$\mathcal{R}$-sectoriality of cylindrical boundary value problems

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Dedicated to Herbert Amann on the occasion of his 70th birthday

Abstract. We prove $\mathcal{R}$-sectoriality or, equivalently, L^p-maximal regularity for a class of operators on cylindrical domains of the form $\mathbb{R}^{n-k} \times V$, where $V \subset \mathbb{R}^k$ is a domain with compact boundary, $\mathbb{R}^k$, or a half-space. Instead of extensive localization procedures, we present an elegant approach via operator-valued multiplier theory which takes advantage of the cylindrical shape of both, the domain and the operator.

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1. Introduction

This note considers the vector-valued L^p-approach to boundary value problems of the type

$$\begin{array}{rcll} \partial_t u + A(x,D)u &=& f & \text{in } \mathbb{R}_+ \times \Omega, \\ B_j(x,D)u &=& 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \quad (j = 1, ..., m), \\ u|_{t=0} &=& u_0 & \text{in } \Omega, \end{array} \tag{1.1}$$

on cylindrical domains $\Omega \subset \mathbb{R}^n$ of the form

$$\Omega = \mathbb{R}^{n-k} \times V, \tag{1.2}$$

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where V is a standard domain (see Definition 2.1) in $\mathbb{R}^k$. Here

$$A(x,D) = \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha$$

is a differential operator in Ω of order $2m$ for $m \in \mathbb{N}$ and

$$B_j(x,D) = \sum_{|\beta|\le m_j} b_\beta(x) D^\beta, \quad m_j < 2m, \quad j = 1,\dots,m,$$

are operators acting on the boundary.

Under the assumption that (1.1) is *parameter-elliptic* and *cylindrical*, we will prove $\mathcal{R}$-sectoriality for the operator A of the corresponding Cauchy problem. Recall that $\mathcal{R}$-sectoriality is equivalent to maximal regularity, cf. [21, Theorem 4.2]. Maximal regularity, in turn, is a powerful tool for the treatment of related nonlinear problems.

Roughly speaking, the assumption 'cylindrical' implies that A resolves into two parts

$$A = A_1 + A_2$$

such that A_1 acts merely on $\mathbb{R}^{n-k}$ and A_2 acts merely on V (see Definition 2.2). Note that many standard systems, such as the heat equation with Dirichlet or Neumann boundary conditions, are of this form. Also note that many physical problems naturally lead to equations in cylindrical domains. We refer to the textbook [10] for an introduction to such type problems. Therefore, boundary value problems of this type are certainly of independent interest. On the other hand, they also naturally appear during localization procedures of boundary value problems on general domains. For instance, if a system of equations via localization is reduced to a half-space or a layer problem, then one is usually faced to a problem in the domain

$$\Omega = \mathbb{R}^{n-1} \times V,$$

where $V = (0,d)$ and $d \in (0,\infty]$. Such reduced problems are often of the above type.

Of course, also problem (1.1) could be treated by a localization procedure employing an infinite partition of the unity (note that the boundary is non-compact). However, such procedures are generally extensive and take quite some pages of exhausting calculations and estimations. For this reason, here we pursue a different strategy. In fact, we essentially take advantage of the cylindrical structure of the domain and the operator and employ operator-valued multiplier theory. Roughly speaking, by this method $\mathcal{R}$-sectoriality of (1.1) in Ω is reduced to the corresponding result on the cross-section V, for which it is well-known (see e.g. [11]). This approach reveals a much shorter and more elegant way to prove the important maximal regularity for boundary value problems of type (1.1) on cylindrical domains of the form (1.2). The chosen approach also demonstrates the strength of operator-valued multiplier theory and its significance in the treatment of partial differential equations in general.

We remark that the idea of such a splitting of the variables and operators is already performed by Guidotti in [15] and [16]. In these papers the author constructs semiclassical fundamental solutions for a class of elliptic operators on cylindrical domains. This proves to be a strong tool for the treatment of related elliptic and parabolic ([15] and [16]), as well as of hyperbolic ([16]) problems. In particular, this approach leads to semiclassical representation formulas for solutions of related elliptic and parabolic boundary value problems. Based on these formulas and on a multiplier result of Amann [6] the author derives a couple of interesting results for these problems in a Besov space setting. In particular, the given applications include asymptotic behavior in the large, singular perturbations, exact boundary conditions on artificial boundaries, and the validity of maximum principles. Very recently in [13] the wellposedness of a class of parabolic boundary value problems in a vector-valued Hölder space setting is proved, when $\Omega = [0, L] \times V$, the first part is given by $A_1 = a(x_n)\partial_n^{2m}$, $x_n \in [0, L]$, and when A_2 is uniformly elliptic.

In contrast to [15], [16], and [13], here we present the L^p-approach to cylindrical boundary value problems. Therefore the notion of $\mathcal{R}$-boundedness comes into play, which is not required in the framework of Besov or Hölder spaces. Also note that in [15] and [16] $A_1 = -\Delta$ is assumed, with a remark on possible generalizations. Here we explicitly consider a wider class of first parts A_1 including higher order operators with variable coefficients. Moreover, with a Banach space E, we consider E-valued solutions and allow the coefficients of the second part A_2 to be operator-valued. Applications for equations with operator-valued coefficients are, for instance, given by coagulation-fragmentation systems (cf. [8]), spectral problems of parametrized differential operators in hydrodynamics (cf. [12]), or (homogeneous) systems in general. Albeit in this note we concentrate on the proof of maximal regularity for problems of type (1.1), we remark that further applications similar to the ones given in [15] and [16] also in the L^p-framework considered here are possible.

Note that E-valued boundary value problems in standard domains, such as $\mathbb{R}^n$, a half-space, and domains with a compact boundary were extensively studied in [11]. There a bounded H^∞-calculus and hence maximal regularity for the operator of the associated Cauchy problem is proved in the situation when E is of class $\mathcal{HT}$. The results obtained in the paper at hand also extend the maximal regularity results proved in [11] to a class of domains with non-compact boundary. For classical papers on scalar-valued boundary value problems we refer to [14], [1], [2], and [20] in the elliptic case and to [4] and [3] in the parameter-elliptic case. (For a more comprehensive list see also [11].) For an approach to a class of elliptic differential operators with Dirichlet boundary conditions in uniform C^2-domains we refer to [17] and [9]. We want to remark that all cited results above are based on standard localization procedures for the domain, contrary to the approach presented in this paper. Here we only localize a certain part of the coefficients but not the domain.

This paper is structured as follows. In Section 2 we define the notion of a cylindrical boundary value problem and give the precise statement of our main result. In Section 3 we recall the notion of parameter-ellipticity and of $\mathcal{R}$-bounded operator families. The proof of our main result Theorem 2.3 then is given in Section 4. The main steps are split in three subsections. In Subsection 4.1 we treat the corresponding operator-valued model problem, that is, here we assume (partly) constant coefficients. By a perturbation argument, in Subsection 4.2 we extend the $\mathcal{R}$-sectoriality of the model problem to slightly varying coefficients. The general case then is handled in Subsection 4.3. The statement of the main result is restricted to the case that the two parts A_1 and A_2 are of the same order. However, the same proof works for mixed order systems. This will be briefly outlined in Section 5.

2. Main result

We proceed with the precise statement of our main result.

Definition 2.1. Let $k \in \mathbb{N}$. We call $V \subset \mathbb{R}^k$ a standard domain, if it is $\mathbb{R}^k$, the half-space $\mathbb{R}^k_+ = \{x = (x_1, \dots, x_k) \in \mathbb{R}^k : \ x_k > 0\}$, or if it has compact boundary.

Let F be a Banach space and let $\Omega := \mathbb{R}^{n-k} \times V \subset \mathbb{R}^n$, where V is a standard domain in $\mathbb{R}^k$. For $x \in \Omega$ we write $x = (x^1, x^2) \in \mathbb{R}^{n-k} \times V$, whenever we want to refer to the cylindrical geometry of Ω. Accordingly, we write $\alpha = (\alpha^1, \alpha^2) \in \mathbb{N}_0^{n-k} \times \mathbb{N}_0^k$ for a multiindex $\alpha \in \mathbb{N}_0^n$. In the sequel we consider the vector-valued boundary value problem

$$\begin{aligned} \lambda u + A(x, D)u &= f \text{ in } \Omega, \\ B_j(x, D)u &= 0 \text{ on } \partial\Omega \quad (j = 1, ..., m), \end{aligned} \tag{2.1}$$

with $A(x, D) = \sum_{|\alpha| \le 2m} a_\alpha(x) D^\alpha$, $m \in \mathbb{N}$, a differential operator in the interior and operators $B_j(x, D) = \sum_{|\beta| < 2m} b_\beta(x) D^\beta$ on the boundary. Vector-valued in this context means that u is F-valued, hence derivatives have to be understood in appropriate F-valued spaces. Accordingly the coefficients of $A(\cdot, D)$ and $B_j(\cdot, D)$ are operator-valued, that is $\mathcal{L}(F)$-valued. In particular, we will consider the following class of operators.

Definition 2.2. The boundary value problem (2.1) is called *cylindrical* if the operator $A(\cdot, D)$ is represented as

$$\begin{aligned} A(x, D) &= A_1(x^1, D) + A_2(x^2, D) \\ &:= \sum_{|\alpha^1| \le 2m} a^1_{\alpha^1}(x^1) D^{(\alpha^1, 0)} + \sum_{|\alpha^2| \le 2m} a^2_{\alpha^2}(x^2) D^{(0, \alpha^2)} \end{aligned}$$

and the boundary operator is given as

$$B_j(x, D) = B_{2,j}(x^2, D) := \sum_{|\beta^2| \le m_j} b^2_{j, \beta^2}(x^2) D^{(0, \beta^2)} \quad (m_j < 2m, \ j = 1, ..., m).$$

Thus the differential operators $A(x, D)$ and $B_j(x, D)$ resolve completely into parts of which each one acts just on $\mathbb{R}^{n-k}$ or just on V.

