Strong solutions for the Navier-Stokes equations on bounded and unbounded domains with a moving boundary

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Abstract

It is proved under mild regularity assumptions on the data that the Navier-Stokes equations in bounded and unbounded noncylindrical regions admit a unique local-in-time strong solution. The result is based on maximal regularity estimates for the Stokes equations in spatial regions with a moving boundary obtained in [16] and the contraction mapping principle.

Keywords. Navier-Stokes equations, moving boundary, maximal regularity.

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1 Introduction and main result

For $T > 0$ let $Q_T := \bigcup_{t\in(0,T)} \Omega(t) \times \{t\} \subseteq \mathbb{R}^{n+1}$ be a noncylindrical space-time domain. In this note we consider the Navier-Stokes equations

$$(NSE)^{\Omega(t)}_{f,v_0} \left\{ \begin{array}{rcll} v_t - \Delta v + (v\cdot\nabla)v + \nabla p &=& f & \text{in } Q_T, \\ \operatorname{div} v &=& 0 & \text{in } Q_T, \\ v &=& 0 & \text{on } \bigcup_{t\in(0,T)} \partial\Omega(t)\times\{t\}, \\ v|_{t=0} &=& v_0 & \text{in } \Omega(0) =: \Omega_0, \end{array} \right.$$

with velocity field v and pressure p. Here we assume the moving boundary, i.e. the evolution of the domain $\Omega(t)$ to be determined by the level-preserving diffeomorphism

$$\psi : \overline{\Omega_0 \times (0,T)} \to \overline{Q_T}, \quad (\xi, t) \mapsto (x,t) = \psi(\xi,t) := (\phi(\xi;t), t)$$

such that for each $t \in [0,T)$, $\phi(\cdot;t)$ maps Ω_0 onto $\Omega(t)$. More precisely we assume the following conditions on ϕ respectively ψ.

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Assumption 1.1. Let $T \in (0,\infty)$, $\Omega_0 \subseteq \mathbb{R}^n$ be a domain of class C^3 either bounded, exterior, or a perturbed half-space. Suppose that the domains $\Omega(t)$, $t \in [0,T]$, are all of the same type as Ω_0, i.e. $\{\Omega(t)\}_{t\in[0,T]}$ is either a family of bounded domains, a family of exterior domains, or a family of perturbed half-spaces. Furthermore:

(1) For each $t \in [0,T]$, $\phi(\cdot;t) : \overline{\Omega_0} \to \overline{\Omega(t)}$ is a C^3-diffeomorphism. Its inverse we denote by $\phi^{-1}(\cdot;t)$ (to emphasize that ϕ^{-1} is merely the inverse w.r.t. the space variables we use the semicolon notation $(\xi;t)$ for the argument of ϕ and ϕ^{-1}).

(2) For ϕ regarded as a function from $Q_T^0 := \Omega_0 \times (0,T)$ into $\mathbb{R}^n$ we assume $\phi \in C_b^{3,1}(Q_T^0) := \{f \in C(Q_T^0) : \partial_t^k D_x^\alpha f \in C_b(Q_T^0),\ 1 \le 2k + |\alpha| \le 3, k \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n\}$, where $C_b(Q_T^0)$ denotes the space of all bounded and continuous functions on Q_T^0.

(3) We have $\det \nabla_\xi \phi(\xi,t) \equiv 1$, $(\xi,t) \in \overline{Q_T^0}$, (volume preserving).

Let us remark that in view of realistic physical situations problem $(NSE)_{f,v_0}^{\Omega(t)}$ should be considered with a certain boundary condition $v = b \neq 0$ at $\bigcup_{t\in(0,T)} \partial\Omega(t) \times \{t\}$. On the other hand, by assuming the existence of a solenoidal field β such that $\beta = b$, the problem with $b \neq 0$ can be reduced to the case $b = 0$ as described in [9] and [8]. Therefore we restrict our considerations to the system $(NSE)_{f,v_0}^{\Omega(t)}$ with zero boundary conditions.

Also, note that in certain concrete situations the existence of the diffeormorpism ψ is established. For instance in [8] the authors give as a nice example of a moving domain $\Omega(t)$ a bowl with swimming goldfishes (note that kisses are not allowed). The existence of ψ in such a situation is proved in [12] and [5].

Now define $\mathcal{I}^p(A) := (X, D(A))_{1-\frac{1}{p},p}$, for $1 < p < \infty$, where the latter space denotes the real interpolation space of a Banach space X and the domain $D(A)$ of a closed operator A in X. For $t \in [0,T)$ we denote by

$$\begin{aligned} A_{\Omega(t)} &= -P_{\Omega(t)}\Delta \quad \text{defined on} \\ D(A_{\Omega(t)}) &= W^{2,q}(\Omega(t)) \cap W_0^{1,q}(\Omega(t)) \cap L_\sigma^q(\Omega(t)) \end{aligned}$$

as usual the Stokes operator in the space of solenoidal fields $L^q_\sigma(\Omega(t)) = \overline{C^\infty_{c,\sigma}(\Omega(t))}^{L^q(\Omega(t))}$, where $C^\infty_{c,\sigma}(\Omega(t)) := \{u \in C_c^\infty(\Omega(t)) : \operatorname{div} u = 0\}$. Here $P_{\Omega(t)} : L^q(\Omega(t)) \to L^q_\sigma(\Omega(t))$ denotes the Helmholtz projection associated to the Helmholtz decomposition $L^q(\Omega(t)) = L^q_\sigma(\Omega(t)) \oplus G_q(\Omega(t))$, where $G_q(\Omega(t)) = \{\nabla p \,;\, p \in \widehat{W}^{1,q}(\Omega(t))\}$. Note that the existence of a compatible family $\{P_{\Omega,q}\}_{q\in(1,\infty)}$ of bounded projections $P_\Omega = P_{\Omega,q} : L^q(\Omega) \to L^q_\sigma(\Omega)$ is well known for all types of domains Ω considered in this note, see e.g. [7], [21], [19]. For the above types of moving domains our main result is

