \tag{24}$$

This relation will be of crucial importance to overcome the unboundedness of the Riesz operators R_j in $L^1(\mathbb{R}^{n-1})$ and $L^\infty(\mathbb{R}^{n-1})$ in Sections 3 and 4. Note that in [Saa06] also a corresponing representation for the pressure is derived. It is given by

$$\hat p(\xi',x_n) = -\mathrm{e}^{-|\xi'|x_n}\frac{\omega(|\xi'|)+|\xi'|}{|\xi'|}\alpha m_\lambda(|\xi'|)\hat h^n(\xi'). \tag{25}$$

Remark 2.4 Let $\varphi_0 \in (0, \pi/2)$ and $\varphi \in (0, \varphi_0/2)$. Then there is a constant $C = C(\varphi_0, \varphi)$, such that

(a) $|\arg \omega(z)| \leq \frac{\pi - \varphi_0}{2}$,

(b) Re $\omega(z) \geq C\sqrt{|\lambda|}$,

(c) Re $\omega(z) \geq C|z|$

for all $\lambda \in \Sigma_{\pi-\varphi_0}$ and $z \in \Sigma_\varphi$.

We set

$$G_\lambda(z) := \frac{z}{\omega(z)}. \tag{26}$$

By elementary calculations we can obtain the following estimates.

Lemma 2.5 *Let $\sigma, \rho \geq 0$, $\varphi_0 \in (0, \pi/2)$ and $\varphi \in (0, \varphi_0/4)$. Then there are constants $C, \delta > 0$ such that*

(a) $|G_\lambda(z)| \leq C$,

(b) $|z^{1+\rho} M_{x_n,\lambda}(z)| \leq C \dfrac{e^{-\delta|z|x_n}}{x_n^\rho(1 + \sqrt{|\lambda|}x_n)}$,

(c) $|\omega(z) m_\lambda(z)| \leq C$,

(d) $|\alpha m_\lambda(z)| \leq C \dfrac{\alpha}{\sqrt{|\lambda|} + \alpha}$,

(e) $|z m_\lambda(z)| \leq C$,

(f) $|\omega(z)^\sigma \mathrm{e}^{-\omega(z)x_n}| \leq C \dfrac{\mathrm{e}^{-\delta\sqrt{|\lambda|}x_n}}{x_n^\sigma}$,

for all $z \in \Sigma_\varphi$, $x_n > 0$ and $\lambda \in \Sigma_{\pi-\varphi_0}$.

For a detailed proof of the lemmas above we refer to [Saa06, Lemma 5.3] or [Saa03, Lemma 4.3].

3 The L^1-case

We will start our discussion in this section with the case of Neumann boundary conditions. In this case the problem can be completely reduced to a problem for the Laplacian due to the fact that the crucial terms in the solution formula vanish. Since the generation result for the Laplacian in $L^1(\mathbb{R}^n_+)$ is well-known, this property transfers immediately to the Stokes operator (see Theorem 3.3). This is not the case for all other types of Robin boundary conditions. As long as $\alpha > 0$ the crucial terms do not vanish, which implies that there is no generation result in that case. In fact, in Theorem 3.5 we give a counterexample of a right hand side $f \in L^1_\sigma(\mathbb{R}^n_+)$ such that the corresponding solution u is not an L^1-function. Utilizing the complex estimates in Lemma 2.5 we then will proceed by proving estimates for the remainder term v of the solution formula (15), (16) in the spaces $L^1(\mathbb{R}^{n-1})$ and $L^\infty(\mathbb{R}^{n-1})$. Based on these estimates and Lemma 2.5 we prove in Theorem 3.7 that the solution u satisfies the typical semigroup gradient estimates in $L^1(\mathbb{R}^n_+)$, in spite of the fact that the solution itself does not belong to $L^1(\mathbb{R}^n_+)$.

Setting $\alpha = 0$, in view of (17), (18), and (23) we see that

$$v^n = 0 \quad \text{and} \quad \hat{\phi}' = \frac{1}{\omega}\hat{h}' = \int_0^\infty \frac{\mathrm{e}^{-\omega s}}{\omega}\hat{f}'(\cdot, s)\mathrm{d}s. \tag{27}$$

Furthermore, according to the results in [Saa06] we have the representations

$$\mathcal{F}[(\lambda - \Delta_D)^{-1} f^n](\xi', x_n) \quad = \quad \int_0^\infty k_-(\xi', x_n, s, \lambda)\hat{f}^n(\xi', s)\mathrm{d}s \quad (\xi', x_n) \in \mathbb{R}^n_+, \tag{28}$$

and

$$\mathcal{F}[(\lambda - \Delta_N)^{-1} f'](\xi', x_n) \quad = \quad \int_0^\infty k_+(\xi', x_n, s, \lambda)\hat{f}'(\xi', s)\mathrm{d}s \quad (\xi', x_n) \in \mathbb{R}^n_+, \tag{29}$$

where

$$k_\pm(\xi', x_n, s, \lambda) := \frac{e^{-\omega(|\xi'|)|x_n - s|} \pm e^{-\omega(|\xi'|)(x_n+s)}}{2\omega(|\xi'|)}.$$

Inserting (27) and (28) in (15) we deduce

$$\begin{aligned} \hat{u}' \quad &= \quad \int_0^\infty k_-(\xi', x_n, s, \lambda)\hat{f}'(\xi', s)\mathrm{d}s + \int_0^\infty \frac{\mathrm{e}^{-\omega(x_n+s)}}{\omega}\hat{f}'(\cdot, s)\mathrm{d}s \\ &= \quad \int_0^\infty k_+(\xi', x_n, s, \lambda)\hat{f}'(\xi', s)\mathrm{d}s. \end{aligned}$$

Hence the formula for $u = (u', u^n)$ simplifies considerably. Namely, in this case we have that

$$u' \quad = \quad (\lambda - \Delta_N)^{-1} f', \tag{30}$$

$$u^n \quad = \quad (\lambda - \Delta_D)^{-1} f^n. \tag{31}$$

In view of (10) and (11) we therefore obtain the following result.

Corollary 3.1 *Let $f \in L^1(\mathbb{R}^n_+) \cap L^q_\sigma(\mathbb{R}^n_+)$ for some $q \in (1, \infty)$. For $\alpha = 0$, i.e., in the case of Neumann boundary conditions, there is a constant $C = C(\varphi_0) > 0$, such that the solution u of the Stokes resolvent problem $(SRP)_{f,\lambda,\alpha}$ satisfies*

$$|\lambda|\|u\|_1 + \sqrt{|\lambda|}\|\nabla u\|_1 \leq C\|f\|_1, \quad \lambda \in \Sigma_{\pi - \varphi_0}.$$

Next, let

$$B := \begin{pmatrix} -\Delta_N & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & -\Delta_N & 0 \\ 0 & \dots & 0 & -\Delta_D \end{pmatrix}$$

be defined in $L^1(\mathbb{R}^n_+)$ with domain

$$D(B) := D(\Delta_N)^{n-1} \times D(\Delta_D).$$

The formulas (30) and (31) motivate the following definition.

Definition 3.2 Let $L^1_\sigma(\mathbb{R}^n_+) := \overline{L^1(\mathbb{R}^n_+) \cap D}^{\|\cdot\|_1}$, where $D = \bigcap_{q\in(1,\infty)} L^q_\sigma(\mathbb{R}^n_+)$. We call

$$A_N u := Bu, \quad u \in D(A_N) := D(B) \cap L^1_\sigma(\mathbb{R}^n_+)$$

the *Stokes operator with Neumann boundary conditions* in $L^1_\sigma(\mathbb{R}^n_+)$.

The next result shows that the operator A_N can be regarded as the restriction of B to $L^1_\sigma(\mathbb{R}^n_+)$, i.e.,

$$A_N = B \restriction_{L^1_\sigma(\mathbb{R}^n_+)} .$$

We remark that this is valid for all $q \in [1,\infty]$, i.e., $A_{N,q} = B \restriction_{L^q_\sigma(\mathbb{R}^n_+)}$.

Theorem 3.3 *The operator $-A_N$ is the generator of a bounded holomorphic C_0-semigroup on $L^1_\sigma(\mathbb{R}^n_+)$. Moreover, $\rho(A_N) = \mathbb{C} \setminus [0,\infty)$, and, if $u_f(\lambda)$ is the unique solution of $(SRP)_{f,\lambda,0}$ for $f \in L^1_\sigma(\mathbb{R}^n_+) \cap D$, then $u_f(\lambda) = (\lambda + A_N)^{-1} f$ for $-\lambda \in \rho(A_N)$. The semigroup $(\mathrm{e}^{-tA_N})_{t\geq 0}$ also satisfies the gradient estimate*

$$\|\nabla \mathrm{e}^{-tA_N} f\|_1 \leq C t^{-1/2} \|f\|_1, \quad t > 0,\ f \in L^1_\sigma(\mathbb{R}^n_+). \tag{32}$$

Proof. Since the operator B is the generator of a bounded holomorphic C_0-semigroup on $L^q(\mathbb{R}^n_+)$, $1 \leq q < \infty$, it remains to verify the invariance of the subspace $L^1_\sigma(\mathbb{R}^n_+)$ under the action of $(\lambda+B)^{-1}$ in order to prove the generation result for A_N. To this end, pick $\lambda \in \mathbb{C}\setminus[0,\infty)$ and $f \in L^1(\mathbb{R}^n_+) \cap D$. By virtue of representation (30) and (31) it readily follows that

$$\operatorname{div}(\lambda + B)^{-1} f = \operatorname{div} u_f(\lambda) = 0 \quad \text{and that} \quad ((\lambda + B)^{-1} f)^n \restriction_{\partial\mathbb{R}^n_+} = 0,$$

which implies that $(\lambda + B)^{-1} f \in L^1(\mathbb{R}^n_+) \cap D$. The continuity of $(\lambda + B)^{-1}$ in $L^1(\mathbb{R}^n_+)$ then yields $(\lambda + B)^{-1}(L^1_\sigma(\mathbb{R}^n_+)) \subseteq L^1_\sigma(\mathbb{R}^n_+)$.