As the $L^p(\Omega, F)$-realization of the boundary value problem

$$(A,B) := (A(\cdot,D), B_1(\cdot,D), ..., B_m(\cdot,D))$$

given by (2.1) we define for $1 < p < \infty$,

$$\begin{aligned} D(A) &:= \{u \in W^{2m,p}(\Omega, F);\ B_j(\cdot,D)u = 0 \quad (j = 1,...,m)\} \\ Au &:= A(\cdot,D)u, \quad u \in D(A). \end{aligned}$$

From now on the cross-section V is assumed to be a standard domain with C^{2m}-boundary. Furthermore, the following smoothness assumptions on the coefficients may hold:

$$\left.\begin{aligned} &a^1_{\alpha^1} \in C(\mathbb{R}^{n-k}, \mathbb{C}) \text{ for } |\alpha^1| = 2m, \quad a^1_{\alpha^1}(\infty) := \lim_{|x^1|\to\infty} a^1_{\alpha^1}(x^1) \text{ exists}, \\ &a^2_{\alpha^2} \in C(\overline{V}, \mathcal{L}(F)) \text{ for } |\alpha^2| = 2m, \quad a^2_{\alpha^2}(\infty) := \lim_{|x^2|\to\infty} a^2_{\alpha^2}(x^2) \text{ exists}, \\ &a^1_{\alpha^1} \in [L^\infty + L^{r_\nu}](\mathbb{R}^{n-k}, \mathbb{C}) \text{ for } |\alpha^1| = \nu < 2m,\ r_\nu \geq p,\ \frac{2m-\nu}{n-k} > \frac{1}{r_\nu}, \\ &a^2_{\alpha^2} \in [L^\infty + L^{r_\nu}](V, \mathcal{L}(F)) \text{ for } |\alpha^2| = \nu < 2m,\ r_\nu \geq p,\ \frac{2m-\nu}{k} > \frac{1}{r_\nu}, \\ &b^2_{j,\beta^2} \in C^{2m-m_j}(\partial V, \mathcal{L}(F)) \quad (j = 1,...,m;\ |\beta^2| \leq m_j). \end{aligned}\right\} \tag{2.2}$$

Our main result reads as follows. For a rigorous definition of maximal regularity, $\mathcal{R}$-sectoriality, and parameter-ellipticity we refer to the subsequent section (i.p. Definitions 3.2, 3.4, and 3.10).

Theorem 2.3. *Let $1 < p < \infty$, let F be a Banach space of class $\mathcal{HT}$ enjoying property (α), and let $\Omega := \mathbb{R}^{n-k} \times V \subset \mathbb{R}^n$, where V is a standard domain of class C^{2m} in $\mathbb{R}^k$. For the boundary value problem* (2.1) *we assume that*

(i) *it is cylindrical,*
(ii) *the coefficients of $A(\cdot,D)$ and $B_j(\cdot,D)$, $j = 1,\dots,m$, satisfy* (2.2),
(iii) *it is parameter-elliptic in Ω of angle $\varphi_{(A,B)} \in [0,\pi)$,*
(iv) *the boundary value problem $(A^\#(\infty,D), B_1(\cdot,D), ..., B_m(\cdot,D))$ with the limit operator $A^\#(\infty,D) := \sum_{|\alpha|=2m} a_\alpha(\infty)D^\alpha$ is parameter-elliptic in Ω with angle less or equal to $\varphi_{(A,B)}$.*

Then for each $\phi > \varphi_{(A,B)}$ there exists $\delta = \delta(\phi) \geq 0$ such that $A + \delta$ is $\mathcal{R}$-sectorial in $L^p(\Omega, F)$ with $\phi^{\mathcal{RS}}_{A+\delta} \leq \phi$ and we have

$$\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\alpha(\lambda + A + \delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ \ell \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^n,\ 0 \leq \ell + |\alpha| \leq 2m\}) < \infty. \tag{2.3}$$

By [21, Theorem 4.2] we obtain

Corollary 2.4. *Let the assumptions of Theorem* 2.3 *be given. Then the operator $A + \delta$ has maximal regularity on $L^p(\Omega, F)$.*

Example. It is not difficult to verify that problem (2.1) with $A = -\Delta$ the negative Laplacian in Ω subject to Dirichlet or Neumann boundary conditions satisfies the assumptions of Theorem 2.3.

3. $\mathcal{R}$-sectoriality and parameter-ellipticity

Throughout this article X, Y, E, and F denote Banach spaces. Given any closed operator A acting on a Banach space we denote by $D(A), \ker(A)$, and $R(A)$ domain of definition, kernel, and range of the operator and by $\rho(A)$ and $\sigma(A)$ its resolvent set and spectrum respectively. The symbol $\mathcal{L}(X,Y)$ stands for the Banach space of all bounded linear operators from X to Y equipped with operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$. As an abbreviation we set $\mathcal{L}(X) := \mathcal{L}(X,X)$.

For $p \in [1,\infty)$ and a domain $G \subset \mathbb{R}^n$, $L^p(G,F)$ denotes the F-valued Lebesgue space of all p-Bochner-integrable functions, i.e., of functions $f : G \to F$ satisfying

$$\|f\|_{L^p(G,F)} := \left(\int_G \|f(x)\|_F^p dx \right)^{\frac{1}{p}} < \infty.$$

We also write $L^\infty(G,F)$ for the space consisting of all functions f satisfying $\|f\|_\infty := \operatorname{ess\,sup}_{x\in G} \|f(x)\|_F < \infty$. The F-valued Sobolev space of order $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ is denoted by $W^{m,p}(G,F)$, that is the space of all $f \in L^p(G,F)$ whose F-valued distributional derivatives up to order m are functions in $L^p(G,F)$ again. Its norm is given by

$$\|f\|_{W^{m,p}(G,F)} := \left(\sum_{|\alpha|\le m} \|D^\alpha f\|^p_{L^p(G,F)} \right)^{\frac{1}{p}},$$

where $\alpha \in \mathbb{N}_0^n$ is a multiindex. We write $\|\cdot\|_p := \|\cdot\|_{L^p(G,F)}$ and $\|\cdot\|_{p,m} := \|\cdot\|_{W^{m,p}(G,F)}$, if no confusion seems likely. Finally, for $m \in \mathbb{N}_0 \cup \{\infty\}$, $C^m(G,F)$ denotes the space of all m-times continously differentiable functions. For general facts on vector-valued function spaces we refer to the nice booklet of Amann, [7].

Definition 3.1. A closed linear operator A in a Banach space X is called *sectorial*, if

1. $\overline{D(A)} = X$, $\ker(A) = \{0\}$, $\overline{R(A)} = X$,
2. $(-\infty, 0) \subset \rho(A)$ and there is some $C > 0$ such that $\|t(t+A)^{-1}\|_{\mathcal{L}(X)} \le C$ for all $t > 0$.

In this case it is well-known, see e.g. [11], that there exists a $\phi \in [0,\pi)$ such that the uniform estimate in 2. extends to all

$$\lambda \in \Sigma_{\pi-\phi} := \{z \in \mathbb{C}\backslash\{0\};\ |\arg(z)| < \pi - \phi\}.$$

The number

$$\phi_A := \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{\lambda\in\Sigma_{\pi-\phi}} \|\lambda(\lambda+A)^{-1}\|_{\mathcal{L}(X)} < \infty\}$$

is called *spectral angle* of A. The class of sectorial operators is denoted by $S(X)$.

Observe that $\sigma(A) \subset \overline{\Sigma}_{\phi_A}$. In case $\phi_A < \frac{\pi}{2}$, the sectorial operator A generates a bounded holomorphic C_0-semigroup on X. For a suitable treatment of related nonlinear problems, however, the generation of a holomorphic semigroup might not be enough. Then the stronger property of maximal regularity is required which is defined as follows.

Definition 3.2. Let $1 \le p \le \infty$, let X be a Banach space, and let $A : D(A) \to X$ be closed and densely defined. Then A is said to have (L^p-) *maximal regularity*, if for each $f \in L^p(\mathbb{R}_+, X)$ there is a unique solution $u : \mathbb{R}_+ \to D(A)$ of the Cauchy problem

$$\begin{cases} u' + Au &= f \quad \text{in } \mathbb{R}_+, \\ u(0) &= 0, \end{cases}$$

satisfying the estimate

$$\|u'\|_{L^p(\mathbb{R}_+,X)} + \|Au\|_{L^p(\mathbb{R}_+,X)} \le C\|f\|_{L^p(\mathbb{R}_+,X)}$$

with a $C > 0$ independent of $f \in L^p(\mathbb{R}_+, X)$.

If the Banach space X is of class $\mathcal{HT}$ (see Definition 3.6), by [21, Theorem 4.2] it is well known that the property of having maximal regularity is equivalent to the $\mathcal{R}$-sectoriality of an operator A. This concept is based on the notion of $\mathcal{R}$-bounded operator families, which we introduce next. We refer to [11] and [18] for a comprehensive introduction to the notion of $\mathcal{R}$-bounded operator families and restrict ourselves here to the definition.

Definition 3.3. A familiy $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-*bounded*, if there exist a $C > 0$ and a $p \in [1, \infty)$ such that for all $N \in \mathbb{N}, T_j \in \mathcal{T}, x_j \in X$ and all independent symmetric $\{-1, 1\}$-valued random variables ε_j on a probability space $(G, \mathcal{M}, P)$ for $j = 1, ..., N$, we have that

$$\|\sum_{j=1}^{N} \varepsilon_j T_j x_j\|_{L^p(G,Y)} \le C\|\sum_{j=1}^{N} \varepsilon_j x_j\|_{L^p(G,X)}. \tag{3.1}$$

The smallest $C > 0$ such that (3.1) is satisfied is called $\mathcal{R}$-bound of $\mathcal{T}$ and denoted by $\mathcal{R}(\mathcal{T})$.

Definition 3.4. A closed operator A in X satisfying condition 1. of Definition 3.1 is called $\mathcal{R}$-*sectorial*, if there exist an angle $\phi \in [0, \pi)$ and a constant $C_\phi > 0$ such that

$$\mathcal{R}(\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\phi}\}) \le C_\phi. \tag{3.2}$$

The class of $\mathcal{R}$-sectorial operators is denoted by $\mathcal{RS}(X)$ and we call $\phi_A^{\mathcal{R}S}$ given as the infimum over all angles ϕ such that (3.2) holds the $\mathcal{R}$-*angle* of A.

We remark that in general $\mathcal{R}$-boundedness is stronger than the uniform boundedness with respect to the operator norm. Therefore $\mathcal{R}$-sectoriality always implies the sectoriality of an operator A and we have

$$\phi_A \le \phi_A^{\mathcal{R}S}.$$

We will use the following two results on $\mathcal{R}$-boundedness frequently in subsequent proofs. The first one shows that $\mathcal{R}$-bounds behave as uniform bounds concerning sums and products. This follows as a direct consequence of the definition of $\mathcal{R}$-boundedness. The second one is known as the contraction principle of Kahane. A proof can be found in [18] or [11].

