THE STOKES OPERATOR WITH ROBIN BOUNDARY CONDITIONS IN $L^\infty_\sigma(\mathbb{R}^n_+)$

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We study the initial-boundary value problem for the Stokes equations with Robin boundary conditions in the half-space $\mathbb{R}^n_+$. It is proved that the associated Stokes operator is the generator of a bounded holomorphic semigroup on $L^\infty_\sigma(\mathbb{R}^n_+)$, which is even strongly continuous on $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$. By a counterexample we will show that this assertion is wrong on $L^1_\sigma(\mathbb{R}^n_+)$, except for the special case of Neumann boundary conditions.

1. Introduction and Main Results

Here we consider the Stokes equations

$$(SE)_{\alpha,f}\begin{cases} \partial_t v - \Delta v + \nabla\pi &= 0 \quad \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ \mathrm{div} v &= 0 \quad \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ v(0) &= f \quad \text{in } \mathbb{R}^n_+, \\ T_\alpha v &= 0 \quad \text{in } \partial\mathbb{R}^n_+ \times (0,\infty), \end{cases} \tag{1}$$

with velocity field v and pressure π for some initial value f on the half-space $\mathbb{R}^n_+ := \{x = (x', x_n) = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n : x_n > 0\}$. In the literature this problem is mainly considered on Lebesgue spaces L^q with $1 < q < \infty$. The particular matter here is that we also consider initial values that belong to solenoidal subspaces of $L^1(\mathbb{R}^n_+)$ and $L^\infty(\mathbb{R}^n_+)$. More precisely we define

$$\begin{aligned} L^1_\sigma(\mathbb{R}^n_+) &:= \overline{L^2_\sigma(\mathbb{R}^n_+) \cap L^1(\mathbb{R}^n_+)}^{L^1}, \\ L^\infty_\sigma(\mathbb{R}^n_+) &:= \{v \in L^\infty(\mathbb{R}^n_+) : (v, \nabla\varphi) = 0, \varphi \in \widehat{W}^{1,1}(\mathbb{R}^n_+)\}, \quad \text{and} \\ \mathrm{BUC}_\sigma(\mathbb{R}^n_+) &:= \{u \in \mathrm{BUC}(\mathbb{R}^n_+) : \mathrm{div} u = 0,\ u^n|_{\partial\mathbb{R}^n_+} = 0\}, \end{aligned}$$

where $\widehat{W}^{k,q}(\mathbb{R}^n_+)$, $1 \le q \le \infty$, denotes the usual homogeneous Sobolev space of order $k \in \mathbb{N} \cup \{0\}$ and $\mathrm{BUC}(\mathbb{R}^n_+)$ denotes the space of all bounded uniformly continuous functions. Furthermore, $L^2_\sigma(\mathbb{R}^n_+) := \overline{C^\infty_{c,\sigma}(\mathbb{R}^n_+)}^{L^2}$, where $C^\infty_{c,\sigma}(\mathbb{R}^n_+) := \{u \in C^\infty_c(\mathbb{R}^n_+) : \text{div} u = 0\}$. A further particular matter are the boundary conditions. Here we deal with Robin boundary conditions, i.e. the trace operator T_α is given by

$$T_\alpha v := \begin{pmatrix} \alpha v' - \partial_n v' \\ v^n \end{pmatrix}\Bigg|_{\partial\mathbb{R}^n_+} \tag{2}$$

for $\alpha \in [0,\infty]$[a]. Here $v' = (v^1, \ldots, v^{n-1})$ and v^n, respectively, denote tangential and normal component of v at the boundary $\mathbb{R}^{n-1}$. Observe, that the case $\alpha = 0$ or $\alpha = \infty$ corresponds to the classical Neumann or Dirichlet boundary conditions respectively.