In order to show the gradient estimates observe that for $f \in L^1_\sigma(\mathbb{R}^n_+)$ we may write

$$\mathrm{e}^{-tA_N} f = \frac{1}{2\pi i} \int_\Gamma \mathrm{e}^{-\lambda t} (\lambda - A_N)^{-1} f \mathrm{d}\lambda$$

with $\Gamma = \{t\mathrm{e}^{i\theta} : \infty > t > \varepsilon\} \cup \{\varepsilon \mathrm{e}^{it} : \theta \leq t \leq 2\pi - \theta\} \cup \{t\mathrm{e}^{-i\theta} : \varepsilon < t < \infty\}$ for $\varepsilon > 0$ and $\theta \in (0, \pi/2)$. By Corollary 3.1 and a density argument we deduce

$$\begin{aligned} \|\nabla \mathrm{e}^{-tA_N} f\|_1 &= \|\frac{1}{2\pi i} \int_\Gamma \mathrm{e}^{-\lambda t} \nabla(\lambda - A_N)^{-1} f \mathrm{d}\lambda\|_1 \\ &\leq C \left(\varepsilon^{1/2} \int_\theta^{2\pi-\theta} \mathrm{e}^{\varepsilon t \cos\theta} \mathrm{d}s + \int_\varepsilon^\infty \frac{\mathrm{e}^{-st\cos\theta}}{\sqrt{s}} \mathrm{d}s \right) \|f\|_1 \end{aligned}$$

for all $f \in L^1_\sigma(\mathbb{R}^n_+)$. Letting $\varepsilon \to 0$ yields (32). □

Remark 3.4 Note that $L^1_\sigma(\mathbb{R}^n_+)$ coincides with the space $\overline{L^1(\mathbb{R}^n_+) \cap L^q_\sigma(\mathbb{R}^n_+)}^{\|\cdot\|_1}$ for each $q \in (1,\infty)$. In fact, let $f \in \overline{L^1(\mathbb{R}^n_+) \cap L^q_\sigma(\mathbb{R}^n_+)}^{\|\cdot\|_1}$, $(v_k) \subseteq L^1(\mathbb{R}^n_+) \cap L^q_\sigma(\mathbb{R}^n_+)$ such that $v_k \to f$ in $L^1(\mathbb{R}^n_+)$, and consider the sequence

$$u_k := u_{k,j_k} := \exp(-\frac{1}{j_k} B) v_k, \quad k, j_k \in \mathbb{N},$$

with B defined as above. Choosing j_k for each $k \in \mathbb{N}$ and $\varepsilon > 0$ in a way, such that $\|\exp(-\frac{1}{j_k}B)v_k - v_k\|_1 < \varepsilon/2$, it easily follows that $u_k \to f$ in $L^1(\mathbb{R}^n_+)$. Moreover, since $\exp(-\frac{1}{j_k}B)g = \exp(-\frac{1}{j_k}A_{N,q})g$ for $g \in L^q_\sigma(\mathbb{R}^n_+)$, $1 < q < \infty$, by the $L^p - L^q$-estimates in [Saa06, Corollary 5.8] for $\exp(-\frac{1}{j_k}A_{N,q})$ we see that $(u_k) \subseteq L^p_\sigma(\mathbb{R}^n_+)$ for each $p \in [q, \infty)$. In view of the inclusion $L^q_\sigma(\mathbb{R}^n_+) \cap L^1(\mathbb{R}^n_+) \subseteq L^p_\sigma(\mathbb{R}^n_+)$ for $p \in (1, q]$, this implies that $u_k \in L^1(\mathbb{R}^n_+) \cap D$. Consequently, $f \in L^1_\sigma(\mathbb{R}^n_+)$.

As we observed in the case of Neumann boundary conditions the crucial terms in the solution formula for $u = (u', u^n)$ vanish. That is not the case if $\alpha \in (0, \infty]$. Here the corresponding result to Corollary 3.1 reads as follows.

Theorem 3.5 *Let $\alpha \in (0, \infty]$. For $\lambda \in \mathbb{R}_+$ there is an $f \in L^1(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$ such that $u \notin L^1(\mathbb{R}^n_+)$, where (u, p) is the solution of problem $(SRP)_{f,\lambda,\alpha}$ given in (15), (16), and (25).*

Proof. For the proof we consider representation (16) for the n-th component of the Stokes flow u and denote the two addends of u^n by u^n_1 and u^n_2. Next we define a special function f satisfying the assumptions of the theorem such that the $L^1(\mathbb{R}^n_+)$-norm of u^n_2 is infinite. Since in view of (10), u^n_1 for each $f \in L^1_\sigma(\mathbb{R}^n_+)$ is an L^1-function we deduce

$$\|u\|_{L^1(\mathbb{R}^n_+)} \geq \|u^n\|_{L^1(\mathbb{R}^n_+)} = \infty$$

for this special f.

The construction of our counterexample is similar to [DHP01]. We will see that f is constructed in a way such that the $L^1(\mathbb{R}^n_+)$-norm of the n-th component of the corresponding solution can be estimated from below by the $L^1(\mathbb{R}^{n-1})$-norm of $R_j G_r$, where R_j is a Riesz operator and G_r is the Gauss kernel. However, as an obvious consequence of well-known properties of the Hardy space $\mathcal{H}^1(\mathbb{R}^{n-1})$ we have $R_j G_r \notin L^1(\mathbb{R}^{n-1})$. This will imply the result.

For this purpose, first let us remind the reader to well-known properties of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. It is defined as

$$\mathcal{H}^1(\mathbb{R}^n) := \{f \in L^1(\mathbb{R}^n) : f^* \in L^1(\mathbb{R}^n)\},$$

where $f^*(x) := \sup_{t>0} |(G_t * f)(x))|$, $x \in \mathbb{R}^n$, and G_t denotes the heat kernel given by $G_t(x) = \frac{1}{(4\pi t)^{n/2}} \mathrm{e}^{-|x|^2/4t}$, $x \in \mathbb{R}^n$, $t > 0$. Equipped with the norm $\|f\|_{\mathcal{H}^1} := \|f^*\|_1$ the space $\mathcal{H}^1(\mathbb{R}^n)$ becomes a Banach space and we have that $\int_{\mathbb{R}^n} f = 0$ for $f \in \mathcal{H}^1(\mathbb{R}^n)$ (see [Ste93] or [FS72]). It is well-known that an $L^1(\mathbb{R}^n)$-function f belongs to $\mathcal{H}^1(\mathbb{R}^n)$ if and only if its Riesz transforms $R_j f$ belong to $L^1(\mathbb{R}^n)$ for $j = 1, \dots, n$. This equivalent definition will be of crucial importance in what follows.

Now let $\lambda \in \mathbb{R}_+$. We define γ by

$$\gamma(|\xi'|) := 1 - (\omega(|\xi'|) + |\xi'|) m_\lambda(|\xi'|) = \frac{\alpha}{\omega(|\xi'|) + |\xi'| + \alpha} \tag{33}$$

and consider the function

$$\hat{f}^n(\xi', x_n) := \frac{i\xi_j}{\omega(|\xi'|)^4 \gamma(|\xi'|)} x_n^2 \mathrm{e}^{-\omega(|\xi'|)x_n} \hat{g}_r(\xi'), \quad (\xi', x_n) \in \mathbb{R}^n_+,$$

where $j \in \{1, \dots, n-1\}$ will be fixed later and g_r is defined as

$$g_r := (\lambda - \Delta')^4 G_r \in \mathcal{S}(\mathbb{R}^{n-1}) \subseteq L^q(\mathbb{R}^{n-1}), \quad 1 \leq q \leq \infty,$$

for some fixed $r \in (0,\infty)$. Setting

$$f := (0,\dots,0,f^j,0,\dots,0,f^n)$$

with

$$\hat{f}^j(\xi',x_n) := -\frac{1}{\omega(|\xi'|)^4\gamma(|\xi'|)}\left(2x_n - \omega(|\xi'|)x_n^2\right)\mathrm{e}^{-\omega(|\xi'|)x_n}\hat{g}_r(\xi'), \quad (\xi',x_n) \in \mathbb{R}^n_+,$$

it follows that $\operatorname{div} f = 0$ and $f\restriction_{\partial\mathbb{R}^n_+} = 0$. To see that f fulfills our assumptions it remains to show $f \in L^1(\mathbb{R}^n_+) \cap L^2(\mathbb{R}^n_+)$. To this end observe that

$$\begin{aligned}
\frac{1}{\omega(z)^4\gamma(z)} &= \frac{1}{\omega(z)^4}\frac{1}{\alpha}(\omega(z)+z+\alpha) \\
&= \frac{1}{\omega(z)^3}\left[\frac{1}{\alpha}(1+G(z)) + \frac{1}{\omega(z)}\right], \quad z \in \Sigma_\varphi.
\end{aligned}$$

Hence, for $\varepsilon \in (0,1)$ we may conclude by Lemma 2.5 (a)

$$\begin{aligned}
\left|\frac{z^\varepsilon}{\omega(z)^4\gamma(z)}\right| &= \left|G(z)^\varepsilon \frac{1}{\omega(z)^{3-\varepsilon}}\left[\frac{1}{\alpha}(1+G(z)) + \frac{1}{\omega(z)}\right]\right| \\
&\leq C(\lambda)\left|\frac{1}{\alpha}(1+G(z)) + \frac{1}{\omega(z)}\right| \\
&\leq C(\lambda)\left(\frac{1}{\alpha} + \frac{1}{\sqrt{\lambda}}\right)
\end{aligned}$$

for $z \in \Sigma_\varphi$. Therefore, according to Proposition 2.2, $\xi \mapsto \frac{1}{\omega^4(|\xi|)\gamma(|\xi|)}$ is a Fourier multiplier on $L^q(\mathbb{R}^{n-1})$ for $1 \leq q \leq \infty$ and $\alpha \in (0,\infty]$. By virtue of Lemma 2.5 (f) this leads to

$$\begin{aligned}
\|f^n\|_{L^q(\mathbb{R}^n_+)} &= \left(\int_0^\infty \|\mathcal{F}^{-1}\frac{1}{\omega(|\cdot|)^4\gamma(|\cdot|)}x_n^2\mathrm{e}^{-\omega(|\cdot|)x_n}\mathcal{F}\nabla' g_r\|^q_{L^q(\mathbb{R}^{n-1})}\mathrm{d}x_n\right)^{1/q} \\
&\leq C(\lambda)\left(\int_0^\infty x_n^{2q}\mathrm{e}^{-c_1\sqrt{\lambda}qx_n}\mathrm{d}x_n\right)^{1/q}\|\nabla' g_r\|_{L^q(\mathbb{R}^{n-1})} \\
&\leq C(\lambda)\|\nabla' g_r\|_{L^q(\mathbb{R}^{n-1})}
\end{aligned}$$

for $1 \leq q < \infty$. Analogously we see that $f^j \in L^q(\mathbb{R}^n_+)$, which implies $f \in L^q(\mathbb{R}^n_+)$ for $1 \leq q < \infty$.

Now let us calculate the $L^1(\mathbb{R}^n_+)$-norm of u_2^n, the second addend of u^n. Note that

$$[1-(\omega(|\xi'|)+|\xi'|)m_\lambda(|\xi'|)]\,\hat{h}^n(\xi') = \frac{i\xi_j}{4\omega(|\xi'|)^7}\hat{g}_r(\xi')$$

and

$$-\int_0^\infty M_{x_n,\lambda}(|\xi'|)\mathrm{d}x_n = \int_0^\infty \frac{\mathrm{e}^{-|\xi'|x_n} - \mathrm{e}^{-\omega(|\xi'|)x_n}}{\omega(|\xi'|)-|\xi'|}\mathrm{d}x_n = \frac{1}{\omega(|\xi'|)|\xi'|}.$$

Hence, an application of Remark 2.4 (b), Lemma 2.5 (b), and Corollary 2.3 shows that the function

$$\xi' \mapsto \hat{u}_2^n(\xi',x_n) = -M_{x_n,\lambda}(|\xi'|)\frac{i\xi_j}{4\omega(|\xi'|)^7}\hat{g}_r(\xi'), \quad x_n > 0,$$

belongs to $L^1(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))$ and the function

$$\xi' \mapsto \int_0^\infty \hat{u}_2^n(\xi', x_n)\mathrm{d}x_n = \frac{1}{4\omega(|\xi'|)^8}\frac{i\xi_j}{|\xi'|}\hat{g}_r(\xi') = \frac{1}{4}\frac{i\xi_j}{|\xi'|}\mathrm{e}^{-|\xi'|^2 r}$$

belongs to $L^2(\mathbb{R}^{n-1})$. The continuity of $\mathcal{F}$ on $L^2(\mathbb{R}^{n-1})$ then implies

$$\int_0^\infty u_2^n(x', x_n)\mathrm{d}x_n = \mathcal{F}^{-1}\left(\int_0^\infty \hat{u}_2^n(\cdot, x_n)\mathrm{d}x_n\right)(x') = \frac{1}{4}R_j G_r(x'), \quad x' \in \mathbb{R}^{n-1}.$$

To show that $G_r \notin \mathcal{H}^1(\mathbb{R}^{n-1})$, it can be simply observed that $G_r \in \mathcal{S}(\mathbb{R}^{n-1})$, but $\widehat{G}_r \neq 0$. By the equivalent definition of $\mathcal{H}^1(\mathbb{R}^{n-1})$ we thus may choose $j \in \{1, \dots, n-1\}$, such that $R_j G_r \notin L^1(\mathbb{R}^{n-1})$. This leads to

$$\begin{aligned}\|u_2^n\|_{L^1(\mathbb{R}^n_+)} &\geq \int_{\mathbb{R}^{n-1}} \left|\int_0^\infty u_2^n(x', x_n)\mathrm{d}x_n\right|\mathrm{d}x' = \frac{1}{4}\int_{\mathbb{R}^{n-1}} |R_j G_r(x')|\mathrm{d}x' \\ &= \frac{1}{4}\|R_j G_r\|_{L^1(\mathbb{R}^{n-1})} = \infty\end{aligned}$$

and the assertion follows. □

Note that Theorem 3.3 and Theorem 3.5 imply Theorem 1.2.