Lemma 3.5. *a) Let X,Y, and Z be Banach spaces and let $\mathcal{T},\mathcal{S} \subset \mathcal{L}(X,Y)$ as well as $\mathcal{U} \subset \mathcal{L}(Y,Z)$ be $\mathcal{R}$-bounded. Then $\mathcal{T}+\mathcal{S} \subset \mathcal{L}(X,Y)$ and $\mathcal{U}\mathcal{T} \subset \mathcal{L}(X,Z)$ are $\mathcal{R}$-bounded as well and we have*

$$\mathcal{R}(\mathcal{T}+\mathcal{S}) \leq \mathcal{R}(\mathcal{S}) + \mathcal{R}(\mathcal{T}), \quad \mathcal{R}(\mathcal{U}\mathcal{T}) \leq \mathcal{R}(\mathcal{U})\mathcal{R}(\mathcal{T}).$$

Furthermore, if $\overline{\mathcal{T}}$ denotes the closure of $\mathcal{T}$ with respect to the strong operator topology, then we have $\mathcal{R}(\overline{\mathcal{T}}) = \mathcal{R}(\mathcal{T})$.

b) [contraction principle of Kahane]

Let $p \in [1,\infty)$. Then for all $N \in \mathbb{N}, x_j \in X, \varepsilon_j$ as above, and for all $a_j, b_j \in \mathbb{C}$ with $|a_j| \leq |b_j|$ for $j = 1,\dots,N$,

$$\|\sum_{j=1}^{N} a_j\varepsilon_j x_j\|_{L^p(G,X)} \leq 2\|\sum_{j=1}^{N} b_j\varepsilon_j x_j\|_{L^p(G,X)} \tag{3.3}$$

holds.

Let E be a Banach space and let $\mathcal{S}(\mathbb{R}^n,E)$ denote the Schwartz space of all rapidly decreasing E-valued functions and let $\mathcal{S}'(\mathbb{R}^n,E) := \mathcal{L}(\mathcal{S}(\mathbb{R}^n,\mathbb{C}),E)$. Then the E-valued Fourier transform

$$\mathcal{F}\varphi(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\varphi(x)dx$$

defines an isomorphism of the space $\mathcal{S}(\mathbb{R}^n,E)$ which extends by duality to the larger space $\mathcal{S}'(\mathbb{R}^n,E)$. Given two Banach spaces E_1,E_2 and any operator-valued function $m \in L^\infty(\mathbb{R}^n,\mathcal{L}(E_1,E_2))$, we may define the operator

$$T_m : \mathcal{S}(\mathbb{R}^n,E_1) \to \mathcal{S}'(\mathbb{R}^n,E_2); \quad T_m\varphi := \mathcal{F}^{-1}m\mathcal{F}\varphi.$$

We say m defines an operator-valued Fourier multiplier, if T_m extends to a bounded operator

$$T_m : \ L^p(\mathbb{R}^n,E_1) \to L^p(\mathbb{R}^n,E_2).$$

In order to state the operator-valued multiplier result our approach is based on, two further notions from Banach space geometry are required.

Definition 3.6. a) The Hilbert transform $H : \mathcal{S}(\mathbb{R},E) \to \mathcal{S}'(\mathbb{R},E)$ is given by $Hf := \mathcal{F}^{-1}m\mathcal{F}f$ where $m(\xi) := \frac{i\xi}{|\xi|}$. The Banach space E is of class $\mathcal{HT}$ or, equivalently, a UMD space, if there exists a $q \in (1,\infty)$ such that H extends to a bounded operator on $L^q(\mathbb{R},E)$. In other words, $m_E := m \cdot id_E$ is an operator-valued (one variable) Fourier multiplier.

b) A Banach space E is said to have property (α), if there exists a $C > 0$ such that for all $n \in \mathbb{N}, \alpha_{ij} \in \mathbb{C}$ with $|\alpha_{ij}| \leq 1$, all $x_{ij} \in E$, and all independent symmetric

$\{-1,1\}$-valued random variables ε_i^1 on a probability space $(G_1,\mathcal{M}_1,P_1)$ and ε_j^2 on a probability space $(G_2,\mathcal{M}_2,P_2)$ for $i,j=1,...,N$, we have that

$$\int_{G_1}\int_{G_2}\|\sum_{i,j=1}^N \varepsilon_i^1(u)\varepsilon_j^2(v)\alpha_{ij}x_{ij}\|_E dudv \le C\int_{G_1}\int_{G_2}\|\sum_{i,j=1}^N \varepsilon_i^1(u)\varepsilon_j^2(v)x_{ij}\|_E dudv.$$

By Plancherel's theorem Hilbert spaces are of class $\mathcal{HT}$. Besides that, it is well-known that the spaces $L^p(G,F)$ are of class $\mathcal{HT}$ provided that $1<p<\infty$ and that F is of class $\mathcal{HT}$. Moreover, $\mathbb{C}^n$ and the spaces $L^p(G,F)$ enjoy property (α) for $1\le p<\infty$, if F does so (cf. [18]).

We are now in position to state the mentioned operator-valued Fourier multiplier theorem. For a proof we refer to [18, Proposition 5.2].

Proposition 3.7. *Let E_1,E_2 be Banach spaces of class $\mathcal{HT}$ with property (α), $1<p<\infty$, and set $X_i:=L^p(\mathbb{R}^n,E_i)$, $i=1,2$. Given any set Λ, let $m_\lambda\in C^n(\mathbb{R}^n\backslash\{0\},\mathcal{L}(E_1,E_2))$ for $\lambda\in\Lambda$ and assume that*

$$\mathcal{R}(\{\xi^\alpha D^\alpha m_\lambda(\xi);\ \xi\in\mathbb{R}^n\backslash\{0\},\ \lambda\in\Lambda,\alpha\in\{0,1\}^n\})\le C_m<\infty.$$

Then for all $\lambda\in\Lambda$ we have

$$T_\lambda:=\mathcal{F}^{-1}m_\lambda\mathcal{F}\in\mathcal{L}(X_1,X_2)$$

and that

$$\mathcal{R}(\{T_\lambda;\ \lambda\in\Lambda\})\le C(n,p,E_1,E_2)C_m<\infty.$$

Next we recall the notion of parameter-ellipticity from [11]. Let F be a Banach space, $G\subset\mathbb{R}^n$ be a domain, and set

$$A(x,D):=\sum_{|\alpha|\le 2m}a_\alpha(x)D^\alpha,$$

where $m\in\mathbb{N}$, $\alpha\in\mathbb{N}_0^n$, and $a_\alpha:G\to\mathcal{L}(F)$. For $\lambda\in\mathbb{C}$ and boundary operators

$$B_j(x,D):=\sum_{|\beta|\le m_j}b_{j,\beta}(x)D^\beta,$$

where $m_j<2m$, $\beta\in\mathbb{N}_0^n$, and $b_{j,\beta}:\partial G\to\mathcal{L}(F)$ for $j=1,...,m$, we consider the boundary value problem

$$\begin{aligned}\lambda u+A(x,D)u &= f \text{ in } G,\\ B_j(x,D)u &= 0 \text{ on } \partial G \quad (j=1,...,m).\end{aligned}\tag{3.4}$$

Definition 3.8. Let F be a Banach space, $G\subset\mathbb{R}^n$, $m\in\mathbb{N}$, and $a_\alpha\in\mathcal{L}(F)$. The $\mathcal{L}(F)$-valued homogeneous polynomial

$$a(\xi):=\sum_{|\alpha|=2m}a_\alpha\xi^\alpha\quad(\xi\in\mathbb{R}^n)$$

is called *parameter-elliptic*, if there exists an angle $\phi\in[0,\pi)$ such that the spectrum $\sigma(a(\xi))$ of $a(\xi)$ in $\mathcal{L}(F)$ satisfies

$$\sigma(a(\xi))\subset\Sigma_\phi\quad(\xi\in\mathbb{R}^n,\ |\xi|=1).\tag{3.5}$$

Then

$$\varphi := \inf\{\phi : \text{ (3.5) holds}\}$$

is called *angle of ellipticity* of a.
A differential operator $A(x, D) := \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha$ with coefficients $a_\alpha : G \to \mathcal{L}(F)$ is called parameter-elliptic in G with angle of ellipticity φ, if the principal part of its symbol

$$a^\#(x,\xi) := \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha$$

is parameter-elliptic with this angle of ellipticity for all $x \in \overline{G}$.

Definition 3.9. Let F be a Banach space and let $G \subset \mathbb{R}^n$ be a C^1-domain. Let $a_\alpha : G \to \mathcal{L}(F)$ and $b_{j,\beta} : \partial G \to \mathcal{L}(F)$. Set $B_j^\#(x, D) := \sum_{|\beta|=m_j} b_{\beta,j}(x)D^\beta$ and let $A^\#(x, D) := \sum_{|\alpha|=2m} a_\alpha(x)D^\alpha$ be parameter-elliptic in G of angle of ellipticity $\varphi \in [0, \pi)$. For each $x_0 \in \partial G$ we write the boundary value problem in local coordinates about x_0. The boundary value problem (3.4) is said to satisfy the *Lopatinskii-Shapiro condition*, if for each $\phi > \varphi$ the ODE on $\mathbb{R}_+$

$$\begin{aligned} (\lambda + A^\#(x_0, \xi', D_{x_n}))v(x_n) &= 0, \quad x_n > 0, \\ B_j^\#(x_0, \xi', D_{x_n})v(0) &= h_j, \quad j = 1, ..., m, \\ v(x_n) &\to 0, \quad x_n \to \infty, \end{aligned}$$

has a unique solution $v \in C((0,\infty), F)$ for each $(h_1, ..., h_m)^T \in F^m$ and each $\lambda \in \overline{\Sigma}_{\pi-\phi}$ and $\xi' \in \mathbb{R}^{n-1}$ with $|\xi'| + |\lambda| \neq 0$.

We refer to [22] for an introduction to the Lopatinskii-Shapiro condition for scalar-valued boundary value problems and to [11] for an extensive treatment of the F-valued case. Parameter-ellipticity of a boundary value problem now reads as follows.

Definition 3.10. The boundary value problem (A, B) given through (3.4) is called parameter-elliptic in G of angle $\varphi \in [0, \pi)$, if $A(\cdot, D)$ is parameter-elliptic in G of angle $\varphi \in [0, \pi)$ and if the Lopatinskii-Shapiro condition holds. To indicate that φ is the angle of ellipticity of the boundary value problem (A, B) we use the subscript notation $\varphi_{(A,B)}$.

