A survey of progress on Donovan's conjecture

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Contents

Blocks

- G - finite group, p - prime. $\mathcal{O}$ - complete discrete valuation ring with algebraically closed residue field k characteristic p.
  K - field of fractions of $\mathcal{O}$, characteristic zero. Assume 'large enough'.
- Write $1 = e_1 + \cdots + e_n$, primitive orthogonal idempotents of $Z(\mathcal{O}G)$.
- $B_i := \mathcal{O}Ge_i$ indecomp. two-sided ideal of $\mathcal{O}G$, called a *block* of $\mathcal{O}G$.
- Let M be a $\mathcal{O}G$-module.

  $M = e_1M \oplus \cdots \oplus e_nM$, decomposition into submodules

  If M indecomposable, then $M = e_iM$ for some unique i.

  We say M belongs to B_i. So M is a B_i-module.
- To study $\mathrm{mod}(\mathcal{O}G)$, suffices to study each $\mathrm{mod}(B_i)$ in turn.
- Similarly decompose $kG = kG\bar{e}_1 \oplus \cdots \oplus KG\bar{e}_n$, where $\bar{B}_i := B_i \otimes_{\mathcal{O}} k = kG\bar{e}_i$.
- Fix a block B with identity element e.

Defect groups

- Defect group - p-subgroup D of G maximal amongst vertices of simple $\bar{B}$-modules.
- D is unique up to G-conjugacy.
- The unique block to which the trivial module belongs is called the *principal block*, written $B_1 = B_1(\mathcal{O}G)$.
- The defect groups for $B_1(\mathcal{O}G)$ are the Sylow p-subgroups.
- B is semisimple if and only if the defect groups are trivial.
- We may infer information on the block from the defect group.

Fix a defect group D of B.

Cartan matrix

- If a is a primitive idempotent with $a\bar{e} = a$, then kGa is a projective indecomposable left $\bar{B}$-module (that is, an indecomposable summand of $\bar{B}$ as a $\bar{B}$-module).
- $\bar{P}_1, \dots, \bar{P}_r$ - the projective indecomposable $\bar{B}$-modules (up to $\cong$). So $\bar{P}_i/\mathrm{rad}(\bar{P}_i) \cong \mathrm{soc}(\bar{P}_i) \cong S_i$, simple.
- Write c_{ij} for the mult. of S_i as comp. factor of $\bar{P}_j$.
- $C_B = (c_{ij})$ is the *Cartan matrix* of B.

Basic algebras

- Let $e = f_{1,1} + \cdots f_{1,t_1} + f_{2,1} + \cdots + f_{2,t_2} + \cdots + f_{r,1} + \cdots + f_{r,t_r}$, decomp. into primitive orthogonal idempotents, labeled so that $\mathcal{O}Gf_{i,j} \cong P_i$ for all j.
- ${}_BB \cong P_1 \oplus \cdots \oplus P_1 \oplus \cdots \oplus P_r \oplus \cdots \oplus P_r$.
- Write $f = f_{11} + f_{21} + \cdots + f_{r1}$ (one of each $\cong$ type).
- $f\mathcal{O}Gf = fBf$ is a *basic algebra* associated to B.
- Similarly $\bar{f}\bar{B}\bar{f}$ is a basic algebra associated to $\bar{B}$.
- Every simple $\bar{f}\bar{B}\bar{f}$-module has dimension one. Hence $\dim_k(\bar{f}\bar{B}\bar{f})$ is the sum of the entries of the Cartan matrix.

Morita equivalence

Say $\mathcal{O}$-algebras (or k-algebras) A and B are Morita equivalent if their module categories $\mathrm{mod}\,(A)$ and $\mathrm{mod}\,(B)$ are equivalent.

Equivalently:
$\exists$ B-A-bimodule M and A-B-bimodule N such that $N \otimes_B M \cong A$, $M \otimes_A N \cong B$, giving an equivalence via

$$M \otimes_A - : \mathrm{mod}(A) \to \mathrm{mod}(B), N \otimes_B - : \mathrm{mod}(B) \to \mathrm{mod}(A)$$

Example: B is Morita equivalent to the basic algebra fBf via $M = Bf$ and $N = fB$.