Next we recall a scaling argument, which allows us to confine ourselves to the case $|\lambda| = 1$, $\lambda \in \Sigma_{\pi-\varphi_0}$. For the time being, we set

$$\omega(\lambda, \xi') := \omega(|\xi'|), \quad \hat{h}_{\hat{f}}(\lambda, \xi') := \hat{h}(\xi'),$$

and

$$m_\alpha(\lambda, \xi') := m_\lambda(|\xi'|), \quad \hat{\phi}_{\alpha,\hat{f}}(\lambda, \xi') := \hat{\phi}(\xi'), \quad \text{and} \quad \hat{u}_{\alpha,\hat{f}}(\lambda, \xi', x_n) := \hat{u}(\xi', x_n)$$

for $(\xi', x_n) \in \mathbb{R}^n_+$ and $\lambda \in \Sigma_{\pi-\varphi_0}$, where f is the right hand side of $(SRP)_{f,\lambda,\alpha}$ and ω, $h = (h', h^n)$, $\phi = (\phi', \phi^n)$, and u are defined as above. Further we define

$$\hat{f}_\lambda(\xi', x_n) := \hat{f}(|\lambda|^{1/2}\xi', |\lambda|^{-1/2}x_n), \quad \lambda \in \Sigma_{\pi-\varphi_0},$$

for a function $f \in L^q(\mathbb{R}^n_+)$. We may check that

$$\omega(\lambda, \xi') = |\lambda|^{1/2}\omega\left(\frac{\lambda}{|\lambda|}, |\lambda|^{-1/2}\xi'\right)$$

and

$$\hat{h}_{\hat{f}}(\lambda, \xi') \quad = \quad |\lambda|^{-1/2}\hat{h}_{\hat{f}_\lambda}\left(\frac{\lambda}{|\lambda|}, |\lambda|^{-1/2}\xi'\right),$$

as well as

$$m_\alpha(\lambda, \xi') = |\lambda|^{-1/2}m_\beta(\frac{\lambda}{|\lambda|}, |\lambda|^{-1/2}\xi')$$

with $\beta = \alpha|\lambda|^{-1/2}$. This implies that

$$\hat{\phi}_{\alpha,\hat{f}}(\lambda, \xi') = |\lambda|^{-1}\hat{\phi}_{\beta,\hat{f}_\lambda}\left(\frac{\lambda}{|\lambda|}, |\lambda|^{-1/2}\xi'\right)$$

and

$$\hat{u}_{\alpha,\hat{f}}(\lambda,\xi',x_n) = |\lambda|^{-1}\hat{u}_{\beta,\hat{f}_\lambda}\left(\frac{\lambda}{|\lambda|}, |\lambda|^{-1/2}\xi', |\lambda|^{1/2}x_n\right).$$

Since $\mathbb{R}^{n-1}$ and $\mathbb{R}_+$ are invariant under dilations, by the change of coordinates

$$\xi' \to |\lambda|^{1/2}\xi', \qquad x' \to |\lambda|^{-1/2}x', \qquad x^n \to |\lambda|^{-1/2}x^n,$$

we therefore may suppose $|\lambda| = 1$, $\lambda \in \Sigma_{\pi-\varphi_0}$. Let us mention that the dilated function u_{β,f_λ} now satisfies the boundary condition $\beta u'_{\beta,f_\lambda} - \partial_n u'_{\beta,f_\lambda} = 0$ on $\mathbb{R}^{n-1}$. In other words, we reduced $(SRP)_{f,\lambda,\alpha}$ with arbitrary $\lambda \in \Sigma_{\pi-\varphi_0}$ to the system $(SRP)_{f_\lambda,\mu,\beta}$ with $|\mu| = 1$ and $\beta = \alpha|\lambda|^{-1/2}$, i.e to equations with a boundary parameter depending on λ. However, this will not cause any trouble, in view of the fact that all the constants occuring in the estimates that we prove in the sequel do not depend on the boundary parameter. Therefore they also do not depend on λ, if we have $\beta = \alpha|\lambda|^{-1/2}$.

In the next lemma we provide with the help of Lemma 2.5 estimates for the remainder terms v and $R_j v$ of formula (15),(16). Recall from (17) that v is given by

$$\hat{v}(\xi',x_n) := M_{x_n,\lambda}(|\xi'|)\left[1 - (\omega(|\xi'|) + |\xi'|)m_\lambda(|\xi'|)\right]\hat{h}(\xi'), \quad (\xi',x_n) \in \mathbb{R}^n_+. \tag{34}$$

One problem here is to handle the unboundedness of R_j in $L^1(\mathbb{R}^{n-1})$ and $L^\infty(\mathbb{R}^{n-1})$. The main idea to overcome this difficulty is to write the symbol $i\xi_j/|\xi'|$ of R_j in the form

$$i\xi_j/|\xi'| = i\xi_j|\xi'|^3 \int_0^\infty \mathrm{e}^{-|\xi'|^4 r}\mathrm{d}r,$$

then to split the integral at $r = 1$, and use relation (24) for the part with $r \geq 1$.

Lemma 3.6 *Let $p \in \{1,\infty\}$, $\varphi_0 \in (0,\pi)$, and $\lambda \in \Sigma_{\pi-\varphi_0}$ with $|\lambda| = 1$. Furthermore, if $p = 1$, suppose that $f \in L^1(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$ and, if $p = \infty$, suppose that $f \in L^\infty(\mathbb{R}^n_+)$ so that $\operatorname{div} f = 0$ and that condition (24) for h is satisfied. Then, for $\delta \in [0,1)$ there are constants $C = C(\delta) > 0$ and $\sigma = \sigma(\delta) \in (0,1)$ such that*

(i) $\||\nabla'|^\delta v(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \leq C\|f\|_{L^\infty(\mathbb{R}^n_+)}, \quad x_n > 0,$

(ii) $\||\nabla'|^{1+\delta} v(\cdot,x_n)\|_{L^p(\mathbb{R}^{n-1})} \leq \dfrac{C}{x_n^{\sigma/p}(1+x_n)}\|f\|_{L^p(\mathbb{R}^n_+)}, \quad x_n > 0,$

(iii) $\||\nabla'|^\delta R_j v^n(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \leq C\|f\|_{L^\infty(\mathbb{R}^n_+)}, \quad x_n > 0,\ j = 1,\dots,n-1,$

(iv) $\||\nabla'|^{1+\delta} R_j v^n(\cdot,x_n)\|_{L^p(\mathbb{R}^{n-1})} \leq \dfrac{C}{x_n^{\sigma/p}(1+x_n)}\|f\|_{L^p(\mathbb{R}^n_+)}, \quad x_n > 0,\ j = 1,\dots,n-1,$

(v) $\||\nabla'|^\delta \partial_k v^n(\cdot,x_n)\|_{L^p(\mathbb{R}^{n-1})} \leq \dfrac{C}{x_n^{\sigma/p}(1+x_n)}\|f\|_{L^p(\mathbb{R}^n_+)}, \quad x_n > 0,\ k = 1,\dots,n,$

(vi) $\||\nabla'|^\delta R_j \partial_k v^n(\cdot,x_n)\|_{L^p(\mathbb{R}^{n-1})} \leq \dfrac{C}{x_n^{\sigma/p}(1+x_n)}\|f\|_{L^p(\mathbb{R}^n_+)}, \quad x_n > 0,\ j = 1,\dots,n-1,$
$k = 1,\dots,n.$

Proof. We start with (i). Note that $M_{x_n,\lambda}$ can be represented as

$$M_{x_n,\lambda}(z) = -\int_0^{x_n} e^{-\omega(z)(x_n-s)} e^{-zs} \mathrm{d}s.$$

Hence, for $\delta \in [0,1)$ and $\varepsilon = \varepsilon(\delta) \in (0, 1-\delta)$ we have that

$$\begin{aligned} |\xi'|^\delta \hat{v}(\xi', x_n) &= -\int_0^\infty \int_0^{x_n} \frac{|\xi'|^\delta}{\omega(|\xi'|)^{\delta+\varepsilon}} \mathrm{e}^{-\omega(|\xi'|)(x_n-\rho)} \mathrm{e}^{-|\xi'|\rho} \\ &\quad \cdot [1 - (\omega(|\xi'|) + |\xi'|) m_\lambda(|\xi'|)]\, \omega(|\xi'|)^{\delta+\varepsilon} \mathrm{e}^{-\omega(|\xi'|)s} \hat{f}(\xi', s) \mathrm{d}\rho \mathrm{d}s \end{aligned}$$

for $(\xi', x_n) \in \mathbb{R}^n_+$. The estimates in Lemma 2.5 (a), (c), (e), and (f) imply for the kernel of the integral that

$$|z^\varepsilon \frac{z^\delta}{\omega(z)^{\delta+\varepsilon}} \mathrm{e}^{-\omega(z)(x_n-\rho)} \mathrm{e}^{-z\rho} [1 - (\omega(z) + z) m_\lambda(z)]\, \omega(z)^{\delta+\varepsilon} \mathrm{e}^{-\omega(z)s}| \leq C \mathrm{e}^{-c_1(x_n-\rho)} \frac{\mathrm{e}^{-c_1 s}}{s^{\delta+\varepsilon}}$$

for all $z \in \Sigma_\varphi$, where $\varphi \in (0, \varphi_0/4)$. Hence we may conclude by Corollary 2.3

$$\begin{aligned} \||\nabla'|^\delta v(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} &\leq C \int_0^{x_n} \mathrm{e}^{-c_1(x_n-\rho)} \mathrm{d}\rho \int_0^\infty \frac{\mathrm{e}^{-c_1 s}}{s^{\delta+\varepsilon}} \|f(\cdot, s)\|_{L^\infty(\mathbb{R}^{n-1})} \mathrm{d}s \\ &\leq C(\delta) \|f\|_{L^\infty(\mathbb{R}^n_+)} \end{aligned}$$

for all $x_n > 0$.