4. Proof of the main result

We denote by

$$\begin{aligned} D(A_2) &:= \{u \in W^{2m,p}(V, F);\ B_{2,j}(\cdot, D)u = 0\ (j = 1, ..., m)\} \\ A_2 u &:= A_2(\cdot, D)u, \quad u \in D(A_2), \end{aligned}$$

the $L^p(V,F)$-realization of the induced boundary value problem

$$\begin{aligned} \lambda u + A_2(x^2,D)u &= f \text{ in } V, \\ B_{2,j}(x^2,D)u &= 0 \text{ on } \partial V \quad (j=1,...,m), \end{aligned} \tag{4.1}$$

on the cross-section V of Ω. As the original boundary value problem (2.1) is assumed to be parameter-elliptic with ellipticity angle $\varphi_{(A,B)} \in [0,\pi)$, it is easy to see that the same is valid for the boundary value problem (4.1) and that the corresponding angle $\varphi_{(A_2,B_2)}$ is no larger than $\varphi_{(A,B)}$. By employing finite open coverings of $\overline{V}$, in [11] the following result is proved.

Proposition 4.1. *Let $V \subset \mathbb{R}^k$ be a standard domain of class C^{2m}. Given the assumptions* (2.2) *on the coefficients a^2 and b^2, for each $\phi > \varphi_{(A_2,B_2)}$ there exists a $\delta_2 = \delta_2(\phi) \geq 0$ such that $A_2 + \delta_2 \in \mathcal{R}S(L^p(V,F))$ with $\phi^{\mathcal{R}S}_{A_2+\delta_2} \leq \phi$. Moreover, we have*

$$\mathcal{R}(\{\lambda^{1-\frac{|\gamma|}{2m}} D^\gamma (\lambda + A_2 + \delta_2)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \leq |\gamma| \leq 2m\}) < \infty. \tag{4.2}$$

Remark 4.2. In [11] just the case $k \geq 2$ is treated, whereas the case $k = 1$ is well-known.

From the definition it is clear that the coefficients of the cylindrical parts A_1 and A_2 of A only depend on x^1 or x^2, respectively. For the sake of simplicity we therefore drop the special indications for x, if no confusion seems likely. To be precise we write

$$A_i(x^i,D) = A_i(x,D) = \sum_{|\alpha|\leq 2m} a^i_\alpha(x) D^\alpha$$

for $i = 1,2$, where

$$a^1_\alpha(x) = \begin{cases} 0 & , \quad \alpha_2 \neq 0, \\ a^1_{\alpha^1}(x^1), & \quad \alpha_2 = 0, \end{cases}$$

$$a^2_\alpha(x) = \begin{cases} 0 & , \quad \alpha_1 \neq 0, \\ a^2_{\alpha^2}(x^2), & \quad \alpha_1 = 0. \end{cases}$$

Further we set $E := L^p(\Omega,F)$ and $X := L^p(\mathbb{R}^{n-k},E) \cong L^p(\Omega,F)$. Given an operator $T : D(T) \subset E \to E$, its canonical extension is defined by

$$\begin{aligned} D(\tilde{T}) &:= L^p(\mathbb{R}^{n-k}, D(T)) \\ (\tilde{T}u)(x) &:= T(u(x)), \quad u \in D(\tilde{T}),\ x \in \mathbb{R}^{n-k}. \end{aligned}$$

4.1. Constant coefficients a^1_α

In the first step we consider the model problem for the cylindrical boundary value problem (2.1), i.e., we assume $A_1(x,D)$ on $\mathbb{R}^{n-k}$ to be given as homogeneous differential operator

$$A_1(D) := \sum_{|\alpha|=2m} a^1_\alpha D^\alpha$$

with constant coefficients $a^1_\alpha \in \mathbb{C}$. Let A_1 denote its realization in X with domain $D(A_1) := W^{2m,p}(\mathbb{R}^{n-k}, E)$. We set

$$A_0(\cdot, D) := A_1(D) + A_2(\cdot, D)$$

and

$$A_0 := A_1 + \tilde{A}_2, \quad D(A_0) := D(A_1) \cap D(\tilde{A}_2).$$

Note that no further restrictions on $A_2(x, D)$ have to be assumed.

Since it will always be clear from the context what we mean, from now on we do not distinguish between $\tilde{A}_2$ and A_2. In other words, from this point on we drop again the tilde notation and just write A_2 for simplicity.

Let $\phi > \varphi_{(A_0,B)}$, $\lambda \in \Sigma_{\pi-\phi}$ and $u \in \mathcal{S}(\mathbb{R}^{n-k}, D(A_2)) \subset D(A_0)$. Applying E-valued Fourier transform $\mathcal{F}$ to $f := (\lambda + A_0 + \delta_2)u$ gives us

$$(\lambda + a_1(\cdot) + A_2 + \delta_2)\mathcal{F}u = \mathcal{F}f.$$

Hence we formally have

$$u = \mathcal{F}^{-1} m^0_\lambda \mathcal{F} f,$$

where m^0_λ is given by the operator-valued symbol

$$m^0_\lambda(\xi) := (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1}, \quad \xi \in \mathbb{R}^{n-k}.$$

Note that $m^0_\lambda \in C^\infty(\mathbb{R}^{n-k}, \mathcal{L}(E))$ is well-defined if

$$-(\lambda + a_1(\xi)) \in \rho(A_2 + \delta_2) \quad (\xi \in \mathbb{R}^{n-k}).$$

In view of $\varphi_{(A_2,B_2)} \le \varphi_{(A_0,B)}$ and Proposition 4.1 this is obviously satisfied in case that $\lambda + a_1(\xi) \in \Sigma_{\pi-\phi}$. This, however, follows directly from the parameter-ellipticity of $A_1(D)$, which is obtained as an immediate consequence of the parameter-ellipticity of (A_0, B), and since the ellipticity angle φ_{A_1} of A_1 fullfills $\varphi_{A_1} \le \varphi_{(A_0,B)} < \phi$.

In order to obtain

$$(\lambda + A_0 + \delta_2)^{-1} = \mathcal{F}^{-1} m^0_\lambda \mathcal{F} f \in \mathcal{L}(X),$$

the idea is to apply the operator-valued multiplier result of Proposition 3.7 to m^0_λ. For this purpose, we next establish suitable representation formulas for derivatives of m^0_λ.

Lemma 4.3. *Let $\phi > \varphi_{(A_0,B)}$. Given $\alpha \in \{0,1\}^{n-k}$, let*

$$\mathcal{Z}_\alpha := \left\{ \mathcal{W} = (\omega^1, ..., \omega^r) \in (\{0,1\}^{n-k})^r;\ r \le n-k,\ \omega^j \ne 0, \sum_{j=1}^r \omega^j = \alpha \right\}$$

denote the set of all additive decompositions of α into $r = r_{\mathcal{W}}$ many positive multiindices. Then, with $C_{\mathcal{W}} := (-1)^r r!$, the formula

$$\xi^\alpha D^\alpha m^0_\lambda(\xi) = (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \cdot \sum_{\mathcal{W} \in \mathcal{Z}_\alpha} C_{\mathcal{W}} \left(\prod_{j=1}^r \xi^{\omega^j} D^{\omega^j} a_1(\xi) \right) (\lambda + a_1(\xi) + A_2 + \delta_2)^{-r}$$

holds for all $\lambda \in \Sigma_{\pi-\phi}$ *and* $\xi \in \mathbb{R}^{n-k}$.

Proof. Let $|\alpha| = 1$. Then there exists $i \in \{1, ..., n-k\}$ such that $\alpha = e_i$. In this case $\mathcal{Z}_\alpha = \{(\alpha)\}$ and we get immediatly

$$\begin{aligned}
&\xi_i D_i m_\lambda^0(\xi) \\
&= -\xi_i (D_i a_1)(\xi)(\lambda + a_1(\xi) + A_2 + \delta)^{-2} \\
&= (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1}(-1)\xi^\alpha (D^\alpha a_1)(\xi)(\lambda + a_1(\xi) + A_2 + \delta)^{-1}.
\end{aligned}$$

Now assume the statement to be true for $\alpha \in \{0,1\}^{n-k}$ with $|\alpha| < n-k$. Then for $l \in \{1, ..., n-k\}$ such that $\alpha_l = 0$ we obtain

$$\begin{aligned}
&\xi_l \xi^\alpha D_l D^\alpha m_\lambda^0(\xi) \\
&= \xi_l D_l \sum_{\mathcal{W} \in \mathcal{Z}_\alpha} C_\mathcal{W} \left(\prod_{j=1}^{r_\mathcal{W}} \xi^{\omega^j} D^{\omega^j} a_1(\xi) \right) (\lambda + a_1(\xi) + A_2 + \delta_2)^{-(1+r_\mathcal{W})} \\
&= \xi_l \sum_{\mathcal{W} \in \mathcal{Z}_\alpha} C_\mathcal{W} \\
&\quad \cdot \left[\sum_{i=1}^{r_\mathcal{W}} \xi^{\omega^i} (D_l D^{\omega^i} a_1)(\xi) \left(\prod_{j \neq i} \xi^{\omega^j} D^{\omega^j} a_1(\xi) \right) (\lambda + a_1(\xi) + A_2 + \delta_2)^{-(1+r_\mathcal{W})} \right. \\
&\quad \left. + \left(\prod_j \xi^{\omega^j} D^{\omega^j} a_1(\xi) \right) (-(1 + r_\mathcal{W}))(D_l a_1)(\xi)(\lambda + a_1(\xi) + A_2 + \delta_2)^{-(2+r_\mathcal{W})} \right] \\
&= (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \\
&\quad \cdot \sum_{\mathcal{W} \in \mathcal{Z}_{\alpha+e_l}} C_\mathcal{W} \left(\prod_{j=1}^{r_\mathcal{W}} \xi^{\omega^j} D^{\omega^j} a_1(\xi) \right) (\lambda + a_1(\xi) + A_2 + \delta_2)^{-r_\mathcal{W}}.
\end{aligned}$$

□

In the sequel we denote by $(\beta, \gamma) \in \mathbb{N}_0^{n-k} \times \mathbb{N}_0^k$ a multiindex such that β is the part corresponding to the variables $x^1 \in \mathbb{R}^{n-k}$ and γ corresponding to the variables $x^2 \in V$. In order to obtain the general estimate (2.3) for the full operator A, we also have to consider the more involved symbols

$$m_\lambda(\xi) := \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta D^\gamma m_\lambda^0(\xi) = \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1}$$

for $\lambda \in \Sigma_{\pi-\phi}$, $\xi \in \mathbb{R}^{n-k}$, and $|\beta| + |\gamma| \le 2m$.

