Twisted conjugacy in soluble arithmetic groups

In collaboration with Y. Santos Rego

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1 Outline

1 Twisted conjugacy and R_∞

Given a group G and an automorphism $\varphi \in \mathrm{Aut}(G)$, the *($\varphi$-)Reidemeister class* of $g \in G$ is

$$[g]_\varphi = \{hg\varphi(h)^{-1} \mid h \in G\}.$$

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Reidemeister number:

$$R(\varphi) = |\{[g]_\varphi \mid g \in G\}|.$$

A group G has *property* R_∞ if, for all $\varphi \in \mathrm{Aut}(G)$, one has

$$R(\varphi) = \infty.$$

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Example

$\mathbb{Z}$ is abelian and infinite so that $R(\text{id}) = \infty$.

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$\mathbb{Z}$ is abelian and infinite so that $R(\mathrm{id}) = \infty$.
However, $R(-\mathrm{id}) = 2$:

$$[0]_{-\mathrm{id}} = \{\text{even numbers}\}, \quad [1]_{-\mathrm{id}} = \{\text{odd numbers}\}.$$

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- (Dekimpe, Gonçalves, Wong and others) Certain (but not all) polycylic groups;
- (Nasybullov) Groups of unitriangular matrices over certain integral domains as long as their nilpotency class is large enough.

Goal

Put previous soluble examples in a common framework or generalize them if possible.

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Investigate upper triangular matrices over integral domains.

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Investigate upper triangular matrices over integral domains.
Develop methods to determine R_∞ depending on base ring.

2 Outline

1. Twisted conjugacy and R_∞
2. Upper triangular matrix groups over R
3. Which of those groups have R_∞?
4. Automorphisms of Rings
5. Some examples in positive characteristic

Throughout, R is an integral domain.

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Consider the group

$$\mathbf{B}_n(R) = \begin{bmatrix} * & * & * & * & * \\ & * & \ddots & * & * \\ & & \ddots & \ddots & * \\ & & & \ddots & * \\ & & & & * \end{bmatrix} \leq \mathsf{GL}_n(R).$$

Some variations

- Projective $\mathbb{P}\mathbf{B}_n(R)$

$$\mathbb{P}\mathbf{B}_n(R) = \frac{\mathbf{B}_n(R)}{Z(\mathbf{B}_n(R))},$$

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Similarly $B_n^+(R)$, $\mathbb{A}\mathrm{ff}^+(R)$ and $\mathbb{P}B_n^+(R)$ without torsion on the main diagonal.

Example

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$$\mathbf{B}_n(R) = \left\{ \begin{bmatrix} \pm p^{k_1} & & * \\ & \ddots & \\ & & \pm p^{k_n} \end{bmatrix} : k_1, \ldots, k_n \in \mathbb{Z} \right\},$$

$$\mathbb{A}\mathrm{ff}(R) = \left\{ \begin{bmatrix} \pm p^k & r \\ 0 & 1 \end{bmatrix} : k \in \mathbb{Z},\ r \in \mathbb{Z}[1/p] \right\}.$$

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$$\mathcal{L}_n = C_n \wr \mathbb{Z}$$

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One can show that $\mathcal{L}_p$ is isomorphic to

$$\mathbb{A}\mathrm{ff}^+(\mathbb{F}_p[t, t^{-1}]) = \left\{ \begin{pmatrix} t^k & f \\ 0 & 1 \end{pmatrix} \mid f \in \mathbb{F}_p[t, t^{-1}],\, k \in \mathbb{Z} \right\}.$$

3 Outline

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2. Upper triangular matrix groups over R
3. Which of those groups have R_∞?
4. Automorphisms of Rings
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Question

For which integral domains R the groups

$$\mathbf{B}_n(R),\ \mathbb{A}\mathrm{ff}(R),\quad \mathbb{P}\mathbf{B}_n(R),$$
$$B_n^+(R),\ \mathbb{A}\mathrm{ff}^+(R),\ \mathbb{P}B_n^+(R)$$

have R_∞?

Let

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where $\mathbf{D}_n(R) \leq \mathsf{GL}_n(R)$ is the group of invertible diagonal matrices.

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Fact

Let $\mathbb{K}$ be a field, then $\mathbf{U}_n(\mathbb{K})$ is characteristic on $\mathbf{B}_n(\mathbb{K})$.