Theorem 1.2. *Let $n \geq 2$, $(n+2)/3 < q < \infty$, and $T \in (0,\infty]$. Let the evolution of $\Omega(t)$, $t \in [0,T]$, be determined by a function ψ satisfying Assumptions 1.1. Then, for each $v_0 \in \mathcal{I}^p(A_{\Omega_0})$ and $f \in L^p((0,T);L^q(\Omega(t)))$ there exists a $T^* \in (0,T)$ and a unique solution (v,p) of problem $(NSE)^{\Omega(t)}_{f,0}$, such that*

$$\begin{aligned} v &\in W^{1,q}((0,T^*);L^q(\Omega(t))) \cap L^q((0,T^*);D(A_{\Omega(t)})), \\ p &\in L^q((0,T^*);\widehat{W}^{1,q}(\Omega(t))). \end{aligned}$$

Remark 1.3. By an inspection of the proof one will realize that the assumption on the family $\{\Omega(t)\}_{t\in[0,T]}$, which we intrinsically use, is not the particular geometric shape of the domains, but the property of having maximal regularity of the corresponding Stokes operator $A_{\Omega(t)}$ for each fixed $t \in [0,T]$. Thus, Theorem 2.1 stays true for each family $\{\Omega(t)\}_{t\in[0,T]}$ with $\Omega(t)$ of the "same type" for all $t \in [0,T]$ such that $A_{\Omega(t)}$ has maximal regularity. This property for instance is also known to be valid for families of asymptotically flat layers as examined in [2], [1].

Some special cases of the situation in Theorem 2.1 are considered in several former works. First investigations of the solvability of $(NSE)^{\Omega(t)}_{f,v_0}$ can be found in [18]. However, until now there are only existence results available under the restrictive assumptions that $q = 2$, i.e. in the Hilbert space case, and that $\{\Omega(t)\}_{t\in[0,T]}$ is a family of bounded domains. In that situation for instance in [8] the existence of a unique local-in-time solution of $(NSE)^{\Omega(t)}_{f,v_0}$ is proved. The existence of global weak solutions in $L^2(\Omega(t))$ for the Navier-Stokes equations was already shown in [6] (see also [3]). The periodic case was obtained in [11], i.e. if $t \mapsto \Omega(t)$ is periodic then there exists a periodic weak solution of the Navier-Stokes equations. A result for more regular periodic solutions can be found in [10]. Another existence result of local-in-time strong solutions of the Navier-Stokes equations in $L^2_\sigma(\Omega(t))$ for bounded $\Omega(t)$ is obtained in [14]. There the authors could relax a restrictive decay condition on the right hand side f assumed in [8], but nevertheless they were able to construct a more regular solution.

We want to remark that the assumptions on the evolution and regularity of $\Omega(t)$ differ in the above cited papers. This depends mainly on the method that the authors use in their works. The approach presented here is closely related to the method used in [8]. Therefore we have similar assumptions on ψ (hence also on $\Omega(t)$) as in [8].

Theorem 2.1 generalizes the above cited results on $(NSE)^{\Omega(t)}_{f,v_0}$ in several directions. Firstly, we handle L^q-spaces for the full scale $(n+2)/3 < q < \infty$

and arbitrary space dimension $n \geq 2$, and not only the Hilbert space case if $n = 2, 3$. Moreover, we prove the existence of strong solutions under mild regularity assumptions on the data. Secondly, our approach also covers various families $\{\Omega(t)\}_{t\in[0,T)}$ of unbounded domains.

As we already mentioned, Theorem 2.1 is based on maximal regularity estimates for the corresponding linear Stokes equations obtained in [16]. Therefore, in Section 2 we recall the main results given in [16] in a slightly adapted form suitable for our purposes. Utilizing the maximal regularity for the Stokes equations and the contraction mapping principle in Section 3 we give the proof of our main result, the unique local-in-time strong solutions to the Navier-Stokes equations on noncylindrical space-time domains.

We introduce some notation used in the sequel. By $C^k(\Omega)$ we denote the space of all k-times continuously differentiable functions in an open subset Ω of $\mathbb{R}^n$, and by $C^k_b(\Omega)$ its subspace of k-times bounded continuously differentiable functions. As usual $W^{k,q}(\Omega)$ is the Sobolev space with norm $\|\cdot\|_{k,q} = (\sum_{j=0}^k \|\nabla^j \cdot\|_q^q)^{1/q}$ and $L^q(\Omega) = W^{0,q}(\Omega)$ the Lebesgue space of q-integrable functions. We also make use of the homogeneous Sobolev space $\widehat{W}^{1,q}(\Omega)$, consisting of all locally integrable functions f in Ω with $\|\nabla f\|_q < \infty$, modulo constants. Note that we do not distinguish between scalar and vector valued Sobolev spaces, i.e. we write $L^q(\Omega)$ for $(L^q(\Omega))^n$, $W^{k,q}(\Omega)$ for $(W^{k,q}(\Omega))^n$, etc. Furthermore, $\mathcal{L}(X,Y)$ denotes the class of all bounded operators from X to Y and $\mathcal{I}som(X,Y)$ its subclass of isomorphisms. If $X = Y$ we set $\mathcal{L}(X) := \mathcal{L}(X,X)$ and $\mathcal{I}som(X) := \mathcal{I}som(X,X)$. The domain of an operator A in a Banach space X we denote by $D(A)$, its range by $R(A)$, and its resolvent set by $\rho(A)$.