Lemma 4.4. *Let* $\phi > \varphi_{(A_0,B)}$. *For* $\alpha \in \{0,1\}^{n-k}$ *we have*

$$\begin{aligned}
\xi^\alpha D^\alpha m_\lambda(\xi) &= \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \\
&\quad \cdot \sum_{\alpha' \le \alpha} C_{\alpha',\beta} \sum_{\mathcal{W} \in \mathcal{Z}_{\alpha-\alpha'}} C_\mathcal{W} \prod_{j=1}^{r} \left(\xi^{\omega^j} (D^{\omega^j} a_1)(\xi)(\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \right),
\end{aligned}$$

for all $\lambda \in \Sigma_{\pi-\phi}$, $\xi \in \mathbb{R}^{n-k}$, and $(\beta,\gamma) \in \mathbb{N}_0^{n-k} \times \mathbb{N}_0^k$ such that $|\beta|+|\gamma| \le 2m$, and with certain constants $C_{\alpha',\beta} \in \mathbb{Z}$.

Proof. We first show

$$\xi^\alpha D^\alpha m_\lambda(\xi) = \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta \sum_{\alpha' \le \alpha} (\prod_{i;\alpha'_i \neq 0} \beta_i) \xi^{\alpha-\alpha'} D^{\alpha-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1}. \tag{4.3}$$

Let $\alpha = e_i$ for some $i \in \{1, ..., n-k\}$. Then

$$\begin{aligned}
&\xi_i D_i m_\lambda(\xi) \\
&= \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta \left(\beta_i D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} + \xi_i D_i D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1}\right)
\end{aligned}$$

already proves (4.3) for the case $|\alpha| = 1$. Assume the statement to be true for $\alpha \in \{0,1\}^{n-k}$ with $|\alpha| < n-k$. For $l \in \{1, ..., n-k\}$ such that $\alpha_l = 0$ we have

$$\begin{aligned}
&\xi_l \xi^\alpha D_l D^\alpha m_\lambda(\xi) \\
&= \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi_l D_l \sum_{\alpha' \le \alpha} (\prod_{i;\alpha'_i \neq 0} \beta_i) \xi^\beta \xi^{\alpha-\alpha'} D^{\alpha-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \\
&= \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \Bigg(\sum_{\alpha' \le \alpha} (\prod_{i;\alpha'_i \neq 0} \beta_i) \beta_l \xi^\beta \xi^{\alpha-\alpha'} D^{\alpha-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \\
&\qquad + \sum_{\alpha' \le \alpha} (\prod_{i;\alpha'_i \neq 0} \beta_i) \xi^\beta \xi^{\alpha+e_l-\alpha'} D^{\alpha+e_l-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \Bigg) \\
&= \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta \\
&\quad \Bigg(\sum_{\alpha' \le \alpha+e_l;\alpha'_l=1} (\prod_{i;\alpha'_i \neq 0} \beta_i) \xi^{\alpha+e_l-\alpha'} D^{\alpha+e_l-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \\
&\qquad + \sum_{\alpha' \le \alpha+e_l;\alpha'_l=0} (\prod_{i;\alpha'_i \neq 0} \beta_i) \xi^{\alpha+e_l-\alpha'} D^{\alpha+e_l-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \Bigg) \\
&= \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta \sum_{\alpha' \le \alpha+e_l} (\prod_{i;\alpha'_i \neq 0} \beta_i) \xi^{\alpha-\alpha'} D^{\alpha-\alpha'} D^\gamma (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1}.
\end{aligned}$$

This proves (4.3). Setting $C_{\alpha',\beta} := \prod_{i;\alpha'_i \neq 0} \beta_i$ and applying Lemma 4.3 now yields

$$\xi^\alpha D^\alpha m_\lambda(\xi) = \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta D^\gamma \sum_{\alpha' \le \alpha} C_{\alpha',\beta}$$

$$\cdot \sum_{\mathcal{W}\in\mathcal{Z}_{\alpha-\alpha'}} C_{\mathcal{W}} \left(\prod_{j=1}^{r} \xi^{\omega^j} D^{\omega^j} a_1(\xi)\right) D^\gamma(\lambda + a_1(\xi) + A_2 + \delta_2)^{-(r+1)}$$

$$= \lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta D^\gamma(\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \sum_{\alpha'\le\alpha} C_{\alpha',\beta}$$

$$\cdot \sum_{\mathcal{W}\in\mathcal{Z}_{\alpha-\alpha'}} C_{\mathcal{W}} \prod_{j=1}^{r} \left(\xi^{\omega^j} (D^{\omega^j} a_1)(\xi)(\lambda + a_1(\xi) + A_2 + \delta_2)^{-1}\right).$$

This proves the assertion. □

With the above formulas at hand we can prove $\mathcal{R}$-sectoriality for the model problem.

Proposition 4.5. *For each $\phi > \varphi_{(A_0,B)}$ we have $A_0 + \delta_2 \in \mathcal{R}S(X)$ with $\delta_2 = \delta_2(\phi)$ as in Proposition 4.1. Moreover, $\phi^{\mathcal{R}S}_{A_0+\delta_2} \le \phi$ and it holds that*

$$\mathcal{R}(\{\lambda^{1-\frac{|\beta|+|\gamma|}{2m}} D^\beta D^\gamma(\lambda + A_0 + \delta_2)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le |\beta| + |\gamma| \le 2m\}) < \infty. \quad (4.4)$$

Furthermore, the domain of A_0 is given as

$$D(A_0) = L^p(\mathbb{R}^{n-k}, D(A_2)) \cap \bigcap_{j=1}^{2m} W^{j,p}(\mathbb{R}^{n-k}, W^{2m-j,p}(V, F)).$$

Proof. Let $\phi > \varphi_{(A_0,B)}$. We show that m_λ fulfills the assumptions of the multiplier result Proposition 3.7, i.e., that

$$\mathcal{R}(\{\xi^\alpha D^\alpha m_\lambda(\xi);\ \xi \in \mathbb{R}^{n-k},\ \lambda \in \Sigma_{\pi-\phi},\ \alpha \in \{0,1\}^{n-k}\}) < \infty.$$

As $\mathcal{R}$-boundedness by virtue of Lemma 3.5 is preserved under summation and composition of $\mathcal{R}$-bounded operator families, it suffices by Lemma 4.4 to prove that

$$\mathcal{R}(\{\lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta D^\gamma(\lambda + a_1(\xi) + A_2 + \delta_2)^{-1};$$
$$\xi \in \mathbb{R}^{n-k},\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le |\beta| + |\gamma| \le 2m\}) < \infty$$

and that

$$\mathcal{R}(\{\xi^\alpha (D^\alpha a_1)(\xi)(\lambda + a_1(\xi) + A_2 + \delta_2)^{-1};$$
$$\xi \in \mathbb{R}^{n-k},\ \lambda \in \Sigma_{\pi-\phi},\ \alpha \in \{0,1\}^{n-k}\}) < \infty.$$

Thanks to (4.2) this follows by the contraction principle of Kahane if we can show that both

$$\kappa_1(\lambda, \xi) := \frac{\lambda^{1-\frac{|\beta|+|\gamma|}{2m}} \xi^\beta}{(\lambda + a_1(\xi))^{1-\frac{|\gamma|}{2m}}}$$

and

$$\kappa_2(\lambda, \xi) := \frac{\xi^\alpha D^\alpha a_1(\xi)}{\lambda + a_1(\xi)}$$

are uniformly bounded in $(\lambda,\xi) \in \Sigma_{\pi-\phi} \times \mathbb{R}^{n-k}$. To see this, we first observe that

$$\kappa_1(s^{2m}\lambda, s\xi) = \kappa_1(\lambda,\xi) \quad (s>0),$$

hence that κ_1 is quasi-homogeneous of degree zero. We set

$$K := \{(\lambda,\xi) \in \overline{\Sigma}_{\pi-\phi} \times \mathbb{R}^{n-k} : \ |\lambda| + |\xi|^{2m} = 1\}. \tag{4.5}$$

By the ellipticity condition, we obtain $a_1(\xi) \in \overline{\Sigma}_{\varphi_{A_1}}$ for all $\xi \in \mathbb{R}^{n-k} \setminus \{0\}$. Since $\varphi_{A_1} < \phi$, it therefore easily follows that

$$\lambda + a_1(\xi) \neq 0 \quad \text{on } K.$$

Consequently, κ_1 is a continuous function on the compact set K and we obtain

$$|\kappa_1(\lambda,\xi)| \leq M \quad ((\lambda,\xi) \in K).$$

By the quasi-homogenity of κ_1 this implies

$$|\kappa_1(s^{2m}\lambda, s\xi)| \leq M \quad ((\lambda,\xi) \in K, \ s>0).$$

We have

$$|s^{2m}\lambda| + |s\xi|^{2m} = s^{2m}(|\lambda| + |\xi|^{2m}).$$

Thus, if we set $s = (|\lambda| + |\xi|^{2m})^{-1/2m}$ we deduce

$$(s^{2m}\lambda, s\xi) \in K \quad ((\lambda,\xi) \in \Sigma_{\pi-\phi} \times \mathbb{R}^{n-k})$$

and therefore that

$$|\kappa_1(\lambda,\xi)| = |\kappa_1(s^{2m}\lambda, s\xi)| \leq M \quad ((\lambda,\xi) \in \Sigma_{\pi-\phi} \times \mathbb{R}^{n-k}).$$

The uniform boundedness of κ_2 can be proved in exactly the same way. By applying Proposition 3.7, relation (4.4) follows.

In particular, we have

$$(\lambda + A_0 + \delta_2)^{-1} = \mathcal{F}^{-1} m_\lambda^0 \mathcal{F} f \in \mathcal{L}(X)$$

and

$$D(A_0) \subset \bigcap_{j=1}^{2m} W^{j,p}(\mathbb{R}^{n-k}, W^{2m-j,p}(V,F)).$$

Furthermore, we can represent the resolvent applied to $f \in \mathcal{S}(\mathbb{R}^{n-k}, E)$ as a Bochner integral via

$$(\lambda + A_0 + \delta_2)^{-1} f(x^1) = \frac{1}{(2\pi)^{(n-k)/2}} \int_{\mathbb{R}^{n-k}} e^{ix^1\cdot\xi} (\lambda + a_1(\xi) + A_2 + \delta_2)^{-1} \mathcal{F} f(\xi) d\xi.$$

Since taking the trace acts as a bounded operator on E, it commutes with the integral sign. This yields

$$B_{2,j}(\lambda + A_0 + \delta_2)^{-1} f = 0 \quad (f \in \mathcal{S}(\mathbb{R}^{n-k}, E)).$$

Employing a density argument we conclude that

$$D(A_0) = L^p(\mathbb{R}^{n-k}, D(A_2)) \cap \bigcap_{j=1}^{2m} W^{j,p}(\mathbb{R}^{n-k}, W^{2m-j,p}(V,F)).$$

Assuming that $(A_0 + \delta_2)u = 0$ for $u \in D(A_0)$ next implies that

$$(a_1(\xi) + A_2 + \delta_2)\mathcal{F}u(\xi) = 0 \quad (\xi \in \mathbb{R}^{n-k}).$$

Since $A_2 + \delta_2$ is sectorial and a_1 parameter-elliptic this yields $\mathcal{F}u = 0$, hence $u = 0$. By permanence properties for sectorial operators, i.e. in this case for $A_2 + \delta_2$, we obtain that the same is true for the dual operator of $A_0 + \delta_2$. This implies that $A_0 + \delta_2$ is injective and has dense range. Hence we have proved that $A_0 + \delta_2 \in \mathcal{R}S(X)$. □

4.2. Slightly varying coefficients a^1_α

By a perturbation argument in this paragraph we generalize the $\mathcal{R}$-sectoriality for constant coefficients to the case of slightly varying coefficients of A_1. To this end, we will employ the following perturbation result which is based on a standard Neuman series argument.

