An introduction to arithmetic groups
(via group schemes)

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Content

- Properties of arithmetic groups
- Arithmetic groups as lattices in Lie groups

Last week

Let G be a linear algebraic group over $\mathbb{Q}$.

Definition:

A subgroup $\Gamma \subseteq G(\mathbb{Q})$ is arithmetic if it is commensurable to $G_0(\mathbb{Z})$ for some integral form G_0 of G.

integral form: a group scheme G_0 over $\mathbb{Z}$ with an isomorphism

$$E_{\mathbb{Q}/\mathbb{Z}}(G_0) \cong G.$$

Recall: Here group schemes are affine and of finite type.

S-arithmetic groups

S: a finite set of prime numbers.

$$\mathbb{Z}_S := \mathbb{Z}[\frac{1}{p} \mid p \in S]$$

Definition:

A subgroup $\Gamma \subseteq G(\mathbb{Q})$ is S-arithmetic if it is commensurable to $G_0(\mathbb{Z}_S)$ for some $\underbrace{\text{integral form } G_0}_{\text{better: „a form over } \mathbb{Z}_S\text{"}}$ of G.

Similar: replace $\mathbb{Z}$ by $\mathbb{F}_p[t]$

$\mathbb{Q}$ by $\mathbb{F}_p(t)$

Properties of arithmetic groups

Theorem 2: Let $\Gamma \subseteq G(\mathbb{Q})$ be an arithmetic group.

✓ 1 Γ is residually finite.

2 Γ is virtually torsion-free.

$\Gamma_0 \trianglelefteq_{fi} \Gamma$ with Γ_0 torsion-free

3 Γ has only finitely many conjugacy classes of finite subgroups.

$\boxed{2} \Rightarrow$ finitely many iso classes of finite subgroups

$F \subseteq \Gamma$ finite $\Rightarrow$ $F \cap \Gamma_0 = \{e\}$ F isomorphic to a subgroup Γ/Γ_0

4 Γ is finitely presented.

$\hookrightarrow \Gamma$ is of type F_∞

Proof: Γ virtually torsion-free

Assume $\Gamma = G_0(\mathbb{Z})$.

Claim: $G_0(\mathbb{Z}, b)$ is torsion-free for $b \geq 3$.

$$G_0(\mathbb{Z}, \overset{b}{\cancel{m}}) = \ker\big(G_0(\mathbb{Z}) \to G_0(\mathbb{Z}/b\mathbb{Z})\big)$$

Suppose $g \in G_0(\mathbb{Z}, b)$ has finite order > 1.

wlog $\operatorname{ord}(g) = p$ prime

$g : \mathcal{O}_{G_0} \longrightarrow \mathbb{Z}$

$g = \varepsilon + b\,h$

$h : \mathcal{O}_{G_0} \to \mathbb{Z}$

$\mathbb{Z}$-linear

Assume h is onto

Proof: Γ virtually torsion-free

- $g = \varepsilon + bh$ with $h\colon \mathcal{O}_{G_0} \to \mathbb{Z}$ onto.
- $\operatorname{ord}(g) = p$ prime.

$$G_0 \longrightarrow G_0 \qquad \rightsquigarrow \Delta^{(k)}\colon \mathcal{O}_{G_0} \longrightarrow \mathcal{O}_{G_0}$$
$$x \longmapsto x^k$$

$$\varepsilon = g^p = g \circ \Delta^{(p)} = \varepsilon + pbh + \sum_{k=2}^{p} \binom{p}{k} b^k \Delta^{(k)}(h)$$

mod b^2: $\quad 0 \equiv pbh \bmod b^2 \Rightarrow b \mid p \Rightarrow p = b \geq 3$

mod p^3: $\quad 0 \equiv p^2 h \bmod p^3 \Rightarrow p^2 \equiv 0 \bmod p^3$ contradiction!

$p \mid \binom{p}{k}$ for all $1 \leq k < p$

Group schemes and topological groups

R : commutative unital ring

G : affine group scheme over R

A : an R-algebra which is also a topological ring. $(\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{A} \dots)$

Observation:

$G(A)$ is a topological group with respect the topology induced by coordinates

$$\psi_{c,A} \colon G(A) \xrightarrow{\cong} V_A(I_c) \subseteq A^n$$

product topology

Fact: Does not depend on chosen coordinates.

Group schemes and topological groups

A : an R-algebra which is also a topological ring.

