Decomposition columns labelled by d-balanced partitions

Pavel Turek

University of Birmingham

November 14, 2025

joint work with David Hemmer, Bim Gustavsson and Stacey Law

Young diagrams

Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of size n is a non-increasing sequence of positive integers, called *parts*, which add up to n.

Young diagrams

Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of size n is a non-increasing sequence of positive integers, called *parts*, which add up to n. We can represent partition λ using its *Young diagram*

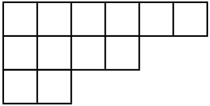


Figure: The Young diagram of $(6, 4, 2)$, a partition of size 12.

Young diagrams

Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of size *n* is a non-increasing sequence of positive integers, called *parts*, which add up to *n*. We can represent partition λ using its *Young diagram*

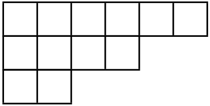


Figure: The Young diagram of $(6, 4, 2)$, a partition of size 12.

We fix an integer $e > 2$.

Young diagrams

Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of size n is a non-increasing sequence of positive integers, called *parts*, which add up to n. We can represent partition λ using its *Young diagram*

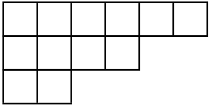


Figure: The Young diagram of $(6, 4, 2)$, a partition of size 12.

We fix an integer $e > 2$. A partition is *e-regular* if it has no e (or more) equal parts.

Young diagrams

Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of size n is a non-increasing sequence of positive integers, called *parts*, which add up to n. We can represent partition λ using its *Young diagram*

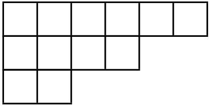


Figure: The Young diagram of $(6, 4, 2)$, a partition of size 12.

We fix an integer $e > 2$. A partition is *e-regular* if it has no e (or more) equal parts.

Example: Partition $(7, 7, 4, 3, 3, 3, 3, 2)$ is 5-regular but not 3-regular.

Irreducible modules

Over a field of characteristic zero, the irreducible modules of the symmetric group S_n are *the Specht modules S^λ* indexed by partitions λ of size n. The same holds for Hecke algebras of quantum characteristic 0.

Irreducible modules

Over a field of characteristic zero, the irreducible modules of the symmetric group S_n are *the Specht modules S^λ* indexed by partitions λ of size n. The same holds for Hecke algebras of quantum characteristic 0.

In (quantum) characteristic e, the Specht modules are not irreducible anymore; the irreducible modules are often denoted by D^μ and indexed by e-regular partitions μ of size n.

Irreducible modules

Over a field of characteristic zero, the irreducible modules of the symmetric group S_n are *the Specht modules S^λ* indexed by partitions λ of size n. The same holds for Hecke algebras of quantum characteristic 0.

In (quantum) characteristic e, the Specht modules are not irreducible anymore; the irreducible modules are often denoted by D^μ and indexed by e-regular partitions μ of size n.

The irreducible composition factors of the Specht modules in (quantum) characteristic e are given by *the decomposition matrix*.

Decomposition matrix

$S^{(\cdot)}$ \ $D^{(\cdot)}$	6	$5,1$	$4,2$	$3,3$	$4,1,1$	$3,2,1$	$2,2,1,1$
6	1						
$5,1$	1	1					
$4,2$			1				
$3,3$		1		1			
$4,1,1$		1			1		
$3,2,1$	1	1		1	1	1	
$2,2,2$	1					1	
$3,1,1,1$					1	1	
$2,2,1,1$							1
$2,1,1,1,1$				1		1	
$1,1,1,1,1,1$				1			

Table: The decomposition matrix of S_6 in characteristic $e = 3$.

Decomposition matrix

$S^{(\cdot)}$ \ $D^{(\cdot)}$	6	$5,1$	$4,2$	$3,3$	$4,1,1$	$3,2,1$	$2,2,1,1$
6	1						
$5,1$	1	1					
$4,2$			1				
$3,3$		1		1			
$4,1,1$		1			1		
$3,2,1$	1	1		1	1	1	
$2,2,2$	1					1	
$3,1,1,1$					1	1	
$2,2,1,1$							1
$2,1,1,1,1$				1		1	
$1,1,1,1,1,1$				1			

Table: The decomposition matrix of S_6 in characteristic $e = 3$.

e-cores

A connected removable group of ae boxes (for some $a > 0$) with no 2×2 square is called an *e-divisible hook*. Its *arm-length* equals the number of columns the e-divisible hook spans minus one.