2 Maximal regularity for the Stokes equations

For the readers convenience we recall here the basic steps that lead to the maximal regularity result for the linearized version of $(NSE)^{\Omega(t)}_{f,v_0}$ obtained in [16]. Also note that we state them in a slightly adapted form as we will need it in Section 3. In this context we restrict the statements here to the case of finite $T > 0$, but note that under suitable additional assumptions all the assertions are still true for $T = \infty$. Employing the notation of the last section, here we are concerned with the linear problem

$$(SE)^{\Omega(t)}_{f,v_0} \left\{ \begin{array}{rcll} v_t - \Delta v + +\nabla p &=& f & \text{in } Q_T, \\ \operatorname{div} v &=& 0 & \text{in } Q_T, \\ v &=& 0 & \text{on } \bigcup_{t\in(0,T)} \partial\Omega(t) \times \{t\}, \\ v|_{t=0} &=& v_0 & \text{in } \Omega(0) =: \Omega_0. \end{array} \right.$$

For this system in [16, Theorem 2.1] it is proved

Theorem 2.1. *Let $n \geq 2$, $1 < q < \infty$, and $T \in (0,\infty)$. Let the evolution of $\Omega(t)$, $t \in [0,T]$, be determined by a function ψ satisfying Assumptions 1.1. Then problem $(SE)^{\Omega(t)}_{f,0}$ has a unique solution $t \mapsto (v(t),p(t)) \in D(A_{\Omega(t)}) \times \widehat{W}^{1,q}(\Omega(t))$, $t \in [0,T]$. Furthermore, this solution satisfies the estimate*

$$\begin{aligned}&\int_0^T \left[\|v_t(t)\|^p_{L^q(\Omega(t))} + \|v(t)\|^p_{W^{2,q}(\Omega(t))} + \|\nabla p(t)\|^p_{L^q(\Omega(t))}\right] \mathrm{d}t \\ \leq\; & C(T)\left(\|v_0\|^p_{\mathcal{I}^p(A_{\Omega_0})} + \int_0^T \|f(t)\|^p_{L^q(\Omega(t))}\mathrm{d}t\right) \qquad (1)\end{aligned}$$

for all $v_0 \in \mathcal{I}^p(A_{\Omega_0})$ and $f \in L^p((0,T);L^q(\Omega(t)))$. If $v_0 = 0$, then the constant $C(T)$ in (1) is uniformly bounded from above on finite intervals, more precisely there is a $T_0 > 0$ and a $C(T_0) > 0$ such that $C(T) \leq C(T_0)$ for all $T \leq T_0$.

The proof of this result relies on a transform of $(SE)^{\Omega(t)}_{f,0}$ via ψ to a problem on the cylindrical domain $\Omega_0 \times (0,T)$. The price we have to pay is that we are then left with a nonautonomous system of partial differential equations, i.e. the coefficients of these transformed equations depend on space and time in general. Here Assumption 1.1 (2) assures that they are at least continuous. Another important point is that the transformed functions belong to the solenoidal space $L^q_\sigma(\Omega_0)$, which relies essentially on Assumption 1.1 (3). More precisely this condition assures that the operator *div* is invariant under the chosen transform.

Similar to the autonomous Stokes equations this will give us the possibility to formulate an associated abstract Cauchy problem with operators acting in $L^q_\sigma(\Omega_0)$. The idea here is to use the family of projections $P_{\Omega_0}(t) : L^q(\Omega_0) \to L^q_\sigma(\Omega_0)$, which are exactly the transformed Helmholtz projections $P_{\Omega(t)}$.

First let us list some obvious consequences of Assumption 1.1. In view of $\det \nabla\phi(\xi,t) \equiv 1$ and $\psi(\xi,t) = (\phi(\xi;t),t)$ we also have $\det \nabla\psi = 1$. Moreover, Assumption 1.1 (2) implies $\psi \in C^{3,1}_b(Q^0_T;\mathbb{R}^{n+1})$. In virtue of the implicit function theorem we therefore have $\psi^{-1} \in C^{3,1}_b(Q_T;\mathbb{R}^{n+1})$ and since $\psi^{-1}(x,t) = (\phi^{-1}(x;t),t)$, $(x,t) \in Q_T$, also $\phi^{-1} \in C^{3,1}_b(Q_T;\mathbb{R}^n)$.