Lemma 4.6. *Let R be a linear operator in X such that $D(A_0) \subset D(R)$ and let δ_2 be given as in Proposition* 4.5. *Assume that there are $\eta > 0$ and $\delta > \delta_2$ such that*

$$\|Rx\|_X \le \eta \|(A_0 + \delta)x\|_X \quad (x \in D(A)).$$

Then $A_0 + R + \delta \in \mathcal{R}S(X)$, $\phi^{\mathcal{R}S}_{A_0+R+\delta} \le \phi^{\mathcal{R}S}_{A_0+\delta_2}$, and for every $\phi > \varphi_{(A_0,B)}$ we have

$$\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma(\lambda + A_0 + R + \delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le \ell + |\beta| + |\gamma| \le 2m\}) < \infty, \quad (4.6)$$

whenever $\eta < \mathcal{R}(\{(A_0 + \delta)(\lambda + A_0 + \delta)^{-1}\})^{-1}$.

Proof. As

$$\begin{aligned} \|R(\lambda + A_0 + \delta)^{-1}\|_{\mathcal{L}(X)} &\le \eta \|(A_0 + \delta)(\lambda + A_0 + \delta)^{-1}\|_{\mathcal{L}(X)} \\ &\le \eta \mathcal{R}(\{(A_0 + \delta)(\lambda + A_0 + \delta)^{-1}\}) \\ &< 1 \end{aligned}$$

by assumption, we see that

$$\lambda + A_0 + R + \delta = \Big(1 + R(\lambda + A_0 + \delta)^{-1}\Big)(\lambda + A_0 + \delta)$$

is invertible. This implies

$$\begin{aligned} &\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma(\lambda + A_0 + R + \delta)^{-1} \\ &= \lambda^{\frac{\ell}{2m}} D^\beta D^\gamma(\lambda + A_0 + \delta)^{-1} \sum_{j=0}^{\infty} (-R(\lambda + A_0 + \delta)^{-1})^j. \end{aligned}$$

By assumption we have $\delta_0 := \delta - \delta_2 > 0$. The fact that

$$|\lambda + \delta_0| \ge c_\phi \delta_0 \quad (\lambda \in \Sigma_{\pi-\phi})$$

for some $c_\phi > 0$ yields the existence of a $M_\phi > 0$ such that

$$\frac{|\lambda^{\ell/2m}|}{|(\lambda + \delta_0)^{1-(|\beta|+|\gamma|)/2m}|} \le M_\phi \quad (\lambda \in \Sigma_{\pi-\phi}).$$

Thanks to the contraction principle of Kahane and Proposition 4.5 we deduce

$$\begin{aligned}&\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma(\lambda+A_0+\delta)^{-1}\})\\ &\leq\ C\mathcal{R}(\{(\lambda+\delta_0)^{1-\frac{|\beta|+|\gamma|}{2m}} D^\beta D^\gamma((\lambda+\delta_0)+A_0+\delta_2)^{-1}\})\ \leq\ C.\end{aligned}$$

Lemma 3.5(a) then yields

$$\begin{aligned}&\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma(\lambda+A_0+\delta)^{-1}(-R(\lambda+A_0+\delta)^{-1})^j\})\\ &\leq\ \mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma(\lambda+A_0+\delta)^{-1}\})\mathcal{R}(\{(R(\lambda+A_0+\delta)^{-1})^j\})\\ &\leq\ C\eta^j\mathcal{R}(\{(A_0+\delta)(\lambda+A_0+\delta)^{-1})\})^j\ \leq\ C\nu^j\quad(j\in\mathbb{N}_0)\end{aligned}$$

with $\nu:=\eta\mathcal{R}(\{(A_0+\delta)(\lambda+A_0+\delta)^{-1}\})<1$. Employing again Lemma 3.5(a), in particular the fact that the $\mathcal{R}$-bound is preserved when taking the closure in the strong operator topology, the assertion follows. □

Corollary 4.7. *Let* $R(x^1,D):=\sum_{|\alpha^1|=2m} r_{\alpha^1}(x^1)D^{(\alpha^1,0)}$ *be given such that the condition* $\sum_{|\alpha^1|=2m}\|r_{\alpha^1}\|_\infty<\eta$ *is satisfied. Set*

$$A^{va}(x,D):=A_0(x^2,D)+R(x^1,D),\quad x\in\Omega,\tag{4.7}$$

and denote its X*-realization by* A^{va} *defined on* $D(A^{va})=D(A_0)$*. Then there exists a* $\delta>0$ *such that* $A^{va}+\delta\in\mathcal{RS}(X)$ *with* $\phi^{\mathcal{RS}}_{A^{va}+\delta}\leq\phi^{\mathcal{RS}}_{A_0+\delta_2}$ *provided that* η *is sufficiently small. In this case for* $\phi>\varphi_{(A_0,B)}$ *we have*

$$\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma(\lambda+A^{va}+\delta)^{-1};\ \lambda\in\Sigma_{\pi-\phi},\ 0\leq l+|\beta|+|\gamma|\leq 2m\})<\infty.\tag{4.8}$$

Proof. By Proposition 4.5, in particular by relation (4.4), there exists a $C>0$ such that

$$\|D^{(\alpha^1,0)}(A_0+\delta)^{-1}\|_{\mathcal{L}(X)}\leq C\quad(\alpha^1\in\mathbb{N}_0^{n-k},\ |\alpha^1|=2m)$$

for each $\delta>\delta_2$. For a fixed $\delta>\delta_2$ this implies

$$\begin{aligned}\|Ru\|_p\ &\leq\ \sum_{|\alpha^1|=2m}\|r_{\alpha^1}\|_\infty\|D^{(\alpha^1,0)}(A_0+\delta)^{-1}(A_0+\delta)u\|_p\\ &\leq\ C\eta\|(A_0+\delta)u\|_p\quad(u\in D(A_0)).\end{aligned}$$

Thus, if we assume that $\eta<1/C\mathcal{R}(\{(A_0+\delta)(\lambda+A_0+\delta)^{-1}\})$, the assertion follows from Lemma 4.6. □

4.3. Variable coefficients a^1_α

In the next lemma we establish estimates that will turn out to be crucial for the localization procedure.

Lemma 4.8. *Let* $1<p<\infty$, $(\beta^1,0)\in\mathbb{N}_0^{n-k}\times\mathbb{N}_0^k$, $|(\beta^1,0)|=\nu<2m$, *and* $r_\nu\geq p$ *such that* $2m-\nu>\frac{n-k}{r_\nu}$*. Let* $b\in[L^\infty+L^{r_\nu}](\mathbb{R}^{n-k})$, A^{va} *be the operator as defined in* (4.7)*, and assume that* $\phi>\varphi_{(A,B)}$.

(a) *For every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that*

$$\|bD^{(\beta^1,0)}u\|_p \le \varepsilon\|u\|_{p,2m} + C(\varepsilon)\|u\|_p \quad (u \in W^{2m,p}(\mathbb{R}^{n-k}, E)).$$

(b) *For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that*

$$\mathcal{R}(\{bD^{(\beta^1,0)}(\lambda + A^{va} + \delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi}\}) \le \varepsilon.$$

Proof. (a) Let $\varepsilon > 0$ be arbitrary. For simplicity we set $\beta = (\beta^1, 0)$. For $b \in L^\infty(\mathbb{R}^{n-k})$ we obtain by Hölder's inequality and vector-valued complex interpolation (see e.g. [5]) that

$$\|bD^\beta u\|_p \le \|b\|_\infty \|u\|_{p,\nu} \le C\|b\|_\infty \|u\|_{p,2m}^{\frac{\nu}{2m}} \|u\|_p^{1-\frac{\nu}{2m}} \quad (u \in W^{2m,p}(\mathbb{R}^{n-k}, E)).$$

With the help of Young's inequality we then can achieve that

$$\|bD^\beta u\|_p \le \varepsilon\|u\|_{p,2m} + C(\varepsilon)\|u\|_p \quad (u \in W^{2m,p}(\mathbb{R}^{n-k}, E)).$$

Now let $b \in L^{r_\nu}(\mathbb{R}^{n-k})$, $r := \frac{r_\nu}{p}$, and $\frac{1}{r} + \frac{1}{r'} = 1$. Then Hölder's inequality and the vector-valued version of the Gagliardo-Nirenberg inequality (see [19]) imply

$$\|bD^\beta u\|_p \le C\|b\|_{pr}\|D^\beta u\|_{pr'} \le C\|b\|_{r_\nu}\|u\|_{p,2m}^\tau \|u\|_p^{1-\tau},$$

where $\tau = \frac{n-k}{r_\nu(2m-\nu)} \in (0,1)$ by our assumption on r_ν. Again an application of Young's inequality yields

$$\|bD^\beta u\|_p \le \varepsilon\|u\|_{p,2m} + C(\varepsilon)\|u\|_p \quad (u \in W^{2m,p}(\mathbb{R}^{n-k}, E)).$$

(b) Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be a family of independent symmetric $\{-1,1\}$-valued random variables on a probability space $([0,1], \mathcal{M}, P)$, $\lambda_j \in \Sigma_{\pi-\phi}$, and $f_j \in X$. For $b \in L^\infty(\mathbb{R}^{n-k})$, $\delta_0 > 0$, and arbitrary $t \in [0,1]$ we have

$$\begin{aligned}&\|\sum_{j=1}^N \varepsilon_j(t) bD^\beta(\lambda_j + \delta_0 + A^{va} + \delta)^{-1} f_j\|_p \\ &\le \|b\|_\infty \|\sum_{j=1}^N \varepsilon_j(t) D^\beta(\lambda_j + \delta_0 + A^{va} + \delta)^{-1} f_j\|_p.\end{aligned}$$

Note that there is a $c_\phi > 0$ such that

$$|\lambda + \delta_0| \ge c_\phi \delta_0 \quad (\lambda \in \Sigma_{\pi-\phi},\ \delta_0 > 0).$$

Taking L^p-norm with respect to t and applying the contraction principle of Kahane therefore yields

$$\begin{aligned}&\|\sum_{j=1}^N \varepsilon_j(\cdot) bD^\beta(\lambda_j + \delta_0 + A^{va} + \delta)^{-1} f_j\|_{L^p([0,1],X)} \\ &\le C\|b\|_\infty \|\sum_{j=1}^N \varepsilon_j(\cdot)\left(\frac{\lambda_j + \delta_0}{\delta_0}\right)^{1-\frac{|\beta|}{2m}} D^\beta(\lambda_j + \delta_0 + A^{va} + \delta)^{-1} f_j\|_{L^p([0,1],X)}.\end{aligned}$$

Thanks to (4.8) this implies

$$\begin{aligned}
&\|\sum_{j=1}^{N}\varepsilon_j(\cdot)bD^{\beta}(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\|_{L^p([0,1],X)}\\
&\le C\|b\|_\infty\delta_0^{-(1-\frac{|\beta|}{2m})}\|\sum_{j=1}^{N}\varepsilon_j(\cdot)f_j\|_{L^p([0,1],X)}.
\end{aligned}$$

Thus for $\delta_0>(C\|b\|_\infty/\varepsilon)^{1/(1-|\beta|/2m)}$ the assertion follows.