A connected removable group of ae boxes (for some $a > 0$) with no 2×2 square is called an *e-divisible hook*. Its *arm-length* equals the number of columns the e-divisible hook spans minus one.

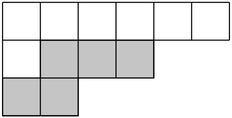


Figure: A 5-divisible hook of $(6, 4, 2)$ of arm length 3.

e-cores

A connected removable group of ae boxes (for some $a > 0$) with no 2×2 square is called an *e-divisible hook*. Its *arm-length* equals the number of columns the e-divisible hook spans minus one.

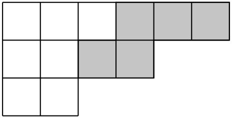


Figure: A 5-divisible hook of $(6, 4, 2)$ of arm length 3.

A connected removable group of ae boxes (for some $a > 0$) with no 2×2 square is called an *e-divisible hook*. Its *arm-length* equals the number of columns the e-divisible hook spans minus one.

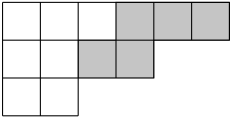


Figure: A 5-divisible hook of $(6, 4, 2)$ of arm length 3.

The e-core of a partition is obtained by successive removals of its e-divisible hooks.

e-cores

A connected removable group of ae boxes (for some $a > 0$) with no 2×2 square is called an *e-divisible hook*. Its *arm-length* equals the number of columns the e-divisible hook spans minus one.

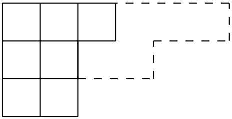


Figure: Finding the 5-core of $(6, 4, 2)$.

The e-core of a partition is obtained by successive removals of its e-divisible hooks.

A connected removable group of ae boxes (for some $a > 0$) with no 2×2 square is called an *e-divisible hook*. Its *arm-length* equals the number of columns the e-divisible hook spans minus one.

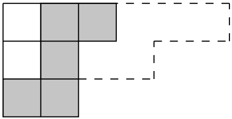


Figure: Finding the 5-core of $(6, 4, 2)$.

The e-core of a partition is obtained by successive removals of its e-divisible hooks.

A connected removable group of ae boxes (for some $a > 0$) with no 2×2 square is called an *e-divisible hook*. Its *arm-length* equals the number of columns the e-divisible hook spans minus one.

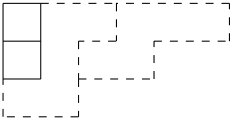


Figure: Finding the 5-core of $(6, 4, 2)$.

The e-core of a partition is obtained by successive removals of its e-divisible hooks.

Odd sequence

The *e-residue* of a box (i, j) of a Young diagram is the remainder of $j - i$ modulo e.

Odd sequence

The *e-residue* of a box (i, j) of a Young diagram is the remainder of $j - i$ modulo e.

0	1	2	3	4	0
4	0	1	2		
3	4				

0	1	2	3
4	0	1	
3	4	0	
2			
1			

Figure: The e-residues of partitions $(6, 4, 2)$ and $(4, 3, 3, 1, 1)$.

Odd sequence

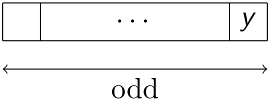
The *e-residue* of a box (i, j) of a Young diagram is the remainder of $j - i$ modulo e.

0	1	2	3	4	0
4	0	1	2		
3	4				

0	1	2	3
4	0	1	
3	4	0	
2			
1			

Figure: The e-residues of partitions $(6, 4, 2)$ and $(4, 3, 3, 1, 1)$.

The *odd sequence* of λ stores in its yth entry $(0 \leq y \leq e - 1)$ the number of parts of λ of the following form.



Odd sequence

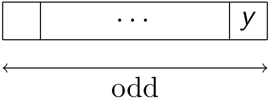
The *e-residue* of a box (i, j) of a Young diagram is the remainder of $j - i$ modulo e.

0	1	2	3	4	0
4	0	1	2		
3	4				

0	1	2	3
4	0	1	
3	4	0	
2			
1			

Figure: The e-residues of partitions $(6, 4, 2)$ and $(4, 3, 3, 1, 1)$.

The *odd sequence* of λ stores in its yth entry $(0 \leq y \leq e - 1)$ the number of parts of λ of the following form.