Let p' be the Hölder conjugated exponent to $p \in \{1, \infty\}$. After applying Fourier transform the term in (ii) can be expressed as

$$\begin{aligned} |\xi'|^{1+\delta} \hat{v}(\xi', x_n) &= \int_0^\infty \frac{|\xi'|^\delta}{\omega(|\xi'|)^{\frac{\delta+\varepsilon}{p'}}} |\xi'| M_{x_n,\lambda}(|\xi'|) \\ &\quad \cdot [1 - (\omega(|\xi'|) + |\xi'|) m_\lambda(|\xi'|)]\, \omega(|\xi'|)^{\frac{\delta+\varepsilon}{p'}} \mathrm{e}^{-\omega(|\xi'|)s} \hat{f}(\xi', s) \mathrm{d}s \qquad (35) \end{aligned}$$

for $(\xi', x_n) \in \mathbb{R}^n_+$. Here Lemma 2.5 (a), (b), (c), (e), and (f) imply

$$\left| z^\varepsilon \frac{z^\delta}{\omega(z)^{\frac{\delta+\varepsilon}{p'}}} z M_{x_n,\lambda}(z) [1 - (\omega(z) + z) m_\lambda(z)]\, \omega(z)^{\frac{\delta+\varepsilon}{p'}} \mathrm{e}^{-\omega(z)s} \right| \leq \frac{C}{x_n^{(\delta+\varepsilon)/p}(1 + x_n)} \frac{\mathrm{e}^{-c_1 s}}{s^{(\delta+\varepsilon)/p'}}$$

for all $z \in \Sigma_\varphi$. Here we applied Lemma 2.5 (b) with $\rho = \delta + \varepsilon$ if $p = 1$ (observe that $\omega(z)^{\frac{\delta+\varepsilon}{p'}} \equiv 1$ in this case), and, if $p = \infty$, we applied Lemma 2.5 (a) on $\frac{z^{\delta+\varepsilon}}{\omega(z)^{\delta+\varepsilon}} = G_\lambda(z)^{\delta+\varepsilon}$ and Lemma 2.5 (b) with $\rho = 0$. Corollary 2.3 then yields

$$\begin{aligned} \||\nabla'|^\delta v(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})} &\leq \frac{C(\delta)}{x_n^{\sigma/p}(1 + x_n)} \int_0^\infty \frac{\mathrm{e}^{-c_1 s}}{s^{\sigma/p'}} \|f(\cdot, s)\|_{L^p(\mathbb{R}^{n-1})} \mathrm{d}s \\ &\leq \frac{C(\delta)}{x_n^{\sigma/p}(1 + x_n)} \|f\|_{L^p(\mathbb{R}^n_+)}, \quad x_n > 0, \end{aligned}$$

where $\sigma := \delta + \varepsilon \in (0, 1)$. This proves (ii).

Let $\delta_1 \in [0,1)$. To see (iii) observe that $\omega\hat{v}^n = -i\xi' \cdot \hat{v}'$ thanks to (24). Inserting $1 = |\xi'|^4 \int_0^\infty \mathrm{e}^{-|\xi'|^4 r}\mathrm{d}r$ in the formula for $\mathcal{F}|\nabla'|^{\delta_1} R_j v^n$ we obtain for $\gamma \in (0, 1-\delta_1)$ that

$$\begin{aligned}
|\xi'|^{\delta_1}\frac{i\xi_j}{|\xi'|}\hat{v}^n(\xi', x_n) &= i\xi_j|\xi'|^{3+\delta_1}\int_0^\infty \mathrm{e}^{-|\xi'|^4 r}\hat{v}^n(\xi', x_n)\mathrm{d}r \\
&= \int_0^1 i\xi_j|\xi'|^{3-\gamma}\mathrm{e}^{-|\xi'|^4 r}|\xi'|^{\delta_1+\gamma}\hat{v}^n(\xi', x_n)\mathrm{d}r \\
&\quad -\sum_{k=1}^{n-1}\int_1^\infty i\xi_j i\xi_k|\xi'|^3\mathrm{e}^{-|\xi'|^4 r}\frac{|\xi'|^{\delta_1}}{\omega(|\xi'|)}\hat{v}^k(\xi', x_n)\mathrm{d}r.
\end{aligned}$$

Now (8) and (9) (with $m = 2$) imply

$$\begin{aligned}
\||\nabla'|^{\delta_1}R_j v^n(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} &\leq C\int_0^1 r^{\frac{\gamma}{4}-1}\||\nabla'|^{\delta_1+\gamma}v^n(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}r \\
&\quad +\sum_{k=1}^{n-1}\int_1^\infty r^{-5/4}\|\omega(|\nabla'|)^{-1}|\nabla'|^{\delta_1}v^k(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}r \\
&\leq C(\delta)\|f\|_{L^\infty(\mathbb{R}^n_+)}
\end{aligned}$$

for $x_n > 0$, where we used the boundedness of the operator $\omega(|\nabla'|)^{-1}$ and applied (i) once with $\delta = \delta_1 + \gamma$ and once with $\delta = \delta_1$.

We will not carry out the proof of (iv), since the term in (iv) can be reduced to the one in (ii) exactly in the same way as (iii) is reduced to (i) .

For $k = 1, \ldots, n-1$ assertion (v) is an immediate consequence of (iv), if we take into account that $|\nabla'|^\delta \partial_k v^n = |\nabla'|^{1+\delta}R_k v^n$. If $k = n$, note that

$$\partial_n M_{x_n,\lambda}(|\xi'|) = -|\xi'|M_{x_n,\lambda}(|\xi'|) - \mathrm{e}^{-\omega(|\xi'|)x_n}, \quad (\xi', x_n) \in \mathbb{R}^n_+.$$

Hence, with $w(\cdot, x_n) := |\nabla'|^\delta \mathrm{e}^{-\omega(|\nabla'|)x_n}\left[1 - (\omega(|\nabla'|) + |\nabla'|)m_\lambda(|\nabla'|)\right]h^n$ we get that

$$|\nabla'|^\delta \partial_n v^n(\cdot, x_n) = -w(\cdot, x_n) - |\nabla'|^{1+\delta}v^n(\cdot, x_n), \; x_n > 0. \tag{36}$$

Similarly to (35) we write the first addend w in the form

$$\begin{aligned}
\hat{w}(\xi', x_n) &= \int_0^\infty \frac{|\xi'|^\delta}{\omega(|\xi'|)^{\delta+\varepsilon}}\omega(|\xi'|)^{\frac{\delta+\varepsilon}{p}}\mathrm{e}^{-\omega(|\xi'|)x_n/2}\mathrm{e}^{-\omega(|\xi'|)x_n/2} \\
&\quad \cdot\left[1 - (\omega(|\xi'|) + |\xi'|)m_\lambda(|\xi'|)\right]\omega(|\xi'|)^{\frac{\delta+\varepsilon}{p'}}\mathrm{e}^{-\omega(|\xi'|)s}\hat{f}^n(\xi', s)\mathrm{d}s
\end{aligned}$$

and obtain analogously to the proof of (ii)

$$\begin{aligned}
\|w(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})} &\leq \frac{C(\delta)\mathrm{e}^{-c_1 x_n/2}}{x_n^{\sigma/p}}\int_0^\infty \frac{\mathrm{e}^{-c_1 s}}{s^{\sigma/p'}}\|f^n(\cdot, s)\|_{L^p(\mathbb{R}^{n-1})}\mathrm{d}s \\
&\leq \frac{C(\delta)}{x_n^{\sigma/p}(1+x_n)}\|f\|_{L^p(\mathbb{R}^n_+)}, \quad x_n > 0,
\end{aligned}$$

where $\sigma = \delta + \varepsilon$. Relation (v) now follows, if we apply (ii) on the second addend of (36).

We omit the proof of (vi) since the same method we used in (iii) applies to (vi). □

Although the Stokes flow u of $(SRP)_{f,\lambda,\alpha}$ does not belong to $L^1(\mathbb{R}^n_+)$, it is known that in the case of Dirichlet boundary conditions the gradient of the velocity of the instationary Stokes equations does, according to a result of Giga, Matsui, and Shimizu (see [GMS99]). As a consequence of the theorem below we will see that this result generalizes to the case of Robin boundary conditions.

Theorem 3.7 *Let $\alpha \in [0,\infty]$ and $\varphi_0 \in (0,\pi)$. Then there exists a constant $C = C(\varphi_0) > 0$ such that the Stokes flow u of $(SRP)_{f,\lambda,\alpha}$ satisfies*

$$\|\nabla u\|_1 \leq \frac{C}{\sqrt{|\lambda|}}\|f\|_1 \tag{37}$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$ and $f \in L^1_\sigma(\mathbb{R}^n_+)$.

Proof. By definition we may assume $f \in L^1(\mathbb{R}^n_+) \cap D$. Again we consider representation (15) and (16) for the components of the Stokes flow u. The estimate for $\nabla(\lambda - \Delta_D)^{-1} f$ is a consequence of (10). Due to Lemma 3.6 (v) and (vi) with $\delta = 0$ we have that

$$\|\nabla v\|_{L^1(\mathbb{R}^n_+)} \leq C \int_0^\infty \frac{1}{x_n^\sigma(1+x_n)}\|f\|_{L^1(\mathbb{R}^n_+)}\mathrm{d}x_n \leq C\|f\|_{L^1(\mathbb{R}^n_+)}$$

and

$$\|\nabla R' v\|_{L^1(\mathbb{R}^n_+)} \leq C\|f\|_{L^1(\mathbb{R}^n_+)}$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$, $|\lambda| = 1$. So, it remains to prove the corresponding estimate for the term $u_2' = \mathrm{e}^{-\omega(|\nabla'|)\cdot}\phi'$.

Let us remark that, if $\alpha = \infty$, then m_λ vanishes, which implies that the term $\mathrm{e}^{-\omega(|\nabla'|)(\cdot)}\phi'$ does not occur in the formula for u. Thus, by rescaling, the claim follows for the case of Dirichlet boundary conditions. As mentioned before, this case is also contained in [GMS99] (see also [SS01]).

If $\alpha \in [0,\infty)$, then in view of (18) the term $i\xi_j\hat{u}_2'$ is represented as

$$i\xi_j\hat{u}_2'(\xi',x_n) = \frac{i\xi_j\mathrm{e}^{-\omega(|\xi'|)x_n}}{\omega(|\xi'|)+\alpha}\left(\hat{h}'(\xi') + \alpha\frac{i\xi'}{|\xi'|}m_\lambda(|\xi'|)\hat{h}^n(\xi')\right), \quad \xi' \in \mathbb{R}^{n-1},\ x_n > 0,$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$, $|\lambda| = 1$, and $j = 1,\dots,n-1$. To evaluate this term we use the same methods as in the proof of Lemma 3.6. By inserting $1 = |\xi'|^4\int_0^\infty \mathrm{e}^{-|\xi'|^4 r}\mathrm{d}r$ we obtain for the first addend of $i\xi_j\hat{u}_2'$ that

$$\begin{aligned}\frac{i\xi_j\mathrm{e}^{-\omega(|\xi'|)x_n}}{\omega(|\xi'|)+\alpha}\hat{h}'(\xi') \quad &= \quad \int_0^\infty\int_0^1 |\xi'|^{5/2} i\xi_j \mathrm{e}^{-|\xi'|^4 r}\frac{|\xi'|^{3/2}}{\omega(|\xi'|)+\alpha}\mathrm{e}^{-\omega(|\xi'|)(x_n+s)}\hat{f}'(\xi',s)\mathrm{d}r\mathrm{d}s \\ &\quad + \int_0^\infty\int_1^\infty |\xi'|^4 i\xi_j\mathrm{e}^{-|\xi'|^4 r}\frac{1}{\omega(|\xi'|)+\alpha}\mathrm{e}^{-\omega(|\xi'|)(x_n+s)}\hat{f}'(\xi',s)\mathrm{d}r\mathrm{d}s.\end{aligned}$$

