An introduction to p-modular representations of p-adic groups

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Part 1 : Warm-up

Introducing the main actors of the play

Local fields and p-adic numbers (I)

Recall: $\mathbb{R}$ is the completion of $\mathbb{Q}$ for the usual absolute value $|\ |_\infty$, which is Archimedean.

Theorem (Ostrowski I, 1916)

Any (non-trivial) Archimedean absolute value on $\mathbb{Q}$ is equivalent to $|\ |_\infty$.

Question: What about non-Archimedean absolute values?

Local fields and p-adic numbers (II)

Definition: Let p be a prime integer.

For any non-zero integer x, set $v_p(x) := \max\{n \in \mathbb{Z}_{\geq 0} \mid p^n \text{ divides } x\}$.

Extend v_p to $\mathbb{Q}$ by setting $v_p(0) := \infty$ and $v_p(a/b) := v_p(a) - v_p(b)$.

Exercise: $|.|_p : x \mapsto p^{-v_p(x)}$ is a non-Archimedean absolute value on $\mathbb{Q}$.

Theorem (Ostrowski II, 1916)

Any non-trivial absolute value on $\mathbb{Q}$ is equivalent to $|\ |_\star$ for a unique $\star \in \mathcal{P} \cup \{\infty\}$, where $\mathcal{P}$ denotes the set of prime integers.

Definition: $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ for the p-adic absolute value $|.|_p$. This is the field of p-adic numbers.

Local fields and p-adic numbers (II)

Some surprising and useful fun facts about p-adic numbers:

(*i*) Set $\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid v_p(x) \geq 0\} = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$.

It is a Discrete Valuation Ring (hence a Principal Ideal Domain) and:

$$\forall\ n \in \mathbb{Z}_{\geq 0},\ \mathbb{Z}_p/p^n\mathbb{Z}_p \simeq \mathbb{Z}/p^n\mathbb{Z}\ .$$

In particular, it is a local ring with finite residue field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

(*ii*) Any finite index subgroup of $\mathbb{Q}_p^\times$ is open and closed.

(*iii*) $\mathbb{Z}_p$ is a compact topological space, and the completion of $(\mathbb{Z}, |.|_p)$.

(*iv*) $\mathbb{Q}_p$ is locally compact (but non compact) and totally disconnected.

(*v*) A series of p-adic numbers is convergent iff its general term converges to 0.

Local fields and p-adic numbers (III)

Definition: A local field is a field endowed with a **discrete** valuation for which it is complete and with **finite** residue field.

Exercise: For any prime integer p, $\mathbb{Q}_p$ is a local field.

Note : $\mathbb{Q}_p$ has many finite field extensions or arbitrarily high degree.

Theorem

Any local field **of characteristic** 0 *is a finite extension of* $\mathbb{Q}_p$ *for a unique prime integer* p.

Note: In characteristic p, local fields are Laurent series with coefficients in finite extensions of $\mathbb{F}_p$.

Absolute Galois group and Weil group of a local field (I)

Let $F/\mathbb{Q}_p$ be a finite extension and $\overline{F}$ be a separable closure of F.

We let $\mathrm{G}_F := \mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group of F.

Let k be the residue field of F.

Then the residue field of $\overline{F}$ is an algebraic closure $\overline{k}$ of k and we have a natural projection homomorphism

$$\mathrm{pr} : \mathrm{G}_F \twoheadrightarrow \mathrm{Gal}(\overline{k}/k) \simeq \widehat{\mathbb{Z}} \ .$$

The Weil group of F is the preimage $W_F := \mathrm{pr}^{-1}\left(\langle \mathrm{Frob_F} \rangle\right)$ of a given topological generator of $\mathrm{Gal}(\overline{k}/k)$.

Absolute Galois group and Weil group of a local field (II)

The Weil group of F is the preimage $W_F := \mathrm{pr}^{-1}(\langle \mathrm{Frob_F} \rangle)$ of a given topological generator of $\mathrm{Gal}(\overline{k}/k)$.

$$
\begin{tikzcd}
1 \arrow[r] & I_F \arrow[r, "\iota"] \arrow[d, equal] & W_F \arrow[r, "\mathrm{pr}"] \arrow[d, "\mathrm{inj}"] & \langle \mathrm{Frob_F} \rangle \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & I_F \arrow[r] & \mathrm{G}_F \arrow[r, "\mathrm{pr}"] & \mathrm{G}_k \arrow[r] & 1
\end{tikzcd}
$$

Warning! The topology of W_F is **NOT** the topology induced by G_F!