We transform $(SE)^{\Omega(t)}_{f,0}$ to a system on a fixed domain as follows. For a function $v : Q_T \to \mathbb{C}^n$ set

$$\tilde{v}(\xi,t) := v(\phi(\xi;t),t), \quad (\xi,t) \in \Omega_0 \times [0,T].$$

Then

$$(\nabla_x v)(\phi(\xi;t),t) = \left[(\nabla_\xi \phi)^{-T} \nabla_\xi \tilde{v}\right](\xi,t), \qquad (2)$$

where M^{-T} denotes $(M^T)^{-1}$ and M^T stands for the transposed Matrix. Now define

$$u(\xi,t) := (\Phi(t)v)(\xi,t) := \left[(\nabla_\xi \phi)^{-1}\tilde{v}\right](\xi,t), \quad (\xi,t) \in \Omega_0 \times [0,T]. \tag{3}$$

Assumption 1.1 (1), (2), and (3) on ϕ imply that

$$\Phi(t) \in Isom(W^{k,q}(\Omega(t)), W^{k,q}(\Omega_0)) \cap Isom(W_0^{k,q}(\Omega(t)), W_0^{k,q}(\Omega_0))$$

for $k = 0,1,2$ and $t \in [0,T]$, and we even have the uniform estimates

$$\|\Phi(t)v\|_{W^{k,p}(\Omega_0)} \le C_1 \|v\|_{W^{k,p}(\Omega(t))} \le C_2 \|\Phi(t)v\|_{W^{k,p}(\Omega_0)} \tag{4}$$

for all $v \in W^{k,p}(\Omega(t))$, $t \in [0,T]$, $k = 0,1,2$. It is also easy to see that $\nu(x,t)$ is the outer normal at $\partial\Omega(t)$ in x if and only if $\mu(\xi,t) = (\nabla\phi)^T(\xi,t)\nu(\phi(\xi,t))$ is the outer normal at $\partial\Omega_0$ in ξ. This implies $\nu \cdot v = 0$ if and only if $\mu \cdot \Phi v = 0$. Furthermore, under Assumption 1.1 (in particular (3)) in [8, Proposition 2.4][1] it is proved that

$$\operatorname{div}_\xi u(\xi,t) = \operatorname{div}_x v(\phi(\xi;t),t), \quad (\xi,t) \in \Omega_0 \times [0,T].$$

This implies that $\Phi(t) : L^q_\sigma(\Omega(t)) \to L^q_\sigma(\Omega_0)$ is an isomorphism as well. This property of Φ, which is essential in what follows, is the reason why we have to choose the special transform given in (3). On the other hand note that this transform is responsible for the fact, that we have to assume C^3 boundary instead of C^2 only.

In view of (2) it is clear that $\Phi(t)\Delta_x\Phi(t)^{-1}$ has a representation as

$$\Phi(t)\Delta_x\Phi(t)^{-1} = \sum_{|\alpha|\le 2} a_\alpha(\cdot,t)D^\alpha \tag{5}$$

with certain matrices $a_\alpha \in C_b^{|\alpha|,\frac{|\alpha|}{2}}(\overline{\Omega \times (0,T)})^{n\times n}$, $|\alpha| \le 2$. Explicitly we have

$$\begin{aligned}
&[\Phi(t)\Delta_x\Phi(t)^{-1}u](\xi,t)\\
&= [(\nabla_\xi\phi)^{-1}(\nabla_\xi\phi)^{-T}\nabla_\xi \cdot (\nabla_\xi\phi)^{-T}\nabla_\xi(\nabla_\xi\phi)u](\xi,t)\\
&= \sum_{i,j,k,\ell,m=1}^{n} \left[(\partial_{x_k}\phi^{-1})(\partial_{x_j}\phi^{-1})^i(\partial_{x_j}\phi^{-1})^\ell\right](\phi(\xi;t),t)\,\cdot\\
&\quad \cdot \Big[(\partial_{\xi_\ell}\partial_{\xi_i}\partial_{\xi_m}\phi^k)u^m + (\partial_{\xi_i}\partial_{\xi_m}\phi^k)\partial_{\xi_\ell}u^m\\
&\quad + (\partial_{\xi_\ell}\partial_{\xi_m}\phi^k)\partial_{\xi_i}u^m + (\partial_{\xi_m}\phi^k)\partial_{\xi_\ell}\partial_{\xi_i}u^m\Big](\xi,t).
\end{aligned} \tag{6}$$

[1] Actually in [8] only bounded Ω_0 are treated. But since it is a pointwise condition the proof given there applies to each $\Omega \subset \mathbb{R}^n$.

We also have

$$\begin{aligned}\partial_t v(x,t) &= \partial_t \left[(\nabla_\xi \phi)u\right](\phi^{-1}(x;t),t)\\ &= \sum_{i,j=1}^{n} (\partial_t \phi^{-1})^j(x;t)\left[(\partial_{\xi_i}\partial_{\xi_j}\phi)u^i + (\partial_{\xi_i}\phi)\partial_{\xi_j}u^i\right](\phi^{-1}(x;t),t)\\ &\quad + \sum_{i=1}^{n}\left[(\partial_{\xi_i}\partial_t\phi)u^i + (\partial_{\xi_i}\phi)\partial_t u^i\right](\phi^{-1}(x;t),t). \qquad (7)\end{aligned}$$

Thus

$$\Phi(t)\partial_t\Phi(t)^{-1} = \partial_t + \sum_{|\beta|\le 1} b_\beta(\cdot,t)D^\beta \qquad (8)$$

with certain $b_\beta \in C_b^{2|\beta|,|\beta|}(\overline{\Omega\times(0,T)})^{n\times n}$, $|\beta|\le 1$. If we set $F := \Phi f$ and $u_0 := \Phi(0)v_0$, as well as $\nabla^\phi(t) := (\nabla_\xi\phi(t))^{-1}(\nabla_\xi\phi(t))^{-T}\nabla_\xi$ and $\tilde p := p\circ\psi$, the transformed equations on $Q_T^0 = \Omega\times(0,T)$ become

$$\begin{aligned} u_t + \textstyle\sum_{|\beta|\le 1} b_\beta D^\beta u - \sum_{|\alpha|\le 2} a_\alpha D^\alpha u + \nabla^\phi(\cdot)\tilde p &= F \quad \text{in } Q_T^0,\\ \operatorname{div} u &= 0 \quad \text{in } Q_T^0,\\ u &= 0 \quad \text{on } \partial\Omega_0\times(0,T),\\ u|_{t=0} &= u_0 \quad \text{in } \Omega_0.\end{aligned}$$