Up to now for the problem considered in the spaces $L^1_\sigma(\mathbb{R}^n_+)$ and $L^\infty_\sigma(\mathbb{R}^n_+)$ there are only results available for the special case of Dirichlet boundary conditions. For instance in [2] is proved, that

$$\|\nabla v\|_{L^1(\mathbb{R}^n_+)} \le Ct^{-1/2}\|f\|_{L^1(\mathbb{R}^n_+)}, \quad t > 0. \tag{3}$$

And in [6] is proved, that the same estimate is valid in $L^\infty(\mathbb{R}^n_+)$. Inequality (3) represents the typical gradient estimates, which can be obtained, if the Stokes operator is the generator of a bounded holomorphic semigroup on the underlying Banach space, as it is the case e.g. on $L^q_\sigma(\mathbb{R}^n_+) := \overline{C^\infty_{c,\sigma}(\mathbb{R}^n_+)}^{L^q}$ for $1 < q < \infty$. Nevertheless, according to a result in [1] the Stokes flow u of the solution (u, p) of the corresponding Stokes resolvent problem

$$(SRP)_{f,\lambda,\alpha} \begin{cases} (\lambda - \Delta)u + \nabla p &= f \quad \text{in } \mathbb{R}^n_+, \\ \text{div} u &= 0 \quad \text{in } \mathbb{R}^n_+, \\ T_\alpha u &= 0 \quad \text{in } \mathbb{R}^{n-1}, \end{cases} \tag{4}$$

for the special case $\alpha = \infty$ and for $\lambda > 0$ in general does not belong to $L^1(\mathbb{R}^n_+)$ for $f \in L^1_\sigma(\mathbb{R}^n_+)$. Thus in the case of Dirichlet boundary conditions there exists no Stokes semigroup on this solenoidal subspace of $L^1(\mathbb{R}^n_+)$. Contrary to that result, in [1] it is also proved that the Stokes operator with Dirichlet boundary conditions is the generator of a bounded holomorphic semigroup on $L^\infty_\sigma(\mathbb{R}^n_+)$, which is even strongly continuous on $\text{BUC}_\sigma(\mathbb{R}^n_+)$.

This surprising behavior of the Stokes operator with Dirichlet boundary conditions in the spaces $L^1(\mathbb{R}^n_+)$ and $L^\infty(\mathbb{R}^n_+)$ attracted the author of the present work to investigate problem (1) with Robin boundary conditions.

[a] The case $\alpha = \infty$ is to understand in the following sense: divide the first line in (2) by α and let $\alpha \to \infty$.

The results stated here can be regarded as generalizations to Robin boundary conditions of the above cited results for Dirichlet boundary conditions. Our main results read as follows

Theorem 1.1. *Let* $n \in \mathbb{N}$ *and* $\varphi_0 \in (0, \pi)$. *Then there exists a constant* $C = C(n, \varphi_0) > 0$ *such that the Stokes flow* u *of problem (4) satisfies*

$$\|\nabla u\|_1 \leq C|\lambda|^{-1/2}\|f\|_1$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$, $\alpha \in [0, \infty]$, *and* $f \in L^1_\sigma(\mathbb{R}^n_+)$. *Here* Σ_θ *denotes the complex sector* $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$ *for* $\theta \in (0, \pi)$.

Theorem 1.2. *Let* $n \in \mathbb{N}$, $\alpha \in (0, \infty]$. *For* $\lambda \in \mathbb{R}_+$ *there is an* $f \in L^1(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$ *such that* $u \notin L^1(\mathbb{R}^n_+)$, *where* (u, p) *is the solution of problem (4).*

Theorem 1.3. *Let* $n \in \mathbb{N}$ *and* $\varphi_0 \in (0, \pi)$. *For each* $f \in L^\infty_\sigma(\mathbb{R}^n_+)$ *and* $\lambda \in \Sigma_{\pi-\varphi_0}$ *there is a unique solution* (u, p) *of (4) such that* $u = u_f(\lambda) \in C^1_b(\mathbb{R}^n_+)$. *Moreover, there is a constant* $C = C(n, \varphi_0) > 0$ *such that* u *satisfies*

$$|\lambda|\|u\|_\infty + \sqrt{|\lambda|}\|\nabla u\|_\infty \leq C\|f\|_\infty \tag{5}$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$, $\alpha \in [0, \infty]$, *and* $f \in L^\infty_\sigma(\mathbb{R}^n_+)$.