BUT the canonical injection inj is continuous with dense image.

Note : For now, can forget W_F and consider the full Galois group G_F

Basic vocabulary of representation theory

Let G be a locally profinite topological group (such as $\mathrm{GL}_n(F)$ for $n \geq 1$).

A representation of G over a field E is a pair (π, V), where V is an E-vector space and $\pi : G \to \mathrm{Aut}_E(V)$ is a group action of G on V.

Such a representation is:

- ⋆ irreducible when V admits exactly two G-stable subvectorspaces (namely $\{0\}$ and V);
- ⋆ smooth when any element of V has open stabilizer in G (under π);
- ⋆ admissible when $V^H := \{v \in V \mid \forall\ h \in H,\ \pi(h)(v) = v\}$ is finite-dimensional over E for any open compact subgroup H of G.

Notes:

- ⋆ One often imposes *smoothness* in the definition of admissibility.
- ⋆ As we will see later, the choice of E really impacts the relations between these notions.

Part 2 : Motivation

Why do we care about p-modular representations ?

From local class field theory to Langlands correspondences

Local class field theory : Given a finite extension $F/\mathbb{Q}_p$, Artin reciprocity map induces a group homomorphism $F^\times \simeq W_F^{\mathrm{ab}}$.

Such an isomorphism naturally induces a bijection of the following form:

$$\left\{ \begin{array}{c} \text{continuous} \\ \text{group homomorphisms} \\ W_F \to \mathbb{C}^\times \end{array} \right\} \stackrel{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{smooth} \\ \text{group homomorphisms} \\ F^\times \to \mathbb{C}^\times \end{array} \right\}$$

Question (Langlands, 1967) : Can we have a higher-dimensional (non abelian) analogue of this natural bijection ?

From local class field theory to Langlands correspondences

Conjecture (Langlands 1967; proven by Harris-Taylor (2001), Henniart (2002), Scholze (2013))

For any integer $n \geq 1$, there is a natural bijection of the following form:

$$\left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{continuous irreducible} \\ \textit{n-dimensional representations} \\ \textit{of } W_F \ (+ \textit{ data}) \textit{ over } \mathbb{C} \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{some irreducible smooth} \\ \textit{representations of} \\ \mathrm{GL}_n(F) \textit{ over } \mathbb{C} \end{array} \right\}$$

Further conjectures and results :

Extension to other groups than GL_n, using L-groups.

Examples of ℓ-adic and ℓ-modular representations

Let $\mathcal{E}$ be an elliptic curve defined over $\mathbb{Q}$.
(Consider an equation of the form $y^2 + axy + by = x^3 + \alpha x^2 + \beta x + \gamma$ with a chosen point at infinity.)

For any $n \in \mathbb{Z}_{\geq 1}$ and any prime ℓ, $\mathrm{G}_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the group $\mathcal{E}[\ell^n]$ of ℓ-torsion points.

- ⋆ For $n = 1$, we get an ℓ-modular representation of $\mathrm{G}_{\mathbb{Q}}$, hence of $\mathrm{G}_{\mathbb{Q}_p}$ for any prime p (via decomposition groups).
- ⋆ Multiplication by ℓ defines transition maps $\mathcal{E}[\ell^{n+1}] \to \mathcal{E}[\ell^n]$, so we can define the Tate module $T_\ell(\mathcal{E}) := \varprojlim \mathcal{E}[\ell^n]$ of $\mathcal{E}$. This $\mathbb{Z}_\ell$-module defines an ℓ-adic representation $V_\ell(\mathcal{E}) := T_\ell(\mathcal{E}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ of $\mathrm{G}_{\mathbb{Q}}$.

Such representations are involved in the proof of Shimura-Taniyama-Weil conjecture (Wiles and al. 1994-2001), which implies Fermat's Last Theorem (Frey 1985, Ribet 1987, Wiles and Taylor-Wiles, 1994-1995).

When non-complex things get more complex... (I)

Following Langlands philosophy, there should exist matching *correspondences* of the following form (p, ℓ arbitrary primes):

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{geometric } 2\text{-dimensional} \\ \ell\text{-adic representations} \\ \text{of } \mathrm{G}_{\mathbb{Q}_p} \text{ over } \overline{\mathbb{Q}}_\ell\end{array}\right\} & \overset{??}{\longleftrightarrow} & \left\{\begin{array}{c}\text{some smooth} \\ \text{representations of} \\ \mathrm{GL}_2(\mathbb{Q}_p) \text{ over } \overline{\mathbb{Q}}_\ell\end{array}\right\} \\
\updownarrow & & \updownarrow \\
\left\{\begin{array}{c}\text{geometric } 2\text{-dimensional} \\ \ell\text{-modular representations} \\ \text{of } \mathrm{G}_{\mathbb{Q}_p} \text{ over } \overline{\mathbb{F}}_\ell\end{array}\right\} & \overset{??}{\longleftrightarrow} & \left\{\begin{array}{c}\text{some smooth} \\ \text{representations of} \\ \mathrm{GL}_2(\mathbb{Q}_p) \text{ over } \overline{\mathbb{F}}_\ell\end{array}\right\}
\end{array}$$

that should

- $\star$ be *natural* enough (w.r.t. L-functions, ε-factors, functoriality...);
- $\star$ make sense and hold for other reductive groups than GL_n (conditions on the image of the geometric representation).

Question: How to cook up such correspondences ?

When non-complex things get more complex... (II)

Statement of the problem :

- ⋆ Describe smooth representations of $\mathrm{GL}_n(F)$ over $\overline{\mathbb{F}}_\ell$, where F is a finite extension of $\mathbb{Q}_p$
  (More generally, of $\mathcal{G}(F)$ for $\mathcal{G}$ a connected reductive group over F)
- ⋆ Connect them to the Galois side in a « natural way »

When $\ell \neq p$, many tools from the $\mathbb{C}$ world transfer more or less easily.

$\rightsquigarrow$ Vignéras 1996 for $\mathrm{GL}_n(F)$ in both ℓ-modular and ℓ-adic settings;

$\rightsquigarrow$ Minguez-Sécherre 2002 for $\mathrm{GL}_n(D)$, with D central division algebra over F, in the ℓ-modular setting;

$\rightsquigarrow$ Ongoing works of many authors for other groups in the ℓ-modular setting, a little less in the ℓ-adic one.

Take $\ell = p$... and everything collapses !

When $\ell = p$:

$\star$ p is not invertible in $\overline{\mathbb{F}}_p$, so forget averaging and counting arguments

$\star$ There is no Haar measure over $\mathrm{GL}_n(F)$ with values in $\overline{\mathbb{F}}_p$, so forget most of harmonic analysis tools... hence of complex methods!

$\star$ Any smooth representation of a pro-p-group is of level 0:

Lemma (Key lemma – Serre 1970's, Barthel-Livné 1994)

Let $\mathcal{P}$ be a pro-p-group and (σ, W) be a smooth non-zero representation of $\mathcal{P}$ over E. Then $W^{\mathcal{P}} := \{w \in W \mid \forall\, g \in \mathcal{P},\ \sigma(g)(w) = w\} \neq \{0\}$.

When ℓ becomes p: what is known so far? (Upshot I)

Full classifications for only 3 groups so far

- $\star$ Barthel-Livné 1994-95 + Breuil 2001 : $\mathrm{GL}_2(\mathbb{Q}_p)$
  $\rightsquigarrow$ associated p-modular LLC (Breuil, Colmez, Paškūnas...)
  $\rightsquigarrow$ associated p-adic LLC (Berger, Colmez, Dospinescu, Paškūnas...)

- $\star$ A. 2010/2011 : $\mathrm{SL}_2(\mathbb{Q}_p)$
  $\rightsquigarrow$ First instance of p-modular LLC *with packets*
  $\rightsquigarrow$ p-adic LLC still in progress (A., Sherman, Ban-Strauch...)

- $\star$ Kozioł 2014 : $U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$
  $\rightsquigarrow$ First instance of p-modular LLC *for a non-split group*
  $\rightsquigarrow$ p-adic LLC also in progress (A.-David-Romano-Wiersema...)

When ℓ becomes p: what is known so far? (Upshot II)

Some partial results... mostly negative, and mostly about GL_2

- ⋆ Abe-Henniart-Herzig-Vignéras 2014: Classification of irreducible admissible smooth representations up to the supersingular ones
- ⋆ Irreducible smooth does not imply admissible (Le 2019)
- ⋆ Restriction to open compact subgroups is not semisimple (A.-Morra 2012 for $\mathrm{SL}_2(\mathbb{Q}_p)$)
- ⋆ Restriction to minimal parabolic subgroups brings control (A.-Hauseux 2019 in rank 1)
- ⋆ Many new conjectures in the $\mathrm{GL}_n(\mathbb{Q}_p)$ case (Breuil et al., 2000-20..)

<u>Part 3 : Some statements</u>
Classifying p-modular representations of (non) reductive p-adic groups

p-modular representations of p-adic groups: the setting (I)

- $\star$ p: prime integer
- $\star$ F: non-Archimedean local field with finite residue field k of char. p ($F/\mathbb{Q}_p$ finite extension or $F = \mathbb{F}_{p^f}((t))$ with $f \in \mathbb{Z}_{\geq 1}$)
- $\star$ $\mathcal{G}$: connected reductive group defined over F (e.g. SL_2 or GL_2) $\rightsquigarrow$ $\mathrm{G} := \mathcal{G}(F)$ is a topological group (« p-adic group » if char $F = 0$)
- $\star$ E: algebraically closed field of characteristic p that contains k

Question : What can we say about isomorphism classes of irreducible (admissible) smooth representations of G over E?

p-modular representations of p-adic groups: the setting (II)

Question : What can we say about isomorphism classes of irreducible (admissible) smooth representations of G over E?

Process in two times:

- $\star$ **Step 1:** Study subquotients of parabolically induced representations $\rightsquigarrow$ *non-supercuspidal* representations
- $\star$ **«Step» 2:** Try to catch supercuspidal representations by other ways

  Note: Such representations always exist (Herzig-Kozioł-Vignéras 19)

Inducing representations of parabolic subgroups

$\mathcal{B}$: parabolic subgroup of $\mathcal{G}$ with Levi decomposition $\mathcal{B} = \mathcal{M}\mathcal{U}$

Example: $\mathcal{B} =$ upper-triangular matrices in $\mathcal{G}$
$\mathcal{M} =$ diagonal matrices in $\mathcal{G}$.

(σ, W) : irreducible **admissible** smooth representation of M over E

Example: For $\mathrm{G} = \mathrm{SL}_2$, we have $\mathcal{M} \simeq \mathbb{G}_m$, hence σ is a continuous group homomorphism $F^\times \to E^\times$.

One extends σ to a smooth representation of B trivial on U and set

$$\mathrm{Ind}_B^G(\sigma) := \left\{ \begin{array}{c} f : G \to W \mid \exists\ K_f \text{ open compact subgroup of } G \text{ s.t.} \\ \forall\ (mu, g, k) \in B \times G \times K_f,\ f(mugk) = \sigma(m) f(g) \end{array} \right\}$$

This defines a functor of parabolic induction that is a right adjoint to restriction to B. Irreducible smooth representations of G that are subquotient of such induced representations are called non supercuspidal.

(Non-)supercuspidal representations of p-adic groups

Theorem (A., 2011)

Assume that $\mathcal{G}$ is of semisimple F-rank 1.

1. $\mathrm{Ind}_B^G(\sigma)$ *is irreducible iff* σ *does not extend to a smooth representation of* G.
2. *Otherwise,* $\mathrm{Ind}_B^G(\sigma)$ *is a non-split representation of* G *of length* 2 *and fits into the following short exact sequence:*

$$1 \longrightarrow \sigma \longrightarrow \mathrm{Ind}_B^G(\sigma) \longrightarrow \mathrm{St}_G \otimes \sigma \longrightarrow 1 \ .$$

3. *There is no non-trivial intertwinning between distinct subquotients of parabolically induced representations.*

Example: For $\mathcal{G} = \mathrm{GL}_2$, the first condition means that σ does factor through det, and we obtain 3 disjoint families of isomorphism classes: principal series representations, special series representations, and characters (= one-dimensional representations).

Supercuspidal representations in the classical setting

Bushnell-Kutzko 1993, 1998: theory of **types** for complex representations

Idea 1 (Bernstein decomposition): Decompose the category of smooth representations of G into blocks indexed by pairs (M, σ) with M Levi factor and σ irreducible cuspidal representation of M.

Idea 2 (Definition of types): Let K be an open compact subgroup of G and η be an irreducible smooth representation of K.

A smooth complex representation ρ of G is of type type (K, η) iff ρ is a quotient of $\mathrm{c\text{-}ind}_K^G(\eta)$. *(The latter functor is defined as* Ind *with an extra assumption on the support of* f*.)*

Motto: Compact induction gives (all) supercuspidal representations.

Supercuspidal representations in the modular setting

When $\ell \neq p$:

Can partially work out this strategy for ℓ-modular representations (Dat 1999, Sécherre-Minguez 2012, Lanard 2019, Cui 2023...)

When $\ell = p$:

There is not even a thing like Bernstein-like decomposition...

... but one can consider compact induction and see how do quotients look like: hopefully, they provide (all) supercuspidal representations.

$\rightsquigarrow$ For $\mathrm{SL}_2(\mathbb{Q}_p)$, we get a **full classification of irreducible smooth** p**-modular representations** :

- the trivial character $\mathbf{1}$;
- the Steinberg representation St;
- the representations *of the principal series* $\mathrm{Ind}_B^G(\chi)$ with $\chi \neq \mathbf{1}$;
- the p supercuspidal representations $\pi_0, \ldots, \pi_{p-1}$.

Supercuspidal representations in the p-modular setting (I)

$\mathcal{O}_F$: the ring of integers of F (as $\mathbb{Z}_p$ is for $\mathbb{Q}_p$).

$K = \mathcal{G}(\mathcal{O}_F)$: a maximal open compact subgroup of G.

(η, W) : irreducible smooth representation of K over E.

$\mathcal{H}(G, K, \eta) := \mathrm{End}_{E[G]}\left(\text{c-ind}_K^G(\eta)\right)$: spherical Hecke algebra

Theorem (Barthel-Livné 1994-1995 for GL_2, A. 2011 for SL_2 and $U(2,1)$)

1. *There exists $T_\eta \in \mathcal{H}(G, K, \eta)$ such that $\mathcal{H}(G, K, \eta) = C[T_\eta]$.*
2. *Let π be an irreducible smooth representation of G over E that admits a central character. Then:*
   - *π is a quotient of $\mathrm{Coker}(T_\eta - \lambda \mathrm{Id})$ for some $\lambda \in E$ and η as above.*
   - *π is supercuspidal iff $\lambda = 0$ in the previous statement.*

Goal: Understand the structure of $\pi(\eta, \lambda) := \mathrm{Coker}(T_\eta - \lambda \mathrm{Id})$.

Classification of p-modular representations for $\mathrm{SL}_2(\mathbb{Q}_p)$

Theorem (A., 2010-2011)

1. *Any irreducible smooth representation of G over $\overline{\mathbb{F}}_p$ is a quotient of some cokernel $\pi(\eta, \lambda)$.*
2. *If $\lambda \neq 0$ and $(\eta, \lambda) \neq (\mathbf{1}, 1)$, then $\pi(r, \lambda)$ is isomorphic to a parabolically induced representation.*
3. *We have the following short exact sequence of $\overline{\mathbb{F}}_p[G]$-modules :*

$$0 \longrightarrow St \longrightarrow \pi(\mathbf{1}, 1) \longrightarrow \mathbf{1} \longrightarrow 0 \ .$$

4. *We have the following non-split short exact sequence of $\overline{\mathbb{F}}_p[G]$-modules for a unique $r \in \{0, \ldots, p-1\}$:*

$$0 \longrightarrow \pi_{p-1-r} \longrightarrow \pi(\eta, 0) \longrightarrow \pi_r \longrightarrow 0 \ .$$

5. *$\{\pi_r,\ 0 \leq r \leq p-1\}$ is a system of representatives of the isomorphism classes of supersingular representations of G.*

What is next ? (Or: towards the second talk of the day)

Beyond the reductive case : the metaplectic group $\widetilde{\mathrm{SL}_2}(F)$

- ⋆ Joint project with **with S. Purkait** (Institute of Science Tokyo)
- Done : Classification of Iwahori-Hecke simple modules and genuine non-supercuspidal representations + connection using Iwahori-typical components; structure of spherical Hecke algebras.
- Ongoing : Classification of genuine supercuspidal representations, variations on Langlands-Shimura correspondence ($F = \mathbb{Q}_p$)

This ends this talk... Are your ready for the next one ?

Vielen Dank !

If you have any question, please feel free to ask !