We call this system $(TSE)^{\Omega_0}_{F,u_0}$. Since $\Phi(t)$ is an isomorphism, clearly $(u,\tilde p)$ satisfies $(TSE)^{\Omega_0}_{F,u_0}$ if and only if (v,p) fulfills $(SE)^{\Omega(t)}_{f,v_0}$. Obviously

$$P_{\Omega_0}(t) := \Phi(t)P_{\Omega(t)}\Phi(t)^{-1} : L^q(\Omega_0)\to L^q_\sigma(\Omega_0), \quad t\in[0,T],$$

is again a projection, where $P_{\Omega(t)}$ denotes the Helmholtz projection on $L^q(\Omega(t))$. Note that

$$G_q(t) := (I - P_{\Omega_0}(t))L^q(\Omega_0) = \{\nabla^\phi(t)(\pi\circ\psi);\pi\in\widehat{W}^{1,q}(\Omega(t))\}.$$

Thus, $P_{\Omega_0}(t)$ is not the Helmholtz projection on $L^q(\Omega_0)$ in general. As $G_q(t)$ depends on t we see that also the projection $P_{\Omega_0}(t)$ does, although its range $L^q_\sigma(\Omega_0)$ is independent of t. Defining

$$\begin{aligned} A_{\Omega_0}(t) &:= -P_{\Omega_0}(t)\sum_{|\alpha|\le 2} a_\alpha(\cdot,t)D^\alpha \quad \text{on} \qquad (9)\\ D(A_{\Omega_0}(t)) &= \Phi(t)D(A_{\Omega(t)}) = W^{2,q}(\Omega_0)\cap W^{1,q}_0(\Omega_0)\cap L^q_\sigma(\Omega_0)\\ &= D(A_{\Omega_0}), \quad t\in[0,T],\end{aligned}$$

and

$$B(t) := P_{\Omega_0}(t)\sum_{|\beta|\le 1} b_\beta(\cdot,t)D^\beta, \quad t\in[0,T], \qquad (10)$$

the system $(TSE)_{F,u_0}^{\Omega_0}$ can be rephrased as the nonautonomous Cauchy problem

$$(CP)_{F,u_0}\left\{\begin{array}{rcll} u'(t)+(A_{\Omega_0}(t)+B(t))u(t) &=& F(t), & t\in(0,T),\\ u(0) &=& u_0, & \end{array}\right.$$

on the space $L^q_\sigma(\Omega_0)$. Observe that $A_{\Omega_0}(t)=\Phi(t)A_{\Omega(t)}\Phi(t)^{-1}$, i.e. it is exactly the transformed Stokes operator on $\Omega(t)$ for $t\in[0,T]$. Moreover, we see that the domain of $A_{\Omega_0}(t)$ does not depend on t and equals the domain of the Stokes operator A_{Ω_0} in $L^q_\sigma(\Omega_0)$.

For $T\in(0,\infty)$ and $p\in(1,\infty)$ we denote by $\mathrm{MR}_p(X,K)$ the class of all operators (and propagators) $A(\cdot)$ having maximal (L^p-) regularity on X with a maximal regularity constant not exceeding K, i.e. there exists a unique solution $t\mapsto u(t)\in D(A(t))$ of the (eventually nonautonomous) Cauchy problem

$$\left\{\begin{array}{rcll} u'+A(\cdot)u &=& f, & \text{in } (0,T),\\ u(0) &=& u_0, & \end{array}\right. \tag{11}$$

satisfying the estimate

$$\|u'\|_{W^{1,p}((0,T);X)}+\|A(\cdot)u\|_{L^p((0,T);X)}\leq K(\|f\|_{L^p((0,T);X)}+\|u_0\|_{\mathcal{I}^p(A(0))})$$

for $f\in L^p((0,T);X)$ and $u_0\in\mathcal{I}^p(A(0))$.

Based on two abstract results for nonautonomous systems (see [16, Teorem 1.4 and Theorem 2.5]) the following result is obtained in [16, Theorem 3.5].

Proposition 2.2. *Let $T\in(0,\infty)$. Let Ω_0, ϕ be as in Assumption 1.1 and the families $\{A_{\Omega_0}(t)\}_{t\in[0,T]}$ and $\{B(t)\}_{t\in[0,T]}$ be defined as in (9) and (10), respectively. Then for $\mu>0$ large enough we have*

$$\|(\mu+A_{\Omega_0}(t)+B(t))(\mu+A_{\Omega_0}(s)+B(s))^{-1}\|_{\mathcal{L}(X)}\leq C,\quad t,s\in[0,T). \tag{12}$$

and $A_{\Omega_0}(\cdot)+B(\cdot)\in\mathrm{MR}(L^q_\sigma(\Omega_0),C(T))$.

We turn to the proof of the maximal regularity result for $(SE)_{f,v_0}^{\Omega(t)}$.