The partition $(6, 4, 2)$ has odd sequence $(0, 0, 0, 0, 0)$.

d-runner matrix

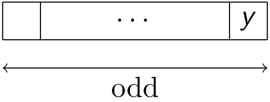
The *e-residue* of a box (i, j) of a Young diagram is the remainder of $j - i$ modulo e.

0	1	2	3	4	0
4	0	1	2		
3	4				

0	1	2	3
4	0	1	
3	4	0	
2			
1			

Figure: The e-residues of partitions $(6, 4, 2)$ and $(4, 3, 3, 1, 1)$.

The *odd sequence* of λ stores in its yth entry $(0 \le y \le e - 1)$ the number of parts of λ of the following form.



The partition $(4, 3, 3, 1, 1)$ has odd sequence $(1, 2, 1, 0, 0)$.

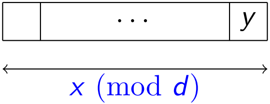
The *e-residue* of a box (i, j) of a Young diagram is the remainder of $j - i$ modulo e.

0	1	2	3	4	0
4	0	1	2		
3	4				

0	1	2	3
4	0	1	
3	4	0	
2			
1			

Figure: The e-residues of partitions $(6, 4, 2)$ and $(4, 3, 3, 1, 1)$.

The *d-runner matrix* of λ stores in its (x, y) entry ($1 \leq x \leq d - 1$ and $0 \leq y \leq e - 1$) the number of parts of λ of the following form.



d-runner matrix

The *e-residue* of a box (i, j) of a Young diagram is the remainder of $j - i$ modulo e.

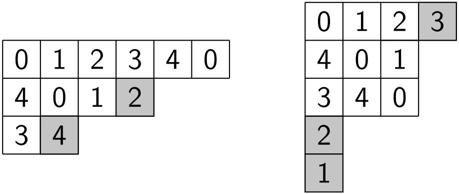
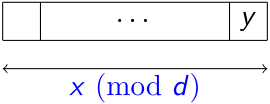


Figure: The e-residues of partitions $(6, 4, 2)$ and $(4, 3, 3, 1, 1)$.

The *d-runner matrix* of λ stores in its (x, y) entry $(1 \leq x \leq d - 1$ and $0 \leq y \leq e - 1)$ the number of parts of λ of the following form.



The partition $(6, 4, 2)$ has 3-runner matrix $\left(\begin{smallmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right)$.

d-runner matrix

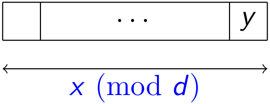
The *e-residue* of a box (i, j) of a Young diagram is the remainder of $j - i$ modulo e.

0	1	2	3	4	0
4	0	1	2		
3	4				

0	1	2	3
4	0	1	
3	4	0	
2			
1			

Figure: The e-residues of partitions $(6, 4, 2)$ and $(4, 3, 3, 1, 1)$.

The *d-runner matrix* of λ stores in its (x, y) entry ($1 \leq x \leq d - 1$ and $0 \leq y \leq e - 1$) the number of parts of λ of the following form.



The partition $(4, 3, 3, 1, 1)$ has 3-runner matrix $\left(\begin{smallmatrix}0&1&1&1&0\\0&0&0&0&0\end{smallmatrix}\right)$.

d-balanced partitions

Definition

Let d be a positive integer. We say that a partition λ is *d-balanced* if d divides all arm lengths of its e-divisible hooks.

d-balanced partitions

Definition

Let d be a positive integer. We say that a partition λ is *d-balanced* if d divides all arm lengths of its e-divisible hooks.

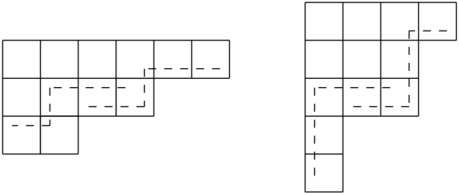


Figure: Partitions $(6, 4, 2)$ and $(4, 3, 3, 1, 1)$ are 3- and 2-balanced, respectively.

d-balanced partitions

Definition

Let d be a positive integer. We say that a partition λ is *d-balanced* if d divides all arm lengths of its e-divisible hooks.

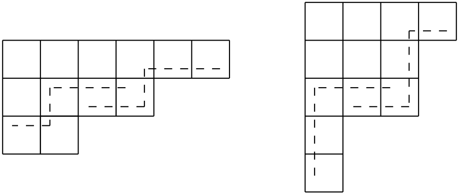


Figure: Partitions $(6, 4, 2)$ and $(4, 3, 3, 1, 1)$ are 3- and 2-balanced, respectively.