Let $\varphi \in (0,\varphi_0/4)$. Applying Remark 2.4 (c) and Lemma 2.5 (a) and (f) we obtain

$$\begin{aligned}&\left|z^{1/4}\frac{z^{3/2}}{(\omega(z)+\alpha)}\mathrm{e}^{-\omega(z)(x_n+s)}\right| = \\ = \quad &\left|\left(z^{1/4}\frac{z^{3/2}}{(\omega(z)+\alpha)\omega(z)^{3/4}}\right)\left(\omega(z)^{3/4}\mathrm{e}^{-\omega(z)x_n}\right)\mathrm{e}^{-\omega(z)s}\right| \\ \leq \quad &\frac{C}{(1+\alpha)x_n^{3/4}}\mathrm{e}^{-c_1(x_n+s)}, \quad x_n, s > 0,\ z \in \Sigma_\varphi,\end{aligned}$$

where we estimated the terms in brackets separately, and

$$\left|\frac{z^{3/4}}{(\omega(z)+\alpha)}\mathrm{e}^{-\omega(z)(x_n+s)}\right| \le \frac{C}{(1+\alpha)x_n^{3/4}}\mathrm{e}^{-c_1(x_n+s)}, \quad x_n, s>0,\ z\in\Sigma_\varphi.$$

Then, Corollary 2.3 in combination with (8) and (9) leads to

$$\begin{aligned}
&\|\partial_j(\omega(|\nabla'|)+\alpha)^{-1}\mathrm{e}^{-\omega(|\nabla'|)x_n}h'\|_{L^1(\mathbb{R}^{n-1})} \le \\
\le\ & \frac{C}{(1+\alpha)}\int_0^\infty\int_0^1 r^{-7/8}x_n^{-3/4}\mathrm{e}^{-c_1(x_n+s)}\|f'(\cdot,s)\|_{L^1(\mathbb{R}^{n-1})}\mathrm{d}r\mathrm{d}s \\
& + \frac{C}{(1+\alpha)}\int_0^\infty\int_1^\infty r^{-5/4}\mathrm{e}^{-c_1(x_n+s)}\|f'(\cdot,s)\|_{L^1(\mathbb{R}^{n-1})}\mathrm{d}r\mathrm{d}s \\
\le\ & C(x_n^{-3/4}+1)\mathrm{e}^{-c_1x_n}\|f\|_{L^1(\mathbb{R}^n_+)}, \quad x_n>0,
\end{aligned}$$

for $j=1,\ldots,n-1$. Consequently we obtain for the first addend of $\nabla' u_2'$

$$\|\nabla'(\omega(|\nabla'|)+\alpha)^{-1}\mathrm{e}^{-\omega(|\nabla'|)\cdot}h'\|_{L^1(\mathbb{R}^n_+)} \le C\|f\|_{L^1(\mathbb{R}^n_+)}$$

for $\lambda\in\Sigma_{\pi-\varphi_0}$, $|\lambda|=1$. Completely analogous calculations lead to the same estimate for the second addend of $\nabla' u_2'$, i.e.,

$$\|\nabla'\mathrm{e}^{-\omega(|\nabla'|)\cdot}(\omega(|\nabla'|)+\alpha)^{-1}\alpha R' m_\lambda h^n\|_{L^1(\mathbb{R}^n_+)} \le C\|f\|_{L^1(\mathbb{R}^n_+)}, \quad \lambda\in\Sigma_{\pi-\varphi_0},\ |\lambda|=1.$$

We also shall omit the details of the estimate for the derivative $\partial_n u_2'$, since this is very similar to the one for $\nabla' u_2'$. Summarizing, we have that

$$\|\nabla u\|_1 \le C\|f\|_1, \quad \lambda\in\Sigma_{\pi-\varphi_0},\ |\lambda|=1. \tag{38}$$

Now, if $\lambda\in\Sigma_{\pi-\varphi_0}$, the claim is proved for the scaled flow $u_{\beta,f_\lambda}(\frac{\lambda}{|\lambda|},\cdot)$ with $\beta=\alpha|\lambda|^{-1/2}$. Because the constant C in (38) does not depend on β (therefore also not on λ) rescaling yields the assertion. □

Remark 3.8 The proof of the above theorem shows that by our methods we can even obtain more regularity for u. Indeed, for $\delta_1\in(0,2),\delta_2\in[0,1)$ we can prove an estimate such as

$$|\lambda|^{(2-\delta_1)/2}\||\nabla'|^{\delta_1}u\|_1 + |\lambda|^{(1-\delta_2)/2}\||\nabla'|^{\delta_2}\nabla u\|_1 \le C(\delta_1,\delta_2)\|f\|_1$$

for $\lambda\in\Sigma_{\pi-\varphi_0}$ and $f\in L^1_\sigma(\mathbb{R}^n_+)$. But, of course the constant $C(\delta_1,\delta_2)$ blows up if $\delta_1\to 0$, $\delta_1\to 2$, or $\delta_2\to 1$.

Corollary 3.9 *Let $\alpha\in[0,\infty]$. The Stokes semigroup $(\mathrm{e}^{-tA_\alpha})_{t\ge 0}$ on $L^q_\sigma(\mathbb{R}^n_+)$, $1<q<\infty$, satisfies the estimate*

$$\|\nabla\mathrm{e}^{-tA_\alpha}f\|_1 \le Ct^{-1/2}\|f\|_1, \quad t>0,\ f\in L^1(\mathbb{R}^n_+)\cap L^q_\sigma(\mathbb{R}^n_+).$$

Proof. We can argue analogously to the proof of (32). For $f\in L^1(\mathbb{R}^n_+)\cap L^q_\sigma(\mathbb{R}^n_+)$ and $\lambda\in\rho(A_\alpha)$ the Stokes flow u is given by $u_f(-\lambda)=-(\lambda-A_\alpha)^{-1}f$. Hence we have $\mathrm{e}^{-tA_\alpha}f=-\int_\Gamma \mathrm{e}^{-\lambda t}u_f(-\lambda)\mathrm{d}\lambda$, and the assertion follows from estimate (37). □

4 The Stokes operator in solenoidal L^∞-spaces

Contrary to the non generation result in $L^1(\mathbb{R}^n_+)$ we will prove in this section the validity of generation results for the Stokes operator with Robin boundary conditions in solenoidal subspaces of $L^\infty(\mathbb{R}^n_+)$. Again by utilizing the estimates in Lemma 3.6 and Lemma 2.5 we will start by proving in Theorem 4.1 resolvent estimates in $L^\infty_\sigma(\mathbb{R}^n_+)$. A crucial point then will be to give a definition of the Stokes operator. Due to the lack of the boundedness of the Helmholtz projection in $L^\infty(\mathbb{R}^n_+)$ it is not possible to define it in the standard way used in reflexive L^q-spaces. Instead, we will give a definition by employing pseudo resolvent methods. To be more precise, we show that the solution u of the Stokes resolvent problem corresponds to a real resolvent. The existence of the resolvent then allows us to define the Stokes operator. By the resolvent estimates in Theorem 4.1 this leads to the generation result in $L^\infty_\sigma(\mathbb{R}^n_+)$, that is Theorem 4.3. Finally, we will turn our attention to the subspaces of continuous functions $C_{0,\sigma}(\mathbb{R}^n_+)$ and $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$. By restricting the Stokes operator in $L^\infty_\sigma(\mathbb{R}^n_+)$ to these spaces the generation result is clear, but here we can also prove that the semigroups are strongly continuous.

First let us justify the definition of $L^\infty_\sigma(\mathbb{R}^n_+)$ given in (3). It is well-known (see e.g. [Gal98]) that we have the following characterizations of $L^q_\sigma(\mathbb{R}^n_+)$ for $1 < q < \infty$:

$$\begin{aligned} L^q_\sigma(\mathbb{R}^n_+) &= \overline{C^\infty_{c,\sigma}(\mathbb{R}^n_+)}^{\|\cdot\|_q} \\ &= \{u \in L^q(\mathbb{R}^n_+) : \ \operatorname{div} u = 0, \ u^n \restriction_{\partial\mathbb{R}^n_+} = 0\} \\ &= \{u \in L^q(\mathbb{R}^n_+) : \ (u, \nabla p) = 0, \ p \in \widehat{W}^{1,q'}(\mathbb{R}^n_+)\}, \end{aligned}$$

where the last one follows by means of the Helmholtz decomposition. To the author the third one seems to be the most appropriate characterization for the definition of $L^\infty_\sigma(\mathbb{R}^n_+)$, since the first one can only be used to define a subspace of continuous functions, whereas the problem with the second one is to give a sense to the trace $u^n \restriction_{\mathbb{R}^{n-1}}$. Thus we define $L^\infty_\sigma(\mathbb{R}^n_+)$ as in (3) that is

$$L^\infty_\sigma(\mathbb{R}^n_+) := \{u \in L^\infty(\mathbb{R}^n_+) : \ (u, \nabla p) = 0, \ p \in \widehat{W}^{1,1}(\mathbb{R}^n_+)\}.$$

Theorem 4.1 *Let $\alpha \in [0, \infty]$ and $\varphi_0 \in (0, \pi)$. For each $f \in L^\infty_\sigma(\mathbb{R}^n_+)$ and $\lambda \in \Sigma_{\pi-\varphi_0}$ there is a unique solution (u, p) of $(SRP)_{f,\lambda,\alpha}$ such that $u = u_f(\lambda) \in C^1_b(\mathbb{R}^n_+)$. Moreover, there is a constant $C = C(\varphi_0) > 0$ such that u satisfies*

$$|\lambda|\|u\|_\infty + \sqrt{|\lambda|}\|\nabla u\|_\infty \le C\|f\|_\infty, \quad f \in L^\infty_\sigma(\mathbb{R}^n_+), \ \lambda \in \Sigma_{\pi-\varphi_0}. \tag{39}$$

Proof. For the proof we also intend to use representation (15), (16) for $u = (u', u^n)$. In the derivation of these formulas in [Saa06, Section 4] we make use of the compatibility conditions $\operatorname{div} f = 0$ and $f^n \restriction_{\partial\mathbb{R}^n_+} = 0$, particularly in the verification of relation (24) for the function $h = (h', h^n)$. By an inspection of (24) one can see that the condition $(f, \nabla p) = 0$, $p \in \widehat{W}^{1,1}(\mathbb{R}^n_+)$, is still sufficient to obtain this relation, and therefore the formulas (15) and (16) are valid for all $f \in L^\infty_\sigma(\mathbb{R}^n_+)$.

Here it also suffices to focus on the second addend u'_2 in (15) since (10) implies (39) for the term $(\lambda - \Delta_D)^{-1} f$, whereas for v^n and $R'v^n$ estimate (39) follows from Lemma 3.6 (i), (iii), (v), and (vi) by setting $\delta = 0$ and $p = \infty$. We sketch a derivation of (39) for u'_2. The calculations are very similar to the proof of Theorem 3.7.

Observe that at this point the proof is completed for the case of Dirchlet boundary conditions, because u'_2 vanishes for $\alpha = \infty$. As mentioned in the introduction the inequality $|\lambda|\|u\|_\infty \le \|f\|_\infty$

is already proved in [DHP01]. However, their proof is based on kernel estimates and their result does not include higher regularity. Estimates for Dirichlet boundary conditions and first order derivatives are given in [Shi99] and [SS01] based on the formula of Ukai [Uka87].