Proof. (of Theorem 2.1)
Observe that in view of (12) and the equivalence of the norms $\|\cdot\|_{2,q}$ and $\|\cdot\|_{D(A_{\Omega_0}(0)+B(0))}$ we have

$$\int_0^T\left(\|u'(t)\|_q^p+\|u(t)\|_{2,q}^p\right)\mathrm{d}t\leq C(T)\left(\int_0^T\|F(t)\|_q^p\mathrm{d}t+\|u_0\|_{\mathcal{I}^p}^p\right). \tag{13}$$

This yields

$$\begin{aligned} & \int_0^T \left(\|(\partial_t + \sum_{|\beta|\le 1} b_\beta(t))u(t)\|_q^p + \|u(t)\|_{2,q}^p + \|\nabla^\phi(t)\tilde{p}(t)\|_q^p \right) \mathrm{d}t \\ \le \;& C(T) \left(\int_0^T \|F(t)\|_q^p \mathrm{d}t + \|u_0\|_{\mathcal{I}^p}^p \right). \end{aligned}$$

for the solution (u,p) of $(TSE)^{\Omega_0}_{F,u_0}$. In view of (4), (8), and since $\{\Phi(t)\}_{t\in[0,T]}$ is a family of isomorphisms, this implies estimate (1) for the solution of the original equations $(SE)^{\Omega(t)}_{f,v_0}$. If $v_0 = 0$ and $f \in L^q((0,T); L^q_\sigma(\Omega(t)))$ we may extent f trivial to the interval $(0,T_0)$, where we denote the extended function by $\bar{f}$. Let (u,p) and $(\bar{u},\bar{p})$ be the solution to problem $(SE)^{\Omega(t)}_{f,0}$ and $(SE)^{\Omega(t)}_{\bar{f},0}$, respectively. The uniqueness of the solution implies $(\bar{u},\bar{p})|_{(0,T)} = (u,p)$. By this fact it easily follows that the constants $C(T)$ in (1) can be dominated by a constant $C(T_0)$ for all $T \le T_0$. This completes the proof. □

3 Strong solutions for the Navier-Stokes equations

Utilizing the maximal regularity for the Stokes equations, in this section we prove our main result Theorem 1.2. In order to estimate the nonlinear term in $(NSE)^{\Omega(t)}_{f,v_0}$, a further main ingredient in the proof will be the following embedding.

Lemma 3.1. *Let $T > 0$, $J = (0,T)$, $a \ge 2$, and $q > \frac{n}{a} + 1$. Then we have*

$$W^{1,q}(J; L^q(\Omega(t))) \cap L^q(J; W^{2,q}(\Omega(t))) \hookrightarrow L^{2q}(J; W^{1,aq/(a-1)}(\Omega(t))).$$

If we replace $W^{1,q}(J; L^q(\Omega(t)))$ by $W^{1,q}_0(J; L^q(\Omega(t)))$ on the left hand side, then there exists a $T_0 > 0$ such that the embedding constant is governed by a constant $C(T_0) > 0$ for all $T \le T_0$.

Proof. Note that (4) and Assumption 1.1 (2) imply that

$$\begin{aligned} \Phi \;\in\; & Isom(W^{\ell,p}(J; W^{k,q}(\Omega(t))), W^{\ell,p}(J; W^{k,q}(\Omega_0))) \\ & \cap Isom(W^{\ell,p}_0(J; W^{k,q}(\Omega(t))), W^{\ell,p}_0(J; W^{k,q}(\Omega_0))) \end{aligned}$$

for $\ell = 0,1$, $k = 0,1,2$, and $1 \le p,q \le \infty$. In particular we have

$$\|\Phi v\|_{W^{\ell,p}(J;W^{k,q}(\Omega_0))} \le C_1 \|v\|_{W^{\ell,p}(J;W^{k,q}(\Omega(t)))} \le C_2 \|\Phi v\|_{W^{\ell,p}(J;W^{k,q}(\Omega_0))} \tag{14}$$

for all $v \in W^{\ell,p}(J; W^{k,q}(\Omega(t)))$, $\ell = 0, 1$, $k = 0, 1, 2$, with C_1, C_2 independent of $T > 0$. Therefore it suffices to prove the embedding

$$W^{1,q}(J; L^q(\Omega_0)) \cap L^q(J; W^{2,q}(\Omega_0)) \hookrightarrow L^{2q}(J; W^{1,aq/(a-1)}(\Omega_0)),$$

and that this embedding is even valid with an embedding constant independent of $T \le T_0$, if we assume zero time trace at $t = 0$.

It is a known fact that

$$W^{1,q}(J; L^q(\Omega_0)) \cap L^q(J; W^{2,q}(\Omega_0)) \hookrightarrow W^{s,q}(J; W^{2(1-s),q}(\Omega_0)), \quad s \in [0, 1]. \tag{15}$$

This follows e.g. by an application of the mixed derivative theorem [20] (see also [15]) for $J = \mathbb{R}$ and $\Omega_0 = \mathbb{R}^n$. Employing suitable extension operators in space and time it can be seen that this embedding is still valid for $J = (0, T)$ and our Ω_0, even with an embedding constant independent of $T \le T_0$, if we assume vanishing time trace at $t = 0$. (see e.g. [15, Proposition 6.1] for the existence of such an extension operator). According to $q > \frac{n}{a} - 1$ we can find an $\varepsilon > 0$ such that $q > \frac{n}{a} - 1 + \varepsilon$. Now set $s := (1 + \varepsilon)/2q$. Since $1 - sq > 0$ and $2q < q/(1 - sq)$ the Sobolev embedding implies

$$\|v\|_{L^{2q}(J; W^{2(1-s),q}(\Omega_0))} \le C\|v\|_{W^{s,q}(J; W^{2(1-s),q}(\Omega_0))}.$$