Theorem (T. 2025)

Let $e > 1$ be an odd integer. The natural map

$$\phi : \{\text{2-balanced partitions}\} \to \{e\text{-cores}\} \times \mathbb{Z}^{e}_{\geq 0}$$

is a bijection. Moreover, μ is e-regular if $\phi_2(\mu)$ contains zero.

d-balanced partitions

Definition

Let d be a positive integer. We say that a partition λ is *d-balanced* if d divides all arm lengths of its e-divisible hooks.

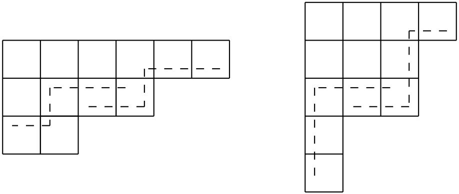


Figure: Partitions $(6,4,2)$ and $(4,3,3,1,1)$ are 3- and 2-balanced, respectively.

Theorem (T. 2025)

Let $e > 1$ be an odd integer. The natural map

$$\phi : \{d\text{-balanced partitions}\} \to \{e\text{-cores}\} \times \mathbb{Z}_{\geq 0}^{(d-1)\times e}$$

is injective. Moreover, μ is e-regular if $\phi_2(\mu)$ contains zero in its first row.

2-balanced columns

Theorem (Hemmer, T. 2025+)

Let e be an odd prime. The decomposition column labelled by a 2-balanced e-regular partition μ contains ones in rows labelled by partitions with the same size, e-core and odd sequence as μ, and zeros elsewhere.

$$\begin{array}{ccc} & & \mu \\ & & 0 \\ & & 1 \\ \begin{array}{c} \text{size} \\ e\text{-core} \\ \text{odd sequence} \end{array} \text{ as } \mu & & 1 \\ & & 0 \\ & & 1 \\ & & 0 \\ & & 0 \end{array}$$

d-balanced columns

Theorem (Gustavsson, Law, T. 2025+)

Let $e > d > 1$ be coprime integers. If the decomposition column labelled by a d-balanced e-regular partition μ contains ones or zeros in rows labelled by partitions with the same size, e-core and d-runner matrix as μ, then these entries are ones, and the other entries in this column are zeros.

$$\begin{array}{c} \text{size} \\ e\text{-core} \\ d\text{-runner matrix} \end{array} \text{ as } \mu \longrightarrow \begin{array}{c} \mu \\ ? \\ \leq 1 \\ \leq 1 \\ ? \\ \leq 1 \\ ? \\ ? \end{array} \quad \Longrightarrow \quad \begin{array}{c} \text{size} \\ e\text{-core} \\ d\text{-runner matrix} \end{array} \text{ as } \mu \longrightarrow \begin{array}{c} \mu \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{array}$$

Ideas behind the proofs

- The previous result uses the Jantzen–Schaper formula and suitable algorithms on partitions.

Ideas behind the proofs

- The previous result uses the Jantzen–Schaper formula and suitable algorithms on partitions.
- For $d = 2$, one can build on the result of Giannelli and Wildon, who found vertices of indecomposable summands of Foulkes modules

$$H^{(2^m)} = \mathbf{1}\uparrow_{S_2 \wr S_m}^{S_{2m}}$$

and 'twisted' Foulkes modules

$$H^{(2^m,k)} = \left(H^{(2^m)} \boxtimes \mathrm{sgn}\right)\uparrow_{S_{2m} \times S_k}^{S_{2m+k}}$$

.

Ideas behind the proofs

- The previous result uses the Jantzen–Schaper formula and suitable algorithms on partitions.
- For $d = 2$, one can build on the result of Giannelli and Wildon, who found vertices of indecomposable summands of Foulkes modules

$$H^{(2^m)} = \mathbf{1}\uparrow_{S_2 \wr S_m}^{S_{2m}}$$

and 'twisted' Foulkes modules

$$H^{(2^m,k)} = \left(H^{(2^m)} \boxtimes \mathrm{sgn}\right)\uparrow_{S_{2m}\times S_k}^{S_{2m+k}}$$

.

- For central orthogonal primitive idempotents a of $\mathbb{F}_e S_{2m}$ and b of $\mathbb{F}_e S_{2m+k}$, partitions in $b\left(\left(aH^{(2^m)} \boxtimes \mathrm{sgn}\right)\uparrow_{S_{2m}\times S_k}^{S_{2m+k}}\right)$ share size, e-core and odd sequence.