$\varphi^\sharp : \mathcal{O}_H \to \mathcal{O}_G$ is onto

Observation:

If $\varphi \colon G \to H$ is a *closed embedding* of affine group schemes over R, then

$$\varphi_A \colon G(A) \to H(A)$$

is a continuous *closed embedding* of topological groups.

(Hint: pick coordinates c for H and push them to G c'

$$I_c \subseteq I_{c'}$$

$$G(A) \cong V_A(I_{c'}) \subseteq V_A(I_c) \cong H(A) \quad)$$

Group schemes and topological groups

G: affine group scheme over $\mathbb{Z}$.

Consequences:

- $G(\mathbb{R})$ is a real Lie group (with finitely many connected components).

$\varphi : E_{\mathbb{R}/\mathbb{Z}}(G) \hookrightarrow GL_n$ closed embedding

$G(\mathbb{R}) \underset{\text{closed}}{\subseteq} GL_n(\mathbb{R})$ Lie group!

Fact: real alg. varieties have finitely many „Euclidean" components!

- $G(\mathbb{Z}) \subseteq G(\mathbb{R})$ is a discrete subgroup.

$$\begin{array}{ccccc} G(\mathbb{R}) & \cong & V_{\mathbb{R}}(I_c) & \subseteq & \mathbb{R}^n \\ \cup & & \cup & & \cup \text{ discrete} \\ G(\mathbb{Z}) & \cong & V_{\mathbb{Z}}(I_c) & \subseteq & \mathbb{Z}^n \end{array}$$

Theorem of Borel and Harish-Chandra

G linear algebraic group over $\mathbb{Q}$

$\Gamma \subseteq G(\mathbb{Q})$ arithmetic subgroup

„lattice"

1. $\Gamma \subseteq G(\mathbb{R})$ has finite covolume

   $\iff$

   there is *no* surjective homomorphism $G \to \mathbb{G}_m$.

   (surjective $G(\mathbb{C}) \to \mathbb{C}^\times$)

2. $\Gamma \subseteq G(\mathbb{R})$ is cocompact

   $\iff$

   there is *no* closed embedding $\mathbb{G}_m \to G$.

   G is anisotropic

Remark: Every surjective $G \to \mathbb{G}_m$ splits.

Examples

$\mathbb{R}/\mathbb{Z} \cong S^1$

- $\mathbb{Z} = \mathbb{G}_a(\mathbb{Z}) \subseteq \mathbb{R} = \mathbb{G}_a(\mathbb{R})$ is cocompact

Exercise: There is no surjective hom of $\mathbb{Q}$-algebras $: \mathbb{Q}[T] \to \mathbb{Q}[T, T^{-1}]$

- $\mathrm{GL}_n(\mathbb{Z}) \subseteq \mathrm{GL}_n(\mathbb{R})$ is not a lattice

$\det : GL_n \longrightarrow \mathbb{G}_m$ is surjective.

- $\mathrm{SL}_n(\mathbb{Z}) \subseteq \mathrm{SL}_n(\mathbb{R})$ is a lattice but is not cocompact

not cocompact: $\mathbb{G}_m \longrightarrow SL_n \qquad a \mapsto \begin{pmatrix} a & & & & \\ & a^{-1} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$

lattice: $\varphi : SL_n \longrightarrow \mathbb{G}_m$

$\varphi_{\mathbb{R}} : SL_n(\mathbb{R}) \longrightarrow \mathbb{R}^\times$

simple

Diagonalization Lemma

Let $\varphi\colon \mathbb{G}_m \to \mathrm{GL}_n$ be a homomorphism of linear algebraic groups over K. There is a matrix $g \in \mathrm{GL}_n(K)$ s.t.

$$g\varphi(\lambda)g^{-1} = \begin{pmatrix} \lambda^{e_1} & & & \\ & \lambda^{e_2} & & \\ & & \ddots & \\ & & & \lambda^{e_n} \end{pmatrix}$$

for certain $e_1, \ldots, e_n \in \mathbb{Z}$ and for all $\lambda \in K^\times$.

Note: If φ is a closed embedding, then $e_i \neq 0$ for some i.

More examples (1)

The Heisenberg group is cocompact:

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\} \quad \subseteq H_3(\mathbb{R})$$

<u>Reason:</u> elements $\neq \mathbb{1}_3$ are not diagonalizable.