If $\alpha \in [0,\infty)$ and $f \in L^\infty_\sigma(\mathbb{R}^n_+)$, by (18) we have that

$$\hat{u}_2'(\xi', x_n) = \frac{\mathrm{e}^{-\omega(|\xi'|)x_n}}{\omega(|\xi'|)+\alpha}\left(\hat{h}'(\xi') + \alpha\frac{i\xi'}{|\xi'|}m_\lambda(|\xi'|)\hat{h}^n(\xi')\right), \quad \xi' \in \mathbb{R}^{n-1},\ x_n > 0,$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$, $|\lambda| = 1$. To the first term of this formula we may apply Remark 2.4 (b), Lemma 2.5 (f), and Corollary 2.3 to the result

$$\begin{aligned}
&\|\mathrm{e}^{-\omega(|\nabla'|)\cdot}(\omega(|\nabla'|)+\alpha)^{-1}h'\|_{L^\infty(\mathbb{R}^n_+)} \leq \\
\leq\ & \sup_{x_n>0}\int_0^\infty \|\mathrm{e}^{-\omega(|\nabla'|)(x_n+s)}(\omega(|\nabla'|)+\alpha)^{-1}f'(\cdot,s)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}s \\
\leq\ & \frac{C}{1+\alpha}\sup_{x_n>0}\mathrm{e}^{-c_1x_n}\int_0^\infty \mathrm{e}^{-c_1 s}\|f'(\cdot,s)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}s \\
\leq\ & C\|f\|_{L^\infty(\mathbb{R}^n_+)}, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ |\lambda| = 1.
\end{aligned}$$

The second addend we write with the help of (24) as

$$\begin{aligned}
&\frac{\mathrm{e}^{-\omega(|\xi'|)x_n}}{\omega(|\xi'|)+\alpha}\alpha m_\lambda(|\xi'|)\frac{i\xi'}{|\xi'|}\hat{h}^n(\xi') = \\
=\ & \int_0^\infty\int_0^1 |\xi'|^{5/2}i\xi'\mathrm{e}^{-|\xi'|^4 r}\frac{|\xi'|^{1/2}}{\omega(|\xi'|)+\alpha}\alpha m_\lambda(|\xi'|)\mathrm{e}^{-\omega(|\xi'|)(x_n+s)}\hat{f}^n(\xi',s)\mathrm{d}r\mathrm{d}s \\
&-\int_0^\infty\int_1^\infty |\xi'|^3 i\xi' i\xi'\cdot\mathrm{e}^{-|\xi'|^4 r}\alpha m_\lambda(|\xi'|)\frac{\mathrm{e}^{-\omega(|\xi'|)(x_n+s)}}{\omega(|\xi'|)(\omega(|\xi'|)+\alpha)}\hat{f}'(\xi',s)\mathrm{d}r\mathrm{d}s.
\end{aligned}$$

Observe that by Remark 2.4 (b) and (c) we have that $|\nabla'|/(\omega(|\nabla'|)+\alpha) \in \mathcal{L}(L^\infty(\mathbb{R}^{n-1}))$. This fact, relations (8), (9), Lemma 2.5 (d), (f), and Corollary 2.3 then lead to the estimate

$$\begin{aligned}
&\|\mathrm{e}^{-\omega(|\nabla'|)\cdot}(\omega(|\nabla'|)+\alpha)^{-1}\alpha m_\lambda(|\nabla'|)R'h^n\|_{L^\infty(\mathbb{R}^n_+)} \leq \\
\leq\ & \frac{C}{1+\alpha}\sup_{x_n>0}\left[\int_0^\infty\int_0^1 r^{-7/8}\mathrm{e}^{-c_1(x_n+s)}\|f^n(\cdot,s)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}r\mathrm{d}s\right. \\
&\left.+\int_0^\infty\int_1^\infty r^{-5/4}\mathrm{e}^{-c_1(x_n+s)}\|f'(\cdot,s)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}r\mathrm{d}s\right] \\
\leq\ & C\|f\|_{L^\infty(\mathbb{R}^n_+)}, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ |\lambda| = 1.
\end{aligned}$$

The estimate for $\nabla u_2'$ in $L^\infty(\mathbb{R}^n_+)$ is analogous to the one in the proof of Theorem 3.7 in the $L^1(\mathbb{R}^n_+)$-norm with the only difference that the roles of the terms $\mathrm{e}^{-\omega x_n}$ and $\mathrm{e}^{-\omega s}$ interchange. More precisely, for $\delta \in (0,1)$ we have to estimate in the present situation expressions of the form $\omega^\delta \mathrm{e}^{-\omega(x_n+s)}$ by

$$|\omega(z)^\delta \mathrm{e}^{-\omega(z)(x_n+s)}| \leq C\frac{\mathrm{e}^{-c_1(x_n+s)}}{s^\delta} \quad x_n, s > 0,\ z \in \Sigma_\varphi.$$

This is due to the fact that we may have a singularity in s but not in x_n, since we estimate in the L^∞-norm. Therefore, estimate (39) is proved for all $\lambda \in \Sigma_{\pi-\varphi_0}$ with $|\lambda| = 1$. Then rescaling yields (39) for all $\lambda \in \Sigma_{\pi-\varphi_0}$.

It remains to prove that $u \in C_b^1(\mathbb{R}^n_+)$. For, observe that

$$\Delta p = \operatorname{div}(f - (\lambda - \Delta)u) = 0,$$

which gives $p \in C^\infty(\mathbb{R}^n_+)$. Then, since $u, f \in L^\infty(\mathbb{R}^n_+)$, we obtain that

$$\Delta u = \lambda u + \nabla p - f \in L^q(K)$$

for some $q > n$ and each smooth compact $K \subseteq \mathbb{R}^n_+$. By elliptic regularity we deduce $u \in W^{2,q}(K)$ and thanks to Sobolev's embedding $u \in C_b^1(K)$ for each smooth compact $K \subseteq \mathbb{R}^n_+$, consequently $u \in C^1(\mathbb{R}^n_+)$. In view of (39) this yields $u \in C_b^1(\mathbb{R}^n_+)$ and the proof is completed. □

Remark 4.2 A similar statement as in Remark 3.8 is valid in $L^\infty_\sigma(\mathbb{R}^n_+)$. By checking the details of the above proof, one will realize that by our methods for each $\delta_1 \in [0, 2), \delta_2 \in [0, 1)$ we also can get an estimate as

$$|\lambda|^{(2-\delta_1)/2} \| |\nabla'|^{\delta_1} u \|_\infty + |\lambda|^{(1-\delta_2)/2} \| |\nabla'|^{\delta_2} \nabla u \|_\infty \le C(\delta_1, \delta_2) \|f\|_\infty$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$ and $f \in L^\infty_\sigma(\mathbb{R}^n_+)$. However, the constant $C(\delta_1, \delta_2)$ blows up if $\delta_1 \to 2$ or $\delta_2 \to 1$. But this is reasonable, since we do not expect that $u \in W^{2,\infty}(\mathbb{R}^n_+)$.

Due to Theorem 4.1 above, for each $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ the mapping $R(\lambda) : L^\infty_\sigma(\mathbb{R}^n_+) \to C_b^1(\mathbb{R}^n_+)$, $R(\lambda)f := u_f(\lambda)$ is well-defined and continuous. Since $C_b^1(\mathbb{R}^n_+) \hookrightarrow C_b(\overline{\mathbb{R}^n_+})$, we may apply the Gauss Theorem to obtain for $u = u_f(\lambda)$

$$(u, \nabla \phi) = \int_{\mathbb{R}^{n-1}} \phi u^n \mathrm{d}x' - (\operatorname{div} u, \phi) = 0$$

for all $\phi \in C_c^\infty(\overline{\mathbb{R}^n_+})$, which shows that we even have $R(\lambda) : L^\infty_\sigma(\mathbb{R}^n_+) \to C_b^1(\mathbb{R}^n_+) \cap L^\infty_\sigma(\mathbb{R}^n_+)$. In order to establish a Stokes operator in $L^\infty_\sigma(\mathbb{R}^n_+)$ we now prove $R(\lambda)$ to be a resolvent. To this end let $f \in L^\infty_\sigma(\mathbb{R}^n_+)$ and set

$$w(\lambda, \mu)f := (R(\lambda) - R(\mu) - (\mu - \lambda)R(\lambda)R(\mu)) f.$$

It is clear that the resolvent identity holds, if we can show

$$(w(\lambda, \mu)f, \varphi) = 0, \quad \varphi \in C_c^\infty(\mathbb{R}^n_+).$$

Pick $\varphi \in C_c^\infty(\mathbb{R}^n_+)$ and let $\{\phi_j\}_{j\in\mathbb{Z}}$ be a Littlewood-Paley decomposition of the unity, i.e., $\sum_{j\in\mathbb{Z}} \widehat{\phi_j}(\xi) = 1$ for $\xi \in \mathbb{R}^n \setminus \{0\}$, where $\widehat{\phi_j}(\xi) := \widehat{\phi_0}(2^{-j}|\xi|)$ and $0 \ne \phi_0(|\cdot|) \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\widehat{\phi_0} \subseteq \{s \in \mathbb{R} : 1/2 \le s \le 2\}$. Then we have that

$$r\varphi_k \to \varphi \quad \text{weakly in } L^1(\mathbb{R}^n_+)$$

for $\varphi_k := \sum_{j=-k}^k \phi_j * E\varphi$. Here r is the restriction to $\mathbb{R}^n_+$ and $Ef := (E^+ f', E^- f^n)$ with

$$E^\pm f(x', x_n) := \begin{cases} f(x', x_n), & x_n > 0, \\ \pm f(x', -x_n), & x_n < 0. \end{cases}$$

Also, note that it is well-known that the Helmholtz projection $P_{\mathbb{R}^n_+}$ in $\mathbb{R}^n_+$ can be represented by

$$P_{\mathbb{R}^n_+} = r P_{\mathbb{R}^n} E,$$

where $P_{\mathbb{R}^n} = \mathcal{F}^{-1}\left[I + \frac{i\xi i\xi^T}{|\xi|^2}\right]\mathcal{F}$ denotes the Helmholtz projection in $\mathbb{R}^n$. A crucial observation now is that $Er\varphi_k = \varphi_k$, since by definition φ_k' is even and φ_k^n is odd with respect to the normal component x_n. This implies that $P_{\mathbb{R}^n_+}\varphi_k = rP_{\mathbb{R}^n}\varphi_k$. Also, note that $\frac{i\xi i\xi^T}{|\xi|^2}\widehat{\varphi}_k \in \mathcal{F}L^1(\mathbb{R}^n)$ thanks to the fact that $\widehat{\varphi}_k$ has compact support away from 0. By the definition of $L^\infty_\sigma(\mathbb{R}^n_+)$ this yields

$$(w(\lambda,\mu)f, (I - P_{\mathbb{R}^n_+})r\varphi_k) = (w(\lambda,\mu)f, \nabla r\mathcal{F}^{-1}[i\xi/|\xi|^2]\cdot\widehat{\varphi}_k) = 0.$$

Next observe that $|\nabla'|^{-\delta}P_{\mathbb{R}^n_+}r\varphi_k = rP_{\mathbb{R}^n}|\nabla'|^{-\delta}\varphi_k \in L^q_\sigma(\mathbb{R}^n_+)$ for $\delta \in [0,1]$ and $1 \le q < \infty$. Thanks to Remark 3.8 this implies that

$$w(\lambda,\mu)P_{\mathbb{R}^n_+}r\varphi_k = |\nabla'|^{1/2}w(\lambda,\mu)(|\nabla'|^{-1/2}P_{\mathbb{R}^n_+}r\varphi_k) \in L^q_\sigma(\mathbb{R}^n_+), \quad 1 \le q < \infty.$$