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Example

Let R be the integral domain $R = \mathbb{Z}[t]$. Consider the homomorphism

$$\varepsilon : (\mathbb{Z}[t], +) \longrightarrow C_2 = \{-1, 1\}$$
$$\textstyle\sum_{i=0}^{N} f_i t^i \longmapsto (-1)^{\sum_{i=0}^{N} f_i}.$$

Example

$\mathbf{U}_2(\mathbb{Z}[t])$ *is not invariant under the automorphism*

$$\begin{aligned} \varphi : \mathbf{B}_2(\mathbb{Z}[t]) &\longrightarrow \mathbf{B}_2(\mathbb{Z}[t]) \\ \begin{pmatrix} u & r \\ 0 & v \end{pmatrix} &\longmapsto \begin{pmatrix} \varepsilon(r) & 0 \\ 0 & \varepsilon(r) \end{pmatrix} \cdot \begin{pmatrix} u & r \\ 0 & v \end{pmatrix} \end{aligned}.$$

Example

In fact

$$\begin{aligned}\varphi\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} &= \begin{pmatrix}\varepsilon(t) & 0\\ 0 & \varepsilon(t)\end{pmatrix}\cdot\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\cdot\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}-1 & -t\\ 0 & -1\end{pmatrix} \notin \mathbf{U}_2(\mathbb{Z}[t]).\end{aligned}$$

Although $\mathbf{U}_n(R)$ is not characteristic in $\mathbf{B}_n(R)$, we have the following.

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Proposition (L. & Santos Rego)

For all $n \in \mathbb{N}_{\geq 2}$, if R is an integral domain, then the subgroup $\mathbf{U}_n(R)$ is characteristic in $\mathbb{P}\mathbf{B}_n(R)$ and $\mathbb{P}B_n^+(R)$.

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$\mathbb{A}\text{ff}(R)$ & $\mathbb{A}\text{ff}^+(R)$

In particular, $\mathbf{U}_2(R)$ is characteristic on $\mathbb{A}\text{ff}(R) = \mathbb{P}\mathbf{B}_2(R)$ and on $\mathbb{A}\text{ff}^+(R)$.

As a consequence, each automorphism ψ of the group

$$\mathbb{A}\mathrm{ff}(R) \cong \mathbf{U}_2(R) \rtimes \left\{ \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} : u \in R^\times \right\}$$

induces an automorphism

$$\overline{\psi} \in \mathrm{Aut}(\mathbb{A}\mathrm{ff}(R)/\mathbf{U}_2(R)).$$

Theorem (L., & Y. Santos)

Let R be an integral domain. Given $\psi \in \mathrm{Aut}(\mathbb{A}\mathrm{ff}(R))$, denote by $\overline{\psi}$ the automorphism induced by ψ on $\mathbb{A}\mathrm{ff}(R)/\mathbf{U}_2(R)$.

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If $R(\overline{\psi}) = \infty$ for all $\psi \in \mathrm{Aut}(\mathbb{A}\mathrm{ff}(R))$, then $\mathbb{A}\mathrm{ff}(R)$, $\mathbb{P}\mathbf{B}_n(R)$ and $\mathbf{B}_n(R)$ have property R_∞ for all $n \geq 2$.

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Example

If $R = \mathbb{Z}[1/p]$, *the groups*

$$\mathbf{B}_n^+(R), \mathbb{P}B_n^+(R) \text{ and } , \mathbb{A}\mathrm{ff}^+(R)$$

all have R_∞ *for* $n \geq 2$.

Example

Let ψ *be an automorphism of*

$$\mathbb{A}\mathrm{ff}^+(\mathbb{Z}[1/p]) \cong \mathbf{U}_2(\mathbb{Z}[1/p]) \rtimes \mathcal{D}_1(\mathbb{Z}[1/p]),$$

where

$$\mathcal{D}_1(\mathbb{Z}[1/p]) = \left\{ \begin{bmatrix} p^k & 0 \\ 0 & 1 \end{bmatrix} : k \in \mathbb{Z} \right\}.$$

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Then the induced automorphism $\overline{\psi}$ on

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satisfies $R(\overline{\psi}) = \infty$.

More precisely, we show that $\overline{\psi}$ *(as a an element of* $\mathsf{GL}_1(\mathbb{Z})$*) has eigenvalue* 1*, i.e. is the identity.*

Example

Fact: *We may assume that*

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Thus, there is $\lambda \in \mathbb{Z}$ *such that*

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There is $r \in \mathbb{Z}[1/p]$ *such that*

$$\psi\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}.$$

Example

Using the equality

$$\begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}^{-1},$$

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we see that

$$\begin{aligned}\begin{bmatrix}1 & rp\\0 & 1\end{bmatrix} &= \psi\left(\begin{bmatrix}1 & p\\0 & 1\end{bmatrix}\right)\\ &= \psi\left(\begin{bmatrix}p & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}p & 0\\0 & 1\end{bmatrix}^{-1}\right)\\ &= \begin{bmatrix}1 & rp^{\lambda}\\0 & 1\end{bmatrix}.\end{aligned}$$

Analogously, one can show that

$$\mathbf{B}_n(\mathbb{Z}[1/p]), (n \geq 2),\ \mathbb{A}\mathrm{ff}(\mathbb{Z}[1/p]),\ \mathbb{P}\mathbf{B}_n(\mathbb{Z}[1/p])$$

all have R_∞.

Analogously, one can show that

$$\begin{aligned} &\mathbf{B}_n(\mathbb{Z}[1/m]),\ \mathbb{A}\mathrm{ff}(\mathbb{Z}[1/m]),\quad \mathbb{P}\mathbf{B}_n(\mathbb{Z}[1/m]),\\ &\mathbf{B}_n^+(\mathbb{Z}[1/m]),\ \mathbb{A}\mathrm{ff}^+(\mathbb{Z}[1/m]),\ \mathbb{P}B_n^+(\mathbb{Z}[1/m]) \end{aligned}$$

all have R_∞.