In case that $b\in L^{r_\nu}(\mathbb{R}^{n-k})$, Hölder's inequality and the Gagliardo-Nirenberg inequality imply for $\tau(2m-\nu)=\frac{n-k}{r_\nu}$ and arbitrary $t\in[0,1]$ that

$$\begin{aligned}
&\|\sum_{j=1}^{N}\varepsilon_j(t)bD^{\beta}(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\|_p\\
&\le \|b\|_{pr}\|\sum_{j=1}^{N}\varepsilon_j(t)D^{\beta}(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\|_{pr'}\\
&\le C\|b\|_{r_\nu}\Bigg(\sum_{|\alpha|=2m}\|\sum_{j=1}^{N}\varepsilon_j(t)D^{\alpha}(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\|_p^p\Bigg)^{\tau/p}\\
&\quad\cdot\|\sum_{j=1}^{N}\varepsilon_j(t)(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\|_p^{1-\tau}.
\end{aligned}$$

Taking L^p-norm with respect to t and applying once more Hölder's inequality we deduce

$$\begin{aligned}
&\|\sum_{j=1}^{N}\varepsilon_j(\cdot)bD^{\beta}(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\|_{L^p([0,1],X)}\\
&\le C\|b\|_{r_\nu}\Bigg(\sum_{|\alpha|=2m}\|\sum_{j=1}^{N}\varepsilon_j(\cdot)D^{\alpha}(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\|_{L^p([0,1],X)}^p\Bigg)^{\tau/p}\\
&\quad\cdot\|\sum_{j=1}^{N}\varepsilon_j(\cdot)(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\|_{L^p([0,1],X)}^{1-\tau}.
\end{aligned}$$

The contraction principle of Kahane then gives us

$$\begin{aligned}
&\|\sum_{j=1}^{N}\varepsilon_j(\cdot)bD^{\beta}(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\|_{L^p([0,1],X)}\\
&\le C\|b\|_{r_\nu}\Bigg(\sum_{|\alpha|=2m}\|\sum_{j=1}^{N}\varepsilon_j(\cdot)D^{\alpha}(\lambda_j+\delta_0+A^{va}+\delta)^{-1}f_j\|_{L^p([0,1],X)}^p\Bigg)^{\tau/p}
\end{aligned}$$

$$\cdot \| \sum_{j=1}^{N} \varepsilon_j(\cdot) \frac{\lambda_j + \delta_0}{\delta_0} (\lambda_j + \delta_0 + A^{va} + \delta)^{-1} f_j \|_{L^p([0,1],X)}^{1-\tau}.$$

Taking into account (4.8) we arrive at

$$\| \sum_{j=1}^{N} \varepsilon_j(\cdot) b D^{\beta} (\lambda_j + \delta_0 + A^{va} + \delta)^{-1} f_j \|_{L^p([0,1],X)}$$

$$\leq C \|b\|_{r_\nu} \delta_0^{\tau-1} \| \sum_{j=1}^{N} \varepsilon_j(\cdot) f_j \|_{L^p([0,1],X)}.$$

Choosing $\delta_0 > (C\|b\|_{r_\nu}/\varepsilon)^{1/(1-\tau)}$ proves the lemma. □

Proof of Theorem 2.3. We denote by

$$A_1^{\#}(x,D) := \sum_{|\alpha|=2m} a_\alpha^1(x) D^\alpha$$

the principal part of $A_1(x,D)$ and by $A_1^{\#}$ its realization in X with domain $D(A_1^{\#}) = W^{2m,p}(\mathbb{R}^{n-k},E)$. Recall that $A_1^{\#}(x,D) = A_1^{\#}(x^1,D)$ does not depend on $x^2 \in V$. Freezing the coefficients at some arbitrary $x_0^1 \in \mathbb{R}^{n-k} \cup \{\infty\}$, Proposition 4.5 applies to $A_1(D) := A_1^{\#}(x_0^1,D)$.

So, we first choose a large ball $B_{r_0}(0) \subset \mathbb{R}^{n-k}$ with a fixed radius $r_0 > 0$ such that

$$|a_{\alpha^1}^1(x^1) - a_{\alpha^1}^1(\infty)| \leq \eta/M_\alpha, \quad \text{for all } |x^1| \geq r_0,\ |\alpha^1| = 2m,$$

and set $U_0 := \mathbb{R}^{n-k} \backslash \overline{B}_{r_0}(0)$. Here $M_\alpha = \left|\{\alpha^1 \in \mathbb{N}_0^{n-k};\ |\alpha^1| = 2m,\ a_{\alpha^1} \neq 0\}\right|$ and $\eta = \eta(\infty)$ is the constant given in Corollary 4.7 for the principal part of the 'limiting operator' $A_1^{\#}(\infty,D) = \sum_{|\alpha|=2m} a_\alpha^1(\infty) D^\alpha$. For every $x_0^1 \in \overline{B}_{r_0}(0)$ let $\eta = \eta(x_0^1)$ be the constant given in Corollary 4.7 for the 'frozen coefficients operator' $A_1(D) := A_1^{\#}(x_0^1,D)$. By our continuity assumptions on the coefficients then there exists a radius $r = r(x_0^1)$ such that

$$|a_{\alpha^1}^1(x^1) - a_{\alpha^1}^1(x_0^1)| \leq \eta(x_0^1)/M_\alpha, \quad \text{for all } |x^1 - x_0^1| \leq r(x_0^1),\ |\alpha^1| = 2m.$$

Obviously the collection $\{B_{r(x_0^1)}(x_0^1)) :\ x_0^1 \in \overline{B}_{r_0}(0)\}$ represents an open covering of $\overline{B}_{r_0}(0)$. Thus, by compactness we have

$$\overline{B}_{r_0}(0) \subseteq \bigcup_{j=1}^{N} B_{r(x_j^1)}(x_j^1)$$

for a certain finite set $(x_j^1)_{j=1}^N$.

For simplicity we set $x_j := (x_j^1, 0)$, $r_j := r(x_j^1)$, and $U_j := B_{r_j}(x_j^1)$ for $j = 1, \ldots, N$, as well as $x_0^1 := \infty$. For each $j = 0, ..., N$ we define coefficients of $A_1^{\#}(x,D)$-localizations

$$A_j^{1,loc}(x,D) := \sum_{|\alpha|=2m} a_{j,\alpha}^1(x) D^\alpha$$

by reflection, i.e., we set

$$a^1_{0,\alpha}(x) = \begin{cases} a^1_\alpha(x) & , \quad x^1 \notin \overline{B}_{r_0}(0), \\ a^1_\alpha(\frac{r_0^2}{|x|^2}x), & x^1 \in \overline{B}_{r_0}(0), \end{cases}$$

and

$$a^1_{j,\alpha}(x) = \begin{cases} a^1_\alpha(x) & , \quad x^1 \in \overline{B}_{r_j}(x^1_j), \\ a^1_\alpha(x_j + \frac{r_j^2}{|x-x_j|^2}(x-x_j)), & x^1 \notin \overline{B}_{r_j}(x^1_j). \end{cases}$$

Then by definition we have

$$\sum_{|\alpha^1|=2m} |a^1_{j,\alpha^1}(x) - a^1_{\alpha^1}(x_j)| \le \eta(x^1_j)$$

for $x = (x^1, 0) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$ and $j = 0, ..., N$, that is, $A^{1,loc}_j(x, D) + A_2(x, D)$ is a small variation of $A^\#(x^1_j, D) := A^\#_1(x^1_j, D) + A_2(x, D)$. Hence Corollary 4.7 applies to

$$A^{loc}_j := A^{1,loc}_j + A_2.$$

In other words, for each $\phi > \varphi_{(A,B)}$ there exists $\delta = \delta(\phi) > 0$ such that $A^{loc}_j + \delta \in \mathcal{RS}(X)$ and we have

$$\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma (\lambda + A^{loc}_j + \delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le \ell + |\beta| + |\gamma| \le 2m\}) \le C_\phi < \infty \tag{4.9}$$

for $j = 0, ..., N$.