Furthermore, we have $n - sq > 0$ and $aq/(a-1) < nq/(n - (1 - 2s))$. Thus, we may apply the Sobolev embedding also in space to the result

$$\begin{aligned} \|v\|_{L^{2q}(J; W^{aq/(a-1),q}(\Omega_0))} &\le C\|v\|_{L^{2q}(J; W^{2(1-s),q}(\Omega_0))} \\ &\le C\|v\|_{W^{s,q}(J; W^{2(1-s),q}(\Omega_0))} \end{aligned}$$

for $v \in W^{s,q}(J; W^{2(1-s),q}(\Omega_0))$. In combination with (14) and (15) this yields the first assertion. The additional assertion follows by the fact that also the embedding constant of the Sobolev embedding in time can be chosen independently of $T \le T_0$, if we assume vanishing time trace at $t = 0$. $\square$

Finally we prove our main result by employing the contraction mapping principle.

Proof. (of Theorem 1.2) First let us introduce some notation. We set

$$\mathbb{E} = \mathbb{E}_1 \times \mathbb{E}_2, \quad \mathbb{F} = \mathbb{F}_1 \times \mathbb{F}_2$$

with

$$\begin{aligned} \mathbb{E}_1 &:= W^{1,q}((0,T); L^q(\Omega(t))) \cap L^q((0,T); D(A_{\Omega(t)})), \\ \mathbb{E}_2 &:= L^q((0,T); \widehat{W}^{1,q}(\Omega(t))), \\ \mathbb{F}_1 &:= L^p((0,T); L^q(\Omega(t))), \\ \mathbb{F}_2 &:= \mathcal{I}^p(A_{\Omega_0}). \end{aligned}$$

Also, denote by $\mathbb{E}_0$ and $\mathbb{F}_0$ the corresponding spaces with vanishing time trace at $t = 0$, that is $\mathbb{E}_0 = \mathbb{E}_{1,0} \times \mathbb{E}_2$ with $\mathbb{E}_{1,0} := W_0^{1,q}((0,T); L^q(\Omega(t))) \cap L^q((0,T); D(A_{\Omega(t)}))$ and $\mathbb{F}_0 := \mathbb{F}_1 \times \{0\}$. Now let L_T be the solution operator of the linear problem $(SE)^{\Omega(t)}_{\cdot,\cdot}$ and observe that according to Theorem 2.1 we have $L_T \in Isom(\mathbb{E}, \mathbb{F})$. Then problem $(NSE)^{\Omega(t)}_{f,v_0}$ can formally be rephrased as

$$L_T(v,p) = (f + F(v), v_0), \tag{16}$$

where $F(v) := -(v \cdot \nabla)v$. For further purposes it will be convenient to split off the part corresponding to the data f and v_0. To do so let (v^*, p^*) be the solution to $(SE)^{\Omega(t)}_{f,v_0}$, i.e.

$$(v^*, p^*) = L_T^{-1}(f, v_0).$$

Moreover, we set $\bar{v} := v - v^*$ and $\bar{p} = p - p^*$. By this notation (16) turns into

$$\begin{aligned} L_T(\bar{v}, \bar{p}) &= (f + F(\bar{v} + v^*), v_0) - L_T(v^*, p^*) \\ &= (F(\bar{v} + v^*), 0) =: H_0(\bar{v}, \bar{p}), \end{aligned}$$

and therefore the fixed point equation reads as

$$(\bar{v}, \bar{p}) = L_T^{-1} H_0(\bar{v}, \bar{p}),$$

where $\bar{v}$ and $H_0(\bar{v}, \bar{p})$ now have zero time trace by construction. Next suppose $a \geq 2$ and that

$$q > \frac{n}{a} + 1. \tag{17}$$

Then Lemma 3.1 implies $\mathbb{E}_1 \hookrightarrow L^{2q}(J, L^{aq/(a-1)}(\Omega(t)))$. For $u, w \in \mathbb{E}_1$ we therefore deduce by applying first the Hölder and then the Sobolev inequality (in space)

$$\begin{aligned} \|(u \cdot \nabla)w\|_{\mathbb{F}_1} &\leq \|u\|_{L^{2q}(J,L^{aq}(\Omega(t)))} \|\nabla w\|_{L^{2q}(J,L^{aq/(a-1)}(\Omega(t)))} \\ &\leq C\|u\|_{L^{2q}(J,W^{1,aq/(a-1)}(\Omega(t)))} \|w\|_{L^{2q}(J,W^{1,aq/(a-1)}(\Omega(t)))}. \end{aligned} \tag{18}$$

Observe that the above application of the Sobolev inequality requires a second condition on q, namely that

$$q > n\frac{a-2}{a}. \tag{19}$$

Since relation (17) is decreasing in a and (19) is increasing in a, the best possible value for q is reached at the intersection point of the graphs of the two equations $y = \frac{n}{a} + 1$ and $y = n\frac{a-2}{a}$, which is

$$(a, y) = \left(\frac{3n}{n-1}, (n+2)/3 \right).$$

Thus, by the assumption $q > (n+2)/3$ and by setting $a = \frac{3n}{n-1}$ the two conditions (17) and (19) are satisfied, which justifies the application of Lemma 3.1 and the Sobolev embedding in estimate (18).