Moreover, since according to the results in [Saa06] we already know that $R(\lambda) : L^q_\sigma(\mathbb{R}^n_+) \to L^q_\sigma(\mathbb{R}^n_+)$ is a resolvent for $1 < q < \infty$, it follows that $w(\lambda,\mu)P_{\mathbb{R}^n_+}r\varphi_k = 0$. This results in

$$\begin{aligned}(w(\lambda,\mu)f,\varphi) &= \lim_{k\to\infty}(w(\lambda,\mu)f, r\varphi_k)\\ &= \lim_{k\to\infty}\Big[(w(\lambda,\mu)f,(I-P_{\mathbb{R}^n_+})r\varphi_k) + (w(\lambda,\mu)f, P_{\mathbb{R}^n_+}r\varphi_k)\Big]\\ &= \lim_{k\to\infty}(f, w(\lambda,\mu)P_{\mathbb{R}^n_+}r\varphi_k)\\ &= 0.\end{aligned}$$

Consequently, $R(\lambda) : L^\infty_\sigma(\mathbb{R}^n_+) \to L^\infty_\sigma(\mathbb{R}^n_+)$ is a pseudo resolvent. To see the injectivity of $R(\lambda)$ let $f \in L^\infty_\sigma(\mathbb{R}^n_+)$ with $R(\lambda)f = 0$. Since $R(\lambda)f = u_f(\lambda)$ solves $(SRP)_{f,\lambda,\alpha}$ we deduce

$$\nabla p = (\lambda - \Delta)u_f(\lambda) + \nabla p = f.$$

Moreover, by $\Delta p = \operatorname{div} f = 0$ in the sense of distributions, p and also $f = \nabla p$ are harmonic in $\mathbb{R}^n_+$. By the Schwarz reflection principle the odd extension of $\partial_n p$ is a bounded and harmonic function on $\mathbb{R}^n$, hence $f^n = \partial_n p = 0$ in view of $\partial_n p \restriction_{\partial\mathbb{R}^n_+} = f^n \restriction_{\partial\mathbb{R}^n_+} = 0$. Observe that we require p to satisfy formula (25). Since this formula does only depend on f^n, but not on the other components of f, we deduce $p = 0$ and therefore also that $f = \nabla p = 0$.

This implies $R(\lambda)$, $\lambda \in \mathbb{C}\setminus(-\infty,0]$ to be a resolvent, hence there exists a closed operator $A_{\alpha,\infty}$ in $L^\infty_\sigma(\mathbb{R}^n_+)$ such that

$$(\lambda + A_{\alpha,\infty})^{-1} = R(\lambda), \quad -\lambda \in \rho(A_{\alpha,\infty}) = \mathbb{C}\setminus[0,\infty).$$

We call $A_\alpha := A_{\alpha,\infty}$ the *Stokes operator in* $L^\infty_\sigma(\mathbb{R}^n_+)$. Theorem 4.1 now implies the following result (where the proof of the gradient estimates is analogous to Theorem 3.3).

Theorem 4.3 *Let $\alpha \in [0,\infty]$. The operator $-A_\alpha$ is the generator of a bounded holomorphic semigroup on $L^\infty_\sigma(\mathbb{R}^n_+)$ (which is not strongly continuous). The semigroup $(\mathrm{e}^{-tA_\alpha})_{t\ge0}$ also satisfies the gradient estimates*

$$\|\nabla \mathrm{e}^{-tA_\alpha}f\|_\infty \le Ct^{-1/2}\|f\|_\infty, \quad t > 0,\ f \in L^\infty_\sigma(\mathbb{R}^n_+). \tag{40}$$

Next we define corresponding Stokes operators in spaces of continuous functions. In the case of Dirichlet boundary conditions we set

$$C_{0,\sigma}(\mathbb{R}^n_+) := \overline{C^\infty_{c,\sigma}(\mathbb{R}^n_+)}^{\|\cdot\|_\infty}. \tag{41}$$

For $-\lambda \in \mathbb{C} \setminus [0,\infty)$ and $f \in C^\infty_{c,\sigma}(\mathbb{R}^n_+)$ we have $R(\lambda)f = u_f(\lambda) \in W^{1,q}_0(\mathbb{R}^n_+) \cap L^q_\sigma(\mathbb{R}^n_+)$, for each $1 < q < \infty$. Since

$$W^{1,q}_0(\mathbb{R}^n_+) \cap L^q_\sigma(\mathbb{R}^n_+) = W^{1,q}_{0,\sigma}(\mathbb{R}^n_+) := \overline{C^\infty_{c,\sigma}(\mathbb{R}^n_+)}^{\|\cdot\|_{1,q}}$$

for $1 < q < \infty$ (see [Gal98] Chapter III.4.) the function $u_f(\lambda)$ can be approximated by a sequence $(u_k) \subseteq C^\infty_{c,\sigma}(\mathbb{R}^n_+)$ in $W^{1,q}(\mathbb{R}^n_+)$. The imbedding $W^{1,q}(\mathbb{R}^n_+) \hookrightarrow C_b(\overline{\mathbb{R}^n_+})$ for $q > n$ now implies

$$\|u_k - u_f\|_\infty \le C\|u_k - u_f\|_{1,q} \to 0, \quad \text{for} \quad k \to \infty,$$

consequently $R(\lambda)f \in C_{0,\sigma}(\mathbb{R}^n_+)$. By the continuity of $R(\lambda)$ we obtain $R(\lambda)(C_{0,\sigma}(\mathbb{R}^n_+)) \subseteq C_{0,\sigma}(\mathbb{R}^n_+)$, and since $C_{0,\sigma}(\mathbb{R}^n_+) \subseteq L^\infty_\sigma(\mathbb{R}^n_+)$, the mapping $R(\lambda) : C_{0,\sigma}(\mathbb{R}^n_+) \to C_{0,\sigma}(\mathbb{R}^n_+)$ is a resolvent. This results in the existence of a closed operator $A_{C_{0,\sigma}}$ in $C_{0,\sigma}(\mathbb{R}^n_+)$ such that

$$(\lambda + A_{C_{0,\sigma}})^{-1} = R(\lambda), \quad -\lambda \in \rho(A_{C_{0,\sigma}}) = \mathbb{C} \setminus [0,\infty),$$

which we call *Stokes operator in* $C_{0,\sigma}(\mathbb{R}^n_+)$. Now $u \in C^\infty_{c,\sigma}(\mathbb{R}^n_+)$ yields $f := (1 + A_2)u \in C^\infty_{c,\sigma}(\mathbb{R}^n_+)$, where A_2 denotes the Stokes operator with Dirichlet boundary conditions in $L^2_\sigma(\mathbb{R}^n_+)$. Hence

$$u = (1 + A_2)^{-1}f = u_f(\lambda) = (1 + A_{C_{0,\sigma}})^{-1}f \in D(A_{C_{0,\sigma}}),$$

and we see, that $A_{C_{0,\sigma}}$ is densely defined. Thus, we have proved the following result, which partly can be found in [DHP01].

Theorem 4.4 *The Stokes operator $-A_{C_{0,\sigma}}$ is the generator of a bounded, holomorphic C_0-semigroup on $C_{0,\sigma}(\mathbb{R}^n_+)$. The gradient estimates (40) are also valid for the semigroup generated by $-A_{C_{0,\sigma}}$.*

A further space of continuous solenoidal functions in the case that $\alpha \in [0,\infty)$ is

$$\mathrm{BUC}_\sigma(\mathbb{R}^n_+) := \{u \in \mathrm{BUC}(\mathbb{R}^n_+) : \ \operatorname{div} u = 0, \ u^n \restriction_{\partial\mathbb{R}^n_+} = 0\},$$

where $\mathrm{BUC}(\mathbb{R}^n_+)$ denotes the space of all bounded uniformly continuous functions in $\mathbb{R}^n_+$. In the case of Dirichlet boundary conditions, i.e., if $\alpha = \infty$, we set

$$\mathrm{BUC}_{\sigma,D}(\mathbb{R}^n_+) := \{u \in \mathrm{BUC}(\mathbb{R}^n_+) : \ \operatorname{div} u = 0, \ u \restriction_{\partial\mathbb{R}^n_+} = 0\}.$$

This is motivated by the following fact. Suppose there exists a stronly continuous semigroup $\{e^{-tA_{\mathrm{BUC}_{\sigma,D}}}\}_{t\ge0}$. Then $e^{-tA_{\mathrm{BUC}_{\sigma,D}}}f \to f$ in $\mathrm{BUC}(\mathbb{R}^n_+)$ for $t \to 0$ and $e^{-tA_{\mathrm{BUC}_{\sigma,D}}}f \restriction_{\partial\mathbb{R}^n_+} = 0$, $t > 0$, imply that $f \restriction_{\partial\mathbb{R}^n_+} = 0$.

Now let $X_\sigma \in \{\mathrm{BUC}_{\sigma,D}(\mathbb{R}^n_+), \mathrm{BUC}_\sigma(\mathbb{R}^n_+)\}$. We will prove that the Stokes operator is the generator of a strongly continuous semigroup on X_σ. For this purpose we need

Lemma 4.5 *Let $\alpha \in [0,\infty]$, $f \in X_\sigma$, and $v_{\alpha,f}(\lambda) = v$ as in (34). Then for each $\delta \in [0,1)$ and $j \in \{1,\dots,n-1\}$ we have that*

$$\lambda^{1-\frac{\delta}{2}}|\nabla'|^\delta v^n_{\alpha,f}(\lambda) \to 0 \quad \textit{and} \quad \lambda R_j v^n_{\alpha,f}(\lambda) \to 0 \quad \textit{in} \quad L^\infty(\mathbb{R}^n_+)$$

if $\lambda \to \infty$, $\lambda > 0$.

Proof. By the scaling argument as described in the previous section and the proof of Lemma 3.6 (i) we deduce

$$\begin{aligned}
\|\lambda^{1-\frac{\delta}{2}}|\nabla'|^{\delta} v^n_{\alpha,f}(\cdot,\cdot,\lambda)\|_\infty &= \lambda^{(n-1)/2} \sup_{x\in\mathbb{R}^n_+} ||\nabla'|^{\delta} v^n_{\beta,f_\lambda}(x',x_n,1)| \\
&\le C(\delta)\lambda^{(n-1)/2} \sup_{x_n>0}(1-\mathrm{e}^{-c_1 x_n}) \int_0^\infty \frac{\mathrm{e}^{-c_1 s}}{s^{\frac{\delta}{2}+\varepsilon}} \|f^n_\lambda(\cdot,s)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}s \\
&= C(\delta)\int_0^\infty \frac{\mathrm{e}^{-c_1 s}}{s^{\frac{\delta}{2}+\varepsilon}} \|f^n(\cdot,s\lambda^{-1/2})\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}s. \qquad (42)
\end{aligned}$$

Here we choose $\delta/2, \varepsilon \in (0,1)$ such that $\frac{\delta}{2}+\varepsilon \in (0,1)$, whereas $\beta = \alpha\lambda^{-1/2}$ is the scaled boundary parameter and f_λ is the scaled function f. Now $f \in X_\sigma$ gives

$$\sup_{x'\in\mathbb{R}^{n-1}} |f^n(x',s\lambda^{-1/2})| = \sup_{x'\in\mathbb{R}^{n-1}} |f^n(x',s\lambda^{-1/2}) - f^n(x',0)| \longrightarrow 0,$$

if $\lambda \to \infty$. Furthermore,

$$\left|\frac{\mathrm{e}^{-c_1 s}}{s^{\frac{\delta}{2}+\varepsilon}} f(x',\lambda^{-1/2}s)\right| \le \frac{\mathrm{e}^{-c_1 s}}{s^{\frac{\delta}{2}+\varepsilon}}\|f^n\|_\infty, \quad \lambda>0,\ (x',s)\in\mathbb{R}^n_+.$$