As a consequence of Theorem 1.1 we obtain the gradient estimates (3) in $L^1_\sigma(\mathbb{R}^n_+)$ also for the Stokes flow v of the initial value problem $(SE)_{\alpha,f}$. This follows easily from the representation $v(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}u(\lambda)\mathrm{d}\lambda, \quad t > 0$, valid for $f \in L^2_\sigma(\mathbb{R}^n_+) \cap L^1(\mathbb{R}^n_+)$, where Γ is a proper path in the complex plane.

From Theorem 1.2 we deduce that there exists no generation result in $L^1_\sigma(\mathbb{R}^n_+)$ for Robin boundary conditions if $\alpha \in (0, \infty]$. In the case of Neumann boundary conditions, i.e. if $\alpha = 0$, it can be easily read of the solution formula stated below that the Stokes flow u of problem (4) is the solution of a resolvent problem for the Laplacien with Neumann boundary conditions. Hence the generation result in this case stays valid, since it is well known for that operator. This implies

Corollary 1.1. *Let* $n \in \mathbb{N}$ *and* $\alpha \in [0, \infty]$. *There exists a Stokes semigroup on* $L^1_\sigma(\mathbb{R}^n_+)$ *if and only if* $\alpha = 0$. *In that case the semigroup is even bounded, holomorphic, and strongly continuous.*

Finally, Theorem 1.3 is the main ingredient for proving

Theorem 1.4. *The Stokes operator is the generator of a bounded holomorphic semigroup on $L^\infty_\sigma(\mathbb{R}^n_+)$, which is even strongly continuous on* $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$.

Moreover, Theorem 1.3 also implies the gradient estimates (3) for the Stokes flow v of problem (1) to be valid in $L^\infty_\sigma(\mathbb{R}^n_+)$.

Among others, the above results can be found in [3] and [5]. Let us remark, that [3] (see also [4]) also includes the treatment of the problem in L^q-spaces for $1 < q < \infty$. For those values of q much more is true than a generation result similar to Theorem 1.4, only. There it is proved, that the Stokes operator with Robin boundary conditions admits a bounded H^∞-calculus on $L^q_\sigma(\mathbb{R}^n_+)$, $1 < q < \infty$, for all boundary parameters $\alpha \in [0, \infty]$.

2. Sketches of the Proofs

We discuss briefly the ideas of the proofs of our results, an refer to [3] and [5] for the details. The proofs are based on an explicit solution formula for problem (4). The construction is as follows. By applying Laplace transform and Fourier transform (here denoted by F) w.r.t. the first $n-1$ spatial variables x', we are left with an ODE in the last spatial component x_n. Via fundamental solution these equations can be solved, which yields for $u = (u', u^n)$

$$\begin{aligned} u' &= (\lambda - \Delta_D)^{-1} f' - R' v_1 + v_2, \\ u^n &= (\lambda - \Delta_D)^{-1} f^n + v_1. \end{aligned}$$

Here Δ_D denotes the Dirichlet Laplacien, $R' = F^{-1}[\frac{i\xi'}{|\xi'|}]F$ the Riesz operator, whereas the Fourier transforms of v_1 and v_2 are given by

$$\begin{aligned} \hat{v}_1(\xi', x_n) = \int_0^\infty & \frac{\mathrm{e}^{-\sqrt{\lambda + |\xi'|^2} x_n} - \mathrm{e}^{-|\xi'| x_n}}{\sqrt{\lambda + |\xi'|^2} - |\xi'|} \frac{\alpha}{\sqrt{\lambda + |\xi'|^2} + |\xi'| + \alpha} \\ & \cdot \mathrm{e}^{-\sqrt{\lambda + |\xi'|^2} s} \hat{f}^n(\xi', s) \mathrm{d}s, \quad (\xi', x_n) \in \mathbb{R}^n_+, \\ \hat{v}_2(\xi', x_n) = \int_0^\infty & \frac{\mathrm{e}^{-\sqrt{\lambda + |\xi'|^2}(x_n + s)}}{\sqrt{\lambda + |\xi'|^2} + \alpha} \big(\hat{f}'(\xi', s) \\ & + \frac{i\xi'}{|\xi'|} \frac{\alpha}{\sqrt{\lambda + |\xi'|^2} + |\xi'| + \alpha} \hat{f}^n(\xi', s) \big) \mathrm{d}s, \quad (\xi', x_n) \in \mathbb{R}^n_+. \end{aligned}$$