More examples (2)

$F = \mathbb{Q}(\sqrt{2})$ quadratic number field

$$\sigma_1\colon F \to \mathbb{R} \qquad \text{with} \quad \sqrt{2} \mapsto \sqrt{2}$$

$$\sigma_2\colon F \to \mathbb{R} \qquad \text{with} \quad \sqrt{2} \mapsto -\sqrt{2}$$

Observation:

$$(\sigma_1, \sigma_2)\colon F \to \mathbb{R} \times \mathbb{R}$$

induces an isomorphism $\mathbb{R} \otimes_{\mathbb{Q}} F \xrightarrow{\cong} \mathbb{R} \times \mathbb{R}$.

More examples (2)

Define:

$$G(A) = \{g \in \mathrm{GL}_{n+1}(A \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]) \mid g^T J g = J\}$$

where $J = \begin{pmatrix} -\sqrt{2} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$.

Observation: $G(\mathbb{R}) \cong \mathrm{O}(n,1) \times \mathrm{O}(n+1)$

$\{g \in GL_{n+1}(\mathbb{R} \otimes_{\mathbb{Q}} F) \mid g^T J g = J\}$

$= \{(g_1, g_2) \in GL_{n+1}(\mathbb{R}) \times GL_{n+1}(\mathbb{R}) \mid g_1^T J g_1 = J,\ g_2^T \sigma_2(J) g_2 = \sigma_2(J)\}$

More examples (2)

Claim: $G(\mathbb{Z}) \subseteq G(\mathbb{R}) \cong \mathrm{O}(n,1) \times \mathrm{O}(n+1)$ is cocompact.

„compact factor trick"

$\varphi : \mathbb{G}_m \to G$

$\tilde{\varphi}_{\mathbb{R}} : \mathbb{G}_m /_{\mathbb{R}} \to E_{\mathbb{R}/\mathbb{Z}}(G) \xhookrightarrow{\text{closed}} GL_{2(n+1)}$

$\tilde{\varphi}_{\mathbb{R}}(t)$ diagonalizable with eigenvalues $t^{e_1}, \dots, t^{e_m}$

$m = 2(n+1)$

More examples (2)

Claim: $G(\mathbb{Z}) \subseteq G(\mathbb{R}) \cong \mathrm{O}(n,1) \times \mathrm{O}(n+1)$ is cocompact.

$t \in \mathbb{Q}^\times$

$$\tilde{\varphi}_{\mathbb{R}}(t) = \left(\begin{array}{c|c} \sigma_1(\varphi(t)) & 0 \\ \hline 0 & \underbrace{\sigma_2(\varphi(t))}_{\mathrm{O}(n+1)} \end{array}\right)$$

complex eigenvalues of abs value 1.

$$\sigma_2(\varphi(t)) = 1$$

$$\Rightarrow \sigma_1(\varphi(t)) = 1$$

φ is not a closed embedding.

An observation

Lemma: Let G, H be real Lie groups with finitely many connected components. Let $\varphi\colon G \to H$ be a surjective homomorphism with compact kernel $K = \ker(\varphi)$.

Assume $\Gamma \subseteq G$ is a discrete subgroup, then the following hold:

1. $\varphi(\Gamma) \subseteq H$ is discrete.
2. Γ torsion-free $\implies \Gamma \cong \varphi(\Gamma)$.
3. G/Γ compact $\iff H/\varphi(\Gamma)$ compact.
4. $\Gamma \subseteq G$ is a lattice $\iff \varphi(\Gamma) \subseteq H$ is a lattice.

Proof

Fact: φ is open and proper. $(\varphi^{-1}(C)$ compact if C is compact$)$

(1) Let $h \in H$, $U \subseteq H$ an open relatively compact neighbourhood.

compact $\varphi^{-1}(\bar{U}) \supseteq \varphi^{-1}(U)$ $\qquad \varphi^{-1}(U) \cap \Gamma$ is finite

$\varphi(\varphi^{-1}(U) \cap \Gamma) = U \cap \varphi(\Gamma)$ is finite

(2) $\Gamma \cap K$ discrete and compact $\Rightarrow$ finite
$\;\;\;\;\|$
$\;\;\{e\}$ $\qquad \varphi|_\Gamma : \Gamma \xrightarrow{\cong} \varphi(\Gamma)$

Proof

Fact: φ is open and proper.