Thus, the function $\frac{\mathrm{e}^{-c_1 s}}{s^{\frac{\delta}{2}+\varepsilon}}\|f^n\|_\infty$ is an integrable majorant for the λ-dependent integrand of the integral in (42). Lebesgue's dominated convergence theorem then implies that

$$\lim_{\lambda\to\infty} \|\lambda^{1-\frac{\delta}{2}}|\nabla'|^{\delta} v^n_{\alpha,f}(\lambda)\|_\infty = 0.$$

Similar calculations as in the proof of Lemma 3.6 (iii) lead to the estimate

$$\begin{aligned}
\|\lambda R_j v^n_{\alpha,f}(\cdot,\cdot,\lambda)\|_\infty &= \lambda^{(n-1)/2}\sup_{x\in\mathbb{R}^n_+} |R_j v_{\beta,f_\lambda}(x',x_n,1)| \\
&\le C\lambda^{(n-1)/2}\sup_{x_n>0}\Bigg(\int_0^R r^{\frac{\gamma}{4}-1}\||\nabla'|^{\gamma} v^n_{\beta,f_\lambda}(\cdot,x_n,1)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}r \\
&\quad + \sum_{k=1}^{n-1}\int_R^\infty r^{-5/4}\|v^k_{\beta,f_\lambda}(\cdot,x_n,1)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}r\Bigg)
\end{aligned}$$

for some $\gamma \in (0,1)$, where we splitted the integral at $r = R$, instead of $r = 1$ as before. Applying Lemma 3.6 (i) to the second addend, we can continue the calculation obtaining

$$\begin{aligned}
\|\lambda R_j v^n_{\alpha,f}(\cdot,\cdot,\lambda)\|_\infty &\le C\lambda^{(n-1)/2}\left(R^{\gamma/4}\||\nabla'|^{\gamma} v^n_{\beta,f_\lambda}(\cdot,x_n,1)\|_{L^\infty(\mathbb{R}^{n-1})} + R^{-1/4}\|f_\lambda\|_\infty\right) \\
&= C\left(R^{\gamma/4}\|\lambda^{1-\frac{\gamma}{2}}|\nabla'|^{\gamma} v^n_{\alpha,f}(\cdot,x_n,\lambda)\|_{L^\infty(\mathbb{R}^{n-1})} + R^{-1/4}\|f\|_\infty\right).
\end{aligned}$$

Choosing $R > \left(\frac{\varepsilon}{2\|f\|_\infty}\right)^{-4}$ and afterwards λ big enough we can achieve

$$\|\lambda R_j v^n_{\alpha,f}(\cdot,\cdot,\lambda)\|_\infty < \varepsilon$$

for arbitrary $\varepsilon > 0$. This yields the assertion. □

In order to obtain densely defined generators we prove that for $f \in X_\sigma$

$$\lambda R(\lambda) f = \lambda u_{\alpha,f}(\lambda) \to f \tag{43}$$

in $L^\infty(\mathbb{R}^n_+)$ if $\lambda \to \infty$. We first consider the case $\alpha \in [0,\infty)$. Observe that the solution $u_{\alpha,f}(\lambda)$ can also be represented by the formula

$$\begin{aligned} u'_{\alpha,f}(\lambda) &= (\lambda - \Delta_N)^{-1} f' - R' v^n_{\alpha,f}(\lambda) + \mathrm{e}^{-\omega(|\nabla'|)\cdot}\phi' - \omega(|\nabla'|)^{-1}\mathrm{e}^{-\omega(|\nabla'|)\cdot}h', \\ u^n_{\alpha,f}(\lambda) &= (\lambda - \Delta_D)^{-1} f^n + v^n_{\alpha,f}(\lambda), \end{aligned}$$

which we will use to prove (43). Since $C^\infty_b(\mathbb{R}^n) \subseteq D(\Delta_{\mathbb{R}^n})$ lies dense in $\mathrm{BUC}(\mathbb{R}^n)$, the Laplacian $\Delta_{\mathbb{R}^n}$ is the generator of a bounded holomorphic C_0-semigroup on $\mathrm{BUC}(\mathbb{R}^n)$. This yields $\lambda(\lambda - \Delta_{\mathbb{R}^n})^{-1} f \to f$ for all $f \in \mathrm{BUC}(\mathbb{R}^n)$. For $f \in \mathrm{BUC}_\sigma(\mathbb{R}^n_+)$ we set $\tilde{f} := (E^+ f', E^- f^n)$, where $E^\pm$ is the extension operator defined in (14). Then obviously $\tilde{f} \in \mathrm{BUC}(\mathbb{R}^n)$. Hence representations (12) and (13) imply that

$$\begin{aligned} \lim_{\lambda\to\infty} \lambda(\lambda - \Delta_N)^{-1} f' &= f' \quad \text{and} \\ \lim_{\lambda\to\infty} \lambda(\lambda - \Delta_D)^{-1} f^n &= f^n, \end{aligned}$$

respectively. Since $\lambda v^n_{\alpha,f}(\lambda)$ and $\lambda R' v^n_{\alpha,f}(\lambda)$ tend to zero if $\lambda \to \infty$ according to Lemma 4.5, it remains to show that

$$\begin{aligned} &\lambda\left(\mathrm{e}^{-\omega(|\nabla'|)\cdot}\phi' - \omega(|\nabla'|)^{-1}\mathrm{e}^{-\omega(|\nabla'|)\cdot}h'\right) = \\ &= \lambda \mathrm{e}^{-\omega(|\nabla'|)\cdot}\alpha(\omega(|\nabla'|) + \alpha)^{-1}\left(-\omega(|\nabla'|)^{-1}h' + R' m_\lambda(|\nabla'|)h^n\right) \longrightarrow 0 \end{aligned}$$

as $\lambda \to \infty$, $\lambda > 0$. By virtue of Lemma 2.5, Corollary 2.3, and since $\alpha < \infty$ the latter term can be estimated by

$$\|\lambda \mathrm{e}^{-\omega(|\nabla'|)\cdot}\alpha(\omega(|\nabla'|) + \alpha)^{-1}\left(-\omega(|\nabla'|)^{-1}h' + R' m_\lambda(|\nabla'|)h^n\right)\|_\infty \le C\lambda^{(\varepsilon-1)/2}\|f\|_\infty$$

for some $\varepsilon \in (0,1)$. This proves (43) for $\alpha \in [0,\infty)$. In the case of Dirichlet boundary conditions the solution $u_{\infty,f}(\lambda)$ is given by

$$\begin{aligned} u'_{\infty,f}(\lambda) &= (\lambda - \Delta_D)^{-1} f' - R' v^n_{\infty,f}(\lambda), \\ u^n_{\infty,f}(\lambda) &= (\lambda - \Delta_D)^{-1} f^n + v^n_{\infty,f}(\lambda). \end{aligned}$$

Thus (43) follows directly from $\lambda(\lambda - \Delta_D) f \to f$, valid for all $f \in \mathrm{BUC}_{\sigma,D}(\mathbb{R}^n_+)$, and Lemma 4.5.

In view of the inclusions $X_\sigma \subseteq L^\infty_\sigma(\mathbb{R}^n_+)$ and $C^1_b(\mathbb{R}^n_+) \subseteq \mathrm{BUC}(\mathbb{R}^n_+)$ we have that $R(\lambda)(X_\sigma) \subseteq X_\sigma$ for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. This implies $R(\lambda) : X_\sigma \to X_\sigma$ to be a resolvent, hence the existence of a closed operator A_{α,X_σ} such that

$$(\lambda + A_{\alpha,X_\sigma})^{-1} = R(\lambda), \quad -\lambda \in \rho(A_{\alpha,X_\sigma}) = \mathbb{C} \setminus [0,\infty).$$

A_{α,X_σ} is also densely defined, due to the validity of (43) for $f \in X_\sigma$. We call A_{α,X_σ} the Stokes operator in X_σ. Summarizing the just proved facts gives

Theorem 4.6 *The operator $-A_{\alpha,X_\sigma}$ is the generator of a bounded holomorphic C_0-semigroup on X_σ. The gradient estimates in (40) are also valid for the semigroup generated by $-A_{\alpha,X_\sigma}$.*

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References

[BG92] F. Brochard and P. G. de Gennes. Shear-dependent slippage at a polymer/solid interface. *Langmuir*, 8:3033–3037, 1992.

[BM88] W. Borchers and T. Miyakawa. L^2 decay for the Navier-Stokes flow in halfspaces. *Math. Ann.*, 282:139–155, 1988.

[DHP01] W. Desch, M. Hieber, and J. Prüss. L^p-theory of the Stokes equation in a half-space. *J. Evol. Equ.*, 1:115–142, 2001.

[DV79] E.B. Dussan V. Spreading of liquids on solid surfaces - static and dynamic contact lines. *Ann. Rev. Fluid Mech.*, 11:371–400, 1979.

[FS72] C. Fefferman and E.M. Stein. H^p spaces of several variables. *Acta Math.*, 129:137–193, 1972.

[Gal98] G.P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations I.* Springer Verlag, 2nd edition, 1998.

[GMS99] Y. Giga, S. Matsui, and Y. Shimizu. On estimates in Hardy spaces for the Stokes flow in a half space. *Math. Z.*, 231(2):383–396, 1999.

[IM] K. Inui and S. Matsui. An operatorial approach to the Navier-Stokes equations in a half-space with non-decaying initial data. In preparation.

[Koz98] H. Kozono. L^1-solutions of the Navier-Stokes equations in exterior domains. *Math. Ann.*, 312:319–340, 1998.

[LBS05] E. Lauga, M.P. Brenner, and H.A. Stone. Microfluids: The no-slip boundary condition. 2005.
http://web.mit.edu/lauga/www/references/LaugaBrennerStone_review_slip_05_pdf.

[McC81] M. McCracken. The resolvent problem for the Stokes equation on halfspaces in L_p. *SIAM J. Math. Anal.*, 12:201–228, 1981.

[Mun89] E. P. Muntz. Rarefied-gas dynamics. *Ann. Rev. Fluid Mech.*, 21:387–417, 1989.

[Saa03] J. Saal. *Robin Boundary Conditions and Bounded H^∞-Calculus for the Stokes Operator.* PhD thesis, TU Darmstadt. Logos Verlag, Berlin, 2003.

[Saa06] J. Saal. Stokes and Navier-Stokes equations with Robin boundary conditions in a half-space. *J. Math. Fluid Mech.*, 8:211–241, 2006.

[Shi99] Y. Shimizu. L^∞-estimate of first-order space derivatives of Stokes flow in a half space. *Funkcialaj Ekvacioj*, 42:291–309, 1999.

[Sol77] V.A. Solonnikov. Estimates for solutions of nonstationary Navier-Stokes equations. *J. Sov. Math.*, 8:467–529, 1977.

[Sol03] V.A. Solonnikov. On nonstationary Stokes problem and Navier-Stokes problem in a half-space with initial data nondecreasing at infinity. *Journal of Mathematical Sciences*, 114:1726–1740, 2003.

[SS01] Y. Shibata and S. Shimizu. A decay property of the Fourier transform and its application to the Stokes problem. *J. Math. Fluid Mech.*, 3:213–230, 2001.

[Ste93] E.M. Stein. *Harmonic Analysis*. Princeton Univ. Press, 1993.

[Tre73] W. Trebels. *Multipliers for (C, α)-bounded Fourier Expansions in Banach Spaces and Approximation Theory*, volume 329 of *Lecture Notes in Math.* Springer Verlag, 1973.

[Uka87] S. Ukai. A solution formula for the Stokes equation in $\mathbf{R}^n_+$. *Comm. Pure Appl. Math.*, 40(5):611–621, 1987.