A corresponding formula can be obtained for the pressure p. To estimate $F^{-1}\hat{v}_1$ and $F^{-1}\hat{v}_2$ we make use of the rotation invariance in $|\xi'|$ of the formulas. If $1 < q < \infty$ this allowed us in [3] and [5] to apply the bounded

H^∞-calculus of the operator $(-\Delta')^{1/2} = F^{-1}[|\xi'|]F$ on $L^q(\mathbb{R}^{n-1})$. This H^∞-calculus is not valid for $q \in \{1,\infty\}$. But here we can provide an appropriate substitute by using the following result of Trebels, cf. [7]. Denote by $BC_0^{k+1}([0,\infty))$ the space of all functions $m : [0,\infty) \to \mathbb{C}$ with bounded and continuous derivatives up to order $k+1$ satisfying $\lim_{t\to\infty} m(t) = 0$.

Lemma 2.1. *Let $N \in \mathbb{N}$, $k > N/2$, and $[t \mapsto m(t)] \in BC_0^{k+1}([0,\infty))$. Morover, let $\|m\|_M := \frac{1}{\Gamma(k+1)} \int_0^\infty t^k |m^{k+1}(t)| \mathrm{d}t < \infty$. Then $[\xi \mapsto m(|\xi|)] \in FL^1(\mathbb{R}^N) = \{\hat{f} : f \in L^1(\mathbb{R}^N)\}$ and $\|F^{-1} m(|\cdot|)\|_{L^1(\mathbb{R}^N)} \leq C\|m\|_M$.*

As a consequence we deduce for holomorphic rotation invariant multipliers

Proposition 2.1. *Let $N \in \mathbb{N}$ and $m \in H^\infty(\Sigma_\vartheta \cup \{0\})$ satisfying $|z^\varepsilon m(z)| \leq K$, $z \in \Sigma_\vartheta$, for some constants $\varepsilon \in (0,1)$, $C_0 > 0$. Then $[\xi \mapsto m(|\xi|)] \in FL^1(\mathbb{R}^N)$ and $\|F^{-1} m(|\cdot|)\|_{L^1(\mathbb{R}^N)} \leq CK$.*

Another problem in estimating v_1 and v_2 in $L^1(\mathbb{R}^n_+)$ and $L^\infty(\mathbb{R}^n_+)$ is to circumvent the unboundedness of the Riesz operators $R_j = F^{-1}[\frac{i\xi_j}{|\xi'|}]F$, $j = 1,\ldots,n-1$, in these spaces. This can be done by rephrasing the formula, such that the Riesz operators vanish. The main idea for this purpose is to replace $1/|\xi'|$ by $|\xi'|^3 \int_0^\infty \mathrm{e}^{-|\xi'|^4 r} \mathrm{d}r$. Then, by applying Proposition 2.1 and well known results for the multiplier $\xi' \mapsto \mathrm{e}^{-|\xi'|^4 r}$, $r > 0$, to the rephrased formulas we can prove the following estimates for the remainder terms v_1, $R'v_1$, and v_2 in the solution formula for u.