Next we choose a partition of unity $(\varphi_j)_{j=0}^N \subset C^\infty(\mathbb{R}^{n-k})$ of $\mathbb{R}^{n-k}$ subordinate to the open covering $(U_j)_{j=0}^N$ such that $0 \le \varphi_j \le 1$. In addition, we fix $\psi_j \in C^\infty(\mathbb{R}^{n-k})$ such that $\psi_j \equiv 1$ on supp φ_j and supp $\psi_j \subset U_j$. We set $\mathcal{B}(x, D) := A(x, D) - A^\#(x, D)$ and pick $\lambda \in \Sigma_{\pi-\phi}$. Then

$$\lambda u + A(\cdot, D)u = f$$

holds if and only if

$$\lambda u + A^\#(\cdot, D)u = f - \mathcal{B}(\cdot, D)u.$$

Multiplying the line above by φ_j we obtain

$$\lambda\varphi_j u + A^\#(\cdot, D)\varphi_j u = \varphi_j f + [A^\#(\cdot, D), \varphi_j]u - \varphi_j \mathcal{B}(\cdot, D)u,$$

where the commutators

$$[A^\#(\cdot, D), \varphi_j] := A^\#(\cdot, D)\varphi_j - \varphi_j A^\#(\cdot, D) = [A^\#_1(\cdot, D), \varphi_j]$$

do only depend on $A^\#_1(\cdot, D)$. Applying the resolvent of A^{loc}_j to the localized equations we deduce

$$\varphi_j u = (\lambda + A^{loc}_j + \delta)^{-1}\varphi_j f + (\lambda + A^{loc}_j + \delta)^{-1}([A^\#(\cdot, D), \varphi_j]u - \varphi_j \mathcal{B}(\cdot, D)u).$$

By multiplying with ψ_j and by summing up over j we gain the representation

$$u = \sum_{j=0}^N \psi_j(\lambda + A^{loc}_j + \delta)^{-1}\varphi_j f + \sum_{j=0}^N \psi_j(\lambda + A^{loc}_j + \delta)^{-1}([A^\#(\cdot, D), \varphi_j]u - \varphi_j \mathcal{B}(\cdot, D))u.$$

Hence we obtain

$$(I - \sum_{j=0}^{N} \psi_j(\lambda + A_j^{loc} + \delta)^{-1} \mathcal{C}_j(\cdot, D))u = \sum_{j=0}^{N} \psi_j(\lambda + A_j^{loc} + \delta)^{-1} \varphi_j f,$$

where

$$\mathcal{C}_j(\cdot, D) := [A_1^{\#}(\cdot, D), \varphi_j] - \varphi_j \mathcal{B}(\cdot, D)$$

is a differential operator in X of lower order whose coefficients fullfill the assumptions of Lemma 4.8. We set

$$R_0(\lambda, \delta) := \sum_{j=0}^{N} \psi_j(\lambda + A_j^{loc} + \delta)^{-1} \varphi_j \tag{4.10}$$

and

$$R_1(\lambda, \delta) := \sum_{j=0}^{N} \psi_j(\lambda + A_j^{loc} + \delta)^{-1} \mathcal{C}_j(\cdot, D).$$

Relation (4.9) and Lemma 4.8(a) now imply that

$$\begin{aligned}
&\|R_1(\lambda, \delta' + \delta_0)u\|_{W^{2m,p}(\Omega, F)} + \delta_0 \|R_1(\lambda, \delta' + \delta_0)u\|_p \\
&\le C \left(\|R_1(\lambda + \delta_0, \delta')u\|_{W^{2m,p}(\Omega, F)} + |\lambda + \delta_0| \|R_1(\lambda + \delta_0, \delta')u\|_p \right) \\
&\le C \|\mathcal{C}_j(\cdot, D)u\|_p \\
&\le C \left(\varepsilon \|u\|_{W^{2m,p}(\mathbb{R}^{n-k}, E)} + C(\varepsilon) \|u\|_p \right) \\
&\le \frac{1}{2} \left(\|u\|_{W^{2m,p}(\mathbb{R}^{n-k}, E)} + \delta_0 \|u\|_p \right) \\
&\le \frac{1}{2} \left(\|u\|_{W^{2m,p}(\Omega, F)} + \delta_0 \|u\|_p \right) \quad (\lambda \in \Sigma_{\pi - \phi})
\end{aligned}$$

for some $\delta' > 0$ and provided that $\delta_0 > 0$ is sufficiently large. Setting $\delta := \delta' + \delta_0$ we see that then

$$L_\lambda := (I - R_1(\lambda, \delta))^{-1} R_0(\lambda, \delta) : L^p(\mathbb{R}^{n-k}, E) \to D(A)$$

is a left inverse of $\lambda + A + \delta$ which admits an estimate

$$|\lambda| \|L_\lambda f\|_p \le C \|f\|_p \quad (\lambda \in \Sigma_{\pi - \phi}).$$

Thus, if we can prove that there exists a right inverse as well, we obtain $A + \delta \in S(X)$ and $\phi_{A+\delta} \le \phi$.

To this end, let $f \in X$ be arbitrary. Then

$$\begin{aligned}
&(\lambda + A(\cdot, D) + \delta) R_0(\lambda, \delta) f = (\lambda + A^{\#}(\cdot, D) + \delta) R_0(\lambda, \delta) f + \mathcal{B}(\cdot, D) R_0(\lambda, \delta) f \\
&= (\lambda + A^{\#}(\cdot, D) + \delta) \sum_{j=0}^{N} \psi_j(\lambda + A_j^{loc} + \delta)^{-1} \varphi_j f \\
&\qquad + \mathcal{B}(\cdot, D) \sum_{j=0}^{N} \psi_j(\lambda + A_j^{loc} + \delta)^{-1} \varphi_j f
\end{aligned}$$

$$= \sum_{j=0}^{N} \psi_j(\lambda + A^{\#}(\cdot, D) + \delta)(\lambda + A_j^{loc} + \delta)^{-1}\varphi_j f$$
$$+ \sum_{j=0}^{N} \mathcal{D}(\cdot, D)(\lambda + A_j^{loc} + \delta)^{-1}\varphi_j f,$$

where

$$\mathcal{D}(\cdot, D) := [A_1^{\#}(\cdot, D), \psi_j] + \mathcal{B}(\cdot, D)\psi_j$$

is again a differential operator in X of lower order whose coefficients fullfill the assumptions of Lemma 4.8. Since supp $\psi_j \subset U_j$ and $\psi \equiv 1$ on supp φ_j, we obtain

$$(\lambda + A(\cdot, D) + \delta)R_0(\lambda, \delta)f = f + R_2(\lambda, \delta)f$$

with

$$R_2(\lambda, \delta) := \sum_{j=0}^{N} \mathcal{D}(\cdot, D)(\lambda + A_j^{loc} + \delta)^{-1}\varphi_j.$$

Lemma 4.8(b) implies $\|R_2(\lambda, \delta)\|_{\mathcal{L}(X)} \leq 1/2$ for large enough $\delta > 0$. Consequently, $R_\lambda := R_0(\lambda, \delta)(I + R_2(\lambda, \delta))^{-1}$ is a right inverse of $\lambda + A + \delta$.

With the help of the Leibniz rule and the contraction principle of Kahane, from representation (4.10) and relation (4.9) we obtain that

$$\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma R_0(\lambda, \delta)\}) \leq C(N + 1).$$

In view of Lemma 4.8(b) and Lemma 3.5 the representation

$$(\lambda + A + \delta)^{-1} = R_0(\lambda, \delta) \sum_{i=0}^{\infty} R_2(\lambda, \delta)^i$$

as a Neumann series finally gives us

$$\mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma (\lambda + A + \delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \leq \ell + |\beta| + |\gamma| \leq 2m\})$$
$$\leq \mathcal{R}(\{\lambda^{\frac{\ell}{2m}} D^\beta D^\gamma R_0(\lambda, \delta)\})\mathcal{R}(\{\sum_{i=0}^{\infty} R_2(\lambda, \delta)^i\})$$
$$\leq (N + 1)C \sum_{i=0}^{\infty} (N + 1)^i (C\varepsilon)^i = \frac{(N + 1)C}{1 - (N + 1)C\varepsilon} < \infty.$$

Hence the proof of Theorem 2.3 is complete. □

5. Mixed orders

All parts of the proof can easily be adjusted to the situation when the differential operators $A_1(\cdot, D)$ and $A_2(\cdot, D)$ have different orders, say $2m_1$ and $2m_2$ respectively. Then a cylindrical boundary value problem is given as

$$\begin{aligned} \lambda u + A(x, D)u &= f \text{ in } \Omega, \\ B_j(x, D)u &= 0 \text{ on } \partial\Omega \quad (j = 1, ..., m), \end{aligned} \tag{5.1}$$

with

$$A(x,D) = A_1(x^1,D) + A_2(x^2,D) := \sum_{|\alpha^1|\le 2m_1} a^1_{\alpha^1}(x^1)D^{(\alpha^1,0)} + \sum_{|\alpha^2|\le 2m_2} a^2_{\alpha^2}(x^2)D^{(0,\alpha^2)}$$

and

$$B_j(x,D) = B_{2,j}(x^2,D) := \sum_{|\beta^2|\le m_{2,j}} b^2_{j,\beta^2}(x^2)D^{(0,\beta^2)} \quad (m_{2,j} < 2m_2,\ j=1,...,m_2).$$

However, then the notion of parameter-ellipticity for the entire cylindrical boundary value problem is no longer appropriate. Instead we assume the differential operator $A_1(\cdot,D)$ to be parameter-elliptic in $\mathbb{R}^{n-k}$ as well as the boundary value problem

$$\begin{aligned} \lambda u + A_2(x,D)u &= f \text{ in } V, \\ B_{2,j}(x,D)u &= 0 \text{ on } \partial V \quad (j=1,...,m), \end{aligned} \tag{5.2}$$

to be parameter-elliptic in the cross-section V of Ω with a joint angle of parameter-ellipticity $\varphi \in [0,\pi)$. The exact same proof as the one of Theorem 2.3 can be used to show the following result.

Theorem 5.1. *Given the assumptions of Theorem* 2.3*, let $A_1(\cdot,D)$ in $\mathbb{R}^{n-k}$ as well as the boundary value problem* (5.2) *in V be parameter-elliptic with a joint angle of parameter-ellipticity $\varphi \in [0,\pi)$. For $\Omega = \mathbb{R}^{n-k}\times V$ we define the $L^p(\Omega,F)$-realization of the cylindrical boundary value problem* (5.1) *by*

$$\begin{aligned} D(A) &= \{u \in L^p(\Omega,F);\ D^\alpha u \in L^p(\Omega,F) \\ &\qquad \textit{for } \frac{|\alpha^1|}{2m_1} + \frac{|\alpha^2|}{2m_2} \le 1 \textit{ and } B_j(\cdot,D)u = 0 \quad (j=1,...,m)\} \\ Au &= A(\cdot,D)u, \quad u \in D(A). \end{aligned}$$

Then for each $\phi > \varphi$ there exists $\delta = \delta(\phi) > 0$ such that $A+\delta \in \mathcal{RS}(L^p(\Omega,F))$ with $\phi^{\mathcal{RS}}_{A+\delta} \le \phi$. Moreover, for $\alpha = (\alpha^1,\alpha^2) \in \mathbb{N}_0^{n-k}\times\mathbb{N}_0^k$ we have

$$\mathcal{R}(\{\lambda^{1-(\frac{|\alpha^1|}{2m_1}+\frac{|\alpha^2|}{2m_2})}D^\alpha(\lambda+A+\delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \le \frac{|\alpha^1|}{2m_1} + \frac{|\alpha^2|}{2m_2} \le 1\}) < \infty.$$

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