4 Outline

We now introduce another way to determine whether $\mathbf{B}_n(R)$ and $\mathbb{P}\mathbf{B}_n(R)$ $(n \geq 5)$ have R_∞ using automorphisms of rings.

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$$\begin{aligned} \alpha_{\mathsf{add}} &\in \mathrm{Aut}(R, +); & \alpha_{\mathsf{add}}(r) &= \alpha(r), \\ \tau_\alpha &\in \mathrm{Aut}((R, +) \times (R, +)); & \tau_\alpha(r, s) &= (\alpha(s), \alpha(r)). \end{aligned}$$

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Assume further that both $R(\alpha_{\mathsf{add}})$ and $R(\tau_\alpha)$ are infinite for all $\alpha \in \mathrm{Aut}_{\mathsf{Ring}}(R)$.

Theorem (L., & Y. Santos)

Let R be an integral domain with a finitely generated group of units $(R^{\times}, \cdot)$.

Assume further that both $R(\alpha_{\textsf{add}})$ and $R(\tau_\alpha)$ are infinite for all $\alpha \in \mathrm{Aut}_{\textsf{Ring}}(R)$.

Then the groups $\mathbf{B}_n(R)$ and $\mathbb{P}\mathbf{B}_n(R)$ have R_∞ for all $n \geq 5$.

Example

Let $R = \mathbb{Z}[t]$, the ring of integer polynomials.

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Example

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$$R(\alpha_{\mathit{add}}) = \infty = R(\tau_\alpha),\ \forall \alpha \in \mathrm{Aut}_{\mathsf{Ring}}(R).$$

In particular, $\mathbf{B}_n(R)$ *and* $\mathbb{P}\mathbf{B}_n(R)$ *have* R_∞ *when* $n \geq 5$*.*

Example

Every $\alpha \in \mathrm{Aut}_{\mathsf{Ring}}(\mathbb{Z}[t])$ *is of the form*

$$\alpha\left(\sum_{i=0}^{d} f_i t^i\right) = \sum_{i=0}^{d} f_i (at+b)^i,$$

for some $a \in \{\pm 1\}$ *and* $b \in \mathbb{Z}$*.*

Example

Claim. *If $i > j$, then $[t^{2i}]_{\alpha_{add}} \neq [t^{2j}]_{\alpha_{add}}$.*

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In fact, $[t^{2i}]_{\alpha_{add}} = [t^{2j}]_{\alpha_{add}}$ if and only if there exists

$$h(t) = \sum_{\ell=0}^{d} h_\ell t^\ell \in \mathbb{Z}[t]$$

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such that $t^{2i} = h(t) + t^{2j} - \alpha_{add}(h(t))$

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such that $t^{2i} = h(t) + t^{2j} - \alpha_{add}(h(t))$, that is,

$$t^{2i} - t^{2j} = \sum_{\ell=0}^{d} h_\ell t^\ell - \sum_{\ell=0}^{d} h_\ell (at+b)^\ell.$$

Example

We can show that the degree d of $h(t)$ cannot be larger than t^{2i}, otherwise

$$t^{2i} - t^{2j} = \sum_{\ell=0}^{d} h_\ell t^\ell - \sum_{\ell=0}^{d} h_\ell (at+b)^\ell. \tag{1}$$

does not hold.

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does not hold.

The leading coefficient of the LHS is 1, whereas on the RHS the leading coefficient is

$$h_{2i} - h_{2i}a^{2i} = (1 - a^{2i})h_{2i} = 0.$$

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does not hold.

The leading coefficient of the LHS is 1, whereas on the RHS the leading coefficient is

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Thus, no $h(t) \in \mathbb{Z}[t]$ satisfies (1).

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Since $R(\alpha_{add}) = \infty$ and $R(\tau_\alpha) = \infty$, the previous theorem assures that

$$\mathbf{B}_n(\mathbb{Z}[t]) \text{ and } \mathbb{P}\mathbf{B}_n(\mathbb{Z}[t])$$

have R_∞ for all $n \geq 5$.

5 Outline

Proposition

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Proposition

Let p be prime. If $R = \mathbb{F}_p[t]$,

$$\mathbf{B}_2(R) \text{ and } \mathbb{A}\mathrm{ff}(R) \text{ do not have } R_\infty.$$

Thus, we cannot apply the first theorem to

$$\mathbf{B}_n(R), \mathbb{P}\mathbf{B}_n(R).$$

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Proposition

The groups

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have R_∞ *for* $n \geq 5$.

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have R_∞ *for* $n \geq 5$*.*

Let $q = p^f$ be a prime power.

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$$f(t) \in \mathbb{F}_p[t] \subseteq \mathbb{F}_q[t]$$

be a non-constant monic polynomial which is irreducible