- Algebras are Morita equivalent if and only if their basic algebras are isomorphic.

Some Morita invariants

- Cartan matrices (up to labelling of rows+columns) Morita invariant. Hence $|D|$ Morita invariant (largest elementary divisor of C).
- Morita equivalence of $\mathcal{O}$-blocks implies perfect isometry. Hence number of irreducible characters of a given height is a Morita invariant.

Morita equivalence w.r.t. k does not preserve isomorphism type of D (Garcia, Margolis, del Rio '22), but still open w.r.t. $\mathcal{O}$.

Conjectures

Conjecture (Donovan)

Fix a p-group P. There are only finitely many Morita equivalence classes amongst blocks of finite groups with defect groups isomorphic to P. [This conjecture may be stated over k or $\mathcal{O}$.]

Conjecture (Weak Donovan)

Fix a p-group P. There is a bound on the dimension of a basic algebra of blocks of finite groups with defect groups isomorphic to P.

The size of a Cartan matrix bounded in terms of P by Brauer-Feit, so Weak Donovan is equivalent to bounding entries of Cartan matrix.

Tame and finite type blocks

Blocks with cyclic defect groups: Donovan's conjecture known (see Dade, Janusz, Kupisch), and further Puig's Conjecture (Linckelmann). Classification of Morita equivalence classes not known in general, even for C_7, for example.

(Potential Morita equiv. classes described, but not always known if they occur).

Tame blocks: Defect groups Q_{2^n}, D_{2^n}, SD_{2^n}. Donovan's Conjecture over k known except for Q_{2^n} where two simple modules (Erdmann).

Known over $\mathcal{O}$ for Q_{2^n} when three simple modules (Eisele).

The complete classification of Morita equivalence classes for SD_{2^n} is not known (even over k). As with cyclic, potential classes described, but not known which occur.

Morita-Frobenius numbers

Approach as by Kessar '04:

- Define ring automorphism σ of kG, $\sigma\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G}(a_g)^p g$.
- This permutes the blocks. Define $\mathrm{mf}_k(\bar{B})$ to be smallest $m \in \mathbb{N}$ such that $\mathrm{mf}_k(\sigma^m(\bar{B}))$ Morita equivalent to $\bar{B}$.
- Control of $\mathrm{mf}_k(\bar{B})$ means control of field of definition of structure constants of basic algebra.
- $\mathrm{mf}_k(\bar{B})$ and dimension of basic algebra bounded in terms of $P \Leftrightarrow$ Donovan's Conjecture for $\bar{B}$.
- Define $\mathrm{mf}_{\mathcal{O}}(B)$ similarly, and also technical (larger) version $\mathrm{sf}_{\mathcal{O}}(B)$ *strong Frobenius number* (E-Livesey).
- Conjecture (Morita Frobenius conjecture): $\mathrm{mf}_k(B)$, $\mathrm{mf}_{\mathcal{O}}(B)$, $\mathrm{sf}_{\mathcal{O}}(B)$ bounded in terms of P.
- Fix P. Donovan Conjecture $\Leftrightarrow$ Weak Donovan+Morita Frobenius Conjecture. (For k or $\mathcal{O}$).
- $\mathrm{mf}_{\mathcal{O}}(\mathrm{B})$ bounded in terms of P for quasisimple groups (Farrell-Kessar '19).

Initial reduction steps

- Külshammer (over k, '95), Eisele (over $\mathcal{O},'21$): To prove Donovan's Conjecture, it suffices to prove it for $G = \langle D^g : g \in G \rangle$.
- Fong-Reynolds/Külshammer-Puig reductions: May assume covered blocks of normal subgroups N are stable, and cannot cover nilpotent blocks unless $N = O_p(G)Z(G)$.

Use these to show suffices to consider only "reduced" pairs (G, B) of a group and a block with defect group $D \cong P$, which have the above constraints. E.g., $O_{p'}(G) \leq Z(G)$, etc.