(3) "$\Rightarrow$": φ induces is a surjective continuous map

$$\overline{\varphi}\colon G/\Gamma \to H/\varphi(\Gamma).$$

$$g\Gamma \mapsto \varphi(g)\varphi(\Gamma)$$

"$\Leftarrow$": If $H/\varphi(\Gamma)$ is compact, there is a compact set $C \subseteq H$ with

$$C\varphi(\Gamma) = H.$$

Then $\varphi^{-1}(C)\Gamma = G$.

and $\varphi^{-1}(C)$ maps onto G/Γ

$\Longrightarrow$ compact.

Back to the example

$$\Gamma = G(\mathbb{Z}) = \left\{ g \in \mathrm{GL}_{n+1}(\mathbb{Z}[\sqrt{2}]) \mid g^T J g = J \right\}$$

is a discrete cocompact subgroup of $G(\mathbb{R}) \cong \mathrm{O}(n,1) \times \mathrm{O}(n+1)$.

Project onto first factor:

Γ is a discrete cocompact subgroup of $\mathrm{O}(n,1)$.

Arithmetically defined groups

Definition:

Let H be a real Lie group with finitely many connected components.

(mostly "arithmetic")

A lattice $\Delta \subseteq H$ is arithmetically defined if

- there is a linear algebraic group G over $\mathbb{Q}$,
- an arithmetic subgroup $\Gamma \subseteq G(\mathbb{Q}) \cap G(\mathbb{R})^0$ and
- a surjective homomorphism $\varphi\colon G(\mathbb{R})^0 \to H^0$ with compact kernel

such that Δ and $\varphi(\Gamma)$ commensurable.

Margulis' arithmeticity

Theorem [Margulis]:

Let H be a connected simple Lie group such that

$H = G(\mathbb{R})^0$ for some linear algebraic $\mathbb{R}$-group G of $\mathbb{R}$-rank ≥ 2.

→ closed embedding $\mathbb{G}_m^2 \hookrightarrow G$ (over $\mathbb{R}$)

Every lattice $\Delta \subseteq H$ is arithmetically defined.

eg. H trivial center, then $H = \mathrm{Aut}(\mathfrak{g})^0$

Simple groups of $\mathbb{R}$-rank ≥ 2

- $\mathrm{SL}_n(\mathbb{R})$ for $n \geq 3$. $\quad rk(SL_n) = n-1$

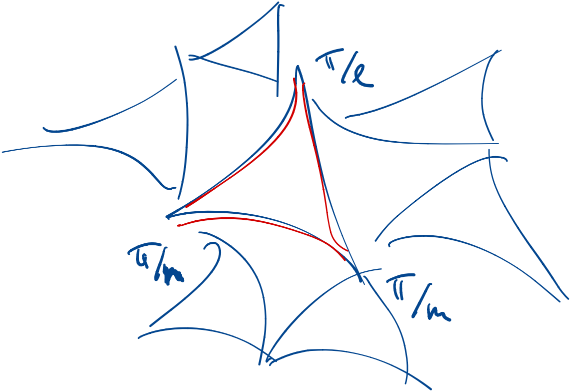
- $\mathrm{Sp}_{2n}(\mathbb{R})$ for $n \geq 2$. $\quad rk(Sp_{2n}) = n$

- $\mathrm{SO}(p,q)$ for $p,q \geq 2$. $\quad rk(SO(p,q)) = \min(p,q)$

- $\mathrm{SU}(p,q)$ for $p,q \geq 2$.

Triangle groups

Hyperbolic triangle group: (ℓ, m, n) with $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} < 1$.



Has a subgroup $\Gamma(\ell, m, n)$ of index 2 which is a lattice in $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$.

Takeuchi's Theorem:

$\Gamma(\ell, m, n)$ is arithmetically defined if and only if all other roots of the minimal polynomial of

$$\lambda(\ell, m, n) = 4c_\ell^2 + 4c_m^2 + 4c_n^2 + 8c_\ell c_m c_n - 4$$

are real and negative (where $c_k = \cos(\frac{\pi}{k})$).

Arithmetic examples: $(2,3,7)$, $(2,8,8)$, $(6,6,6)$, … only finitely many!

Non-arithmetic examples: $(2,5,7)$, $(3,7,7)$, $(4,11,13)$, …

Questions?

$\mathbb{Z}[1/p]$ $\qquad G_0(\mathbb{Z}[1/p]) \subseteq G(\mathbb{R}) \times G(\mathbb{Q}_p)$

Questions?