Now fix $T_0 > 0$. Let $\mathbb{B}_r(0) \subseteq \mathbb{E}_0$ be the ball around 0 with radius r, and $(\bar{v}, \bar{p}) \in \mathbb{B}_r(0)$. Applying (18) to $H_0(\bar{v}, \bar{p})$ yields

$$\begin{aligned} \|H_0(\bar{v}, \bar{p})\|_{\mathbb{F}} &\leq \|F(\bar{v} + v^*)\|_{\mathbb{F}_1} \\ &\leq C \left(\|\bar{v}\|_{a,q}^2 + \|\bar{v}\|_{a,q} \|v^*\|_{a,q} + \|v^*\|_{a,q}^2 \right), \end{aligned}$$

where $\| \cdot \|_{a,q}$ denotes the norm of the space $L^{2q}(J; W^{1,aq/(a-1)}(\Omega(t)))$. Applying Lemma 3.1 to the terms involving $\bar{v}$ results in

$$\|H_0(\bar{v}, \bar{p})\|_{\mathbb{F}} \leq C \left(\|(\bar{v}, \bar{p})\|_{\mathbb{E}_0}^2 + \|(\bar{v}, \bar{p})\|_{\mathbb{E}_0} \|v^*\|_{a,q} + \|v^*\|_{a,q}^2 \right) \tag{20}$$

for all $T \leq T_0$ in view of $(\bar{v}, \bar{p}) \in \mathbb{E}_0$. Note that by definition $H_0 \in \mathcal{L}(\mathbb{E}_0, \mathbb{F}_0)$. According to Theorem 2.1 we have $\|L_T^{-1}\|_{\mathcal{L}(\mathbb{F}_0, \mathbb{E}_0)} \leq C(T_0)$ for all $T \leq T_0$. Hence, there exists a constant $C_0 > 0$ independent of $T \leq T_0$ such that

$$\begin{aligned} \|L_T^{-1} H_0(\bar{v}, \bar{p})\|_{\mathbb{E}_0} &\leq \|L_T^{-1}\|_{\mathcal{L}(\mathbb{F}_0, \mathbb{E}_0)} \|H_0(\bar{v}, \bar{p})\|_{\mathbb{F}} \\ &\leq C_0 \left(\|(\bar{v}, \bar{p})\|_{\mathbb{E}_0}^2 + \|(\bar{v}, \bar{p})\|_{\mathbb{E}_0} \|v^*\|_{a,q} + \|v^*\|_{a,q}^2 \right). \end{aligned}$$

Observe that v^* is a fixed function only depending on the data (f, v_0). Hence we may choose $r > 0$ small so that $r < \max\{1, 1/3C_0\}$ and then $T > 0$ small such that

$$\|v^*\|_{a,q} < \frac{r}{3C_0}.$$

This implies that

$$\|L_T^{-1} H_0(\bar{v}, \bar{p})\|_{\mathbb{E}_0} \leq r,$$

that is $L_T^{-1} H_0(\mathbb{B}_r(0)) \subseteq \mathbb{B}_r(0)$. To see that $L_T^{-1} H_0$ is a contraction observe that

$$\begin{aligned} &\|L_T^{-1} H_0(\bar{v}_1, \bar{p}_1) - L_T^{-1} H_0(\bar{v}_2, \bar{p}_2)\|_{\mathbb{E}_0} \\ &\leq \|L_T^{-1}\|_{\mathcal{L}(\mathbb{F}_0, \mathbb{E}_0)} \|H_0(\bar{v}, \bar{p}) - H_0(\bar{v}, \bar{p})\|_{\mathbb{F}_0} \\ &\leq C(T_0) \Bigg(\|[(\bar{v}_1 - \bar{v}_2) \cdot \nabla] v^*\|_{\mathbb{F}_1} + \|(v^* \cdot \nabla)(\bar{v}_1 - \bar{v}_2)\|_{\mathbb{F}_1} \\ &\qquad + \|[(\bar{v}_1 - \bar{v}_2) \cdot \nabla] \bar{v}_1\|_{\mathbb{F}_1} + \|(\bar{v}_2 \cdot \nabla)(\bar{v}_1 - \bar{v}_2)\|_{\mathbb{F}_1} \Bigg). \end{aligned}$$

By applying (18) and Lemma 3.1 we obtain in a similar way as above that

$$\begin{aligned} &\|L_T^{-1} H_0(\bar{v}_1, \bar{p}_1) - L_T^{-1} H_0(\bar{v}_2, \bar{p}_2)\|_{\mathbb{E}_0} \\ &\leq C_0 \Bigg(\|v^*\|_{a,q} + \|(\bar{v}_1, \bar{p}_1)\|_{\mathbb{E}_0} + \|(\bar{v}_2, \bar{p}_2)\|_{\mathbb{E}_0} \Bigg) \|(\bar{v}_1 - \bar{v}_2, \bar{p}_1 - \bar{p}_2)\|_{\mathbb{E}_0} \end{aligned}$$

with a constant $C_0 > 0$ not depending on $T \leq T_0$. Consequently, if we choose $T, r > 0$ such that $r, \|v^*\|_{a,q} < 1/(6C_0)$, we see that $L_T^{-1} H_0 : \mathbb{B}_r(0) \to \mathbb{B}_r(0)$ is a contraction and the assertion follows by the contraction mapping principle.

□

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