Lemma 2.2. *Let $n \in \mathbb{N}$, $\varphi_0 \in (0,\pi)$, $q \in \{1,\infty\}$ and $w \in \{v_1, R'v_1, v_2\}$. For each $\delta \in [0,1)$ there exists a $C_\delta > 0$, such that*

$$\|w(\cdot,x_n,\lambda)\|_{L^q(\mathbb{R}^{n-1})} \leq \frac{C_\delta}{(1+x_n)^\delta} \|f\|_{L^q(\mathbb{R}^n_+)},$$
$$\|\nabla w(\cdot,x_n,\lambda)\|_{L^q(\mathbb{R}^{n-1})} \leq \frac{C_\delta}{(1+x_n)^{1+\delta}} \|f\|_{L^q(\mathbb{R}^n_+)}$$

for $x_n > 0$, $\alpha \in [0,\infty]$ and $\lambda \in \Sigma_{\pi-\varphi_0}$ with $|\lambda| = 1$.

By a scaling argument in λ these inequalities immediately imply Theorem 1.1 and Theorem 1.3 since the resolvent estimates for the operator Δ_D are well known.

For defining the Stokes operator in $L^\infty_\sigma(\mathbb{R}^n_+)$ we then show that

$$R(\lambda) : L^\infty_\sigma(\mathbb{R}^n_+) \to L^\infty_\sigma(\mathbb{R}^n_+),\ R(\lambda)f := u(\lambda), \quad \lambda \in \Sigma_{\pi-\varphi_0},$$

is injective and satisfies the resolvent identity. This follows by direct methods, using the explicit representation for $u(\lambda)$. This implies $R(\lambda)$ to be a resolvent, hence there is a unique operator $A_{L^\infty_\sigma}$, which we call Stokes operator in $L^\infty_\sigma(\mathbb{R}^n_+)$, such that $(\lambda + A_{L^\infty_\sigma})^{-1} = R(\lambda)$, $\lambda \in \Sigma_{\pi-\varphi_0}$. The resolvent estimates in Theorem 1.3 then imply the generation result in $L^\infty_\sigma(\mathbb{R}^n_+)$ stated in Theorem 1.4. If we additionally assume $f \in \mathrm{BUC}_\sigma(\mathbb{R}^n_+)$ we can show, again by direct methods, that $R(\lambda)f \to f$ in $L^\infty(\mathbb{R}^n_+)$ for $\lambda \to \infty$. Hence $A_{L^\infty_\sigma}|_{\mathrm{BUC}_\sigma(\mathbb{R}^n_+)}$ is even the generator of a strongly continuous bounded holomorphic semigroup on $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$.

To see Theorem 1.2 we give a counterexample of an initial value $f \in L^1(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$, such that $\|v_1\|_1 = \infty$. The n-th component of this counterexample is given by

$$\hat{f}^n(\xi', x_n) = i\xi_j \frac{\sqrt{\lambda + |\xi'|^2} + |\xi'| + \alpha}{\alpha} x_n^2 (\lambda + |\xi'|^2)^2 \mathrm{e}^{-\sqrt{\lambda+|\xi'|^2}x_n} \hat{G}_1(\xi')$$

for $(\xi', x_n) \in \mathbb{R}^n_+$, $\lambda > 0$, where $\hat{G}_r(\xi') = \mathrm{e}^{-|\xi'|^2 r}$, $r > 0$, denotes the heat kernel. It is easy to see that this function is the last component of a vectorfield f satisfying $\mathrm{div} f = 0$. Again by applying Proposition 2.1 it follows $f \in L^1(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$. Inserting this f in the formula for for v_1 by a calculation we deduce $\|v_1\|_{L^1(\mathbb{R}^n_+)} \geq \frac{1}{4}\|R_j G_1\|_{L^1(\mathbb{R}^{n-1})}$. It is well known that G_1 is not an element of the Hardy space $H^1(\mathbb{R}^{n-1})$. Moreover, an $L^1(\mathbb{R}^{n-1})$-function g is known to belong to $H^1(\mathbb{R}^{n-1})$ if and only if $R_j g \in L^1(\mathbb{R}^{n-1})$ for $j = 1, \ldots, n-1$. Hence for an appropriate $j \in \{1, \ldots, n-1\}$, $R_j G_1 \notin L^1(\mathbb{R}^{n-1})$, which yields the assertion of Theorem 1.2.

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