Then investigate structure of such G.

Abelian defect groups

- Let $N \lhd G$ index p. Let $b \in \mathrm{Blk}(N)$ covered by $B \in \mathrm{Blk}(G)$ with abelian defect group D. Then $\mathrm{sf}_{\mathcal{O}}(B) \leq \mathrm{sf}_{\mathcal{O}}(b)$ (E-Livesey '19).
- 2-blocks of quasisimple groups with abelian defect groups are described by E-Kessar-Külshammer-Sambale '14.

Theorem (E-Eisele-Livesey '20)

(i) Fix p. If the Weak Donovan Conjecture holds for blocks of quasisimple groups with abelian defect groups, then Donovan's Conjecture holds for abelian p-groups.
(ii) Donovan's Conjecture holds for abelian 2-groups.

2-blocks with abelian defect groups of rank at most 4 classified up to Morita equivalence (E-Livesey '24), $(C_2)^5$ (Ardito '21)

Nonabelian defect groups

Let P be a nonabelian p-group (or class of p-groups). To prove Donovan's Conjecture for P, we need (at least):

- to understand blocks of quasisimple groups with this defect group
- to understand configurations where blocks with this defect group occur in automorphism groups of simple groups and their central extensions.

Resistant p-groups and controlled blocks: A p-group P is *resistant* if all fusion systems on P are realised by a finite group with P as a normal Sylow p-subgroup. (E.g., abelian groups, Suzuki 2-groups, etc.)

Blocks with resistant defect groups are controlled blocks.

J. An ($p \neq 2$ '11, $p = 2$ '20): description of controlled blocks of quasisimple groups.

Controlled blocks

Fusion trivial p-groups

If P resistant and $\mathrm{Aut}(P)$ is a p-group, then there is a unique fusion system on P. Call P fusion trivial. Then B is nilpotent (see Broué-Puig), and so Morita equivalent to $\mathcal{O}P$ (Puig '88).

Theorem (E-Külshammer-Sambale '11)

Let $P = \langle x, y : x^{2^r} = y^{2^r} = [x, y]^2 = [x, x, y] = [y, x, y] = 1$, one of the classes of minimal nonabelian 2-groups. Then there are two Morita equivalence classes of blocks with defect groups $\cong P$.

Important in proof: Cannot cover non-nilpotent block of subgroup index 2.

Controlled blocks

Theorem (E-Livesey '21)

Donovan's Conjecture holds for $Q_8 \times Q_8$, $Q_8 \times C_{2^n}$ (w.r.t. $\mathcal{O}$).

Important in proof: the results that control $\mathrm{sf}_{\mathcal{O}}(B)$ when passing to normal subgroups $N \lhd G$ of index p in the abelian defect group case generalize to the case $G = C_D(D \cap N)N$, where D is a defect group for B.

Remark: We cannot at present describe the Morita equivalence classes.

Theorem (E '24)

Let P be a Suzuki 2-group. Then blocks with defect groups $\cong P$ described up to Morita equivalence, and Donovan's Conjecture holds (w.r.t. $\mathcal{O}$).

Important in proof: Normal subgroups must be 'large' in some sense.

Extraspecial defect groups

- Blocks of quasisimple groups with extraspecial defect groups described (An-E '11).
- If B block of (central extension of) automorphism group of simple group with defect group $D \cong p_+^{1+2}$ and element of D lies outside of quasisimple, then few cases occur.
  Briefly: if Lie type $G(q)$, where $p \nmid q$ and there is field auto of order p, then almost never occurs that the quasisimple has block with defect group $C_p \times C_p$. If it does, then field auto centralizes the defect group.

Theorem (An-E '25)

Donovan's Conjecture holds for p_+^{1+2} for $p \geq 5$. Further, Morita equivalence classes described (w.r.t. $\mathcal{O}$).

Block library wiki site

I maintain a wiki site recording

- progress on Donovan's Conjecture
- the classification of Morita equivalence classes of blocks with a given defect group
- Picard groups for blocks

https://wiki.manchester.ac.uk/blocks/index.php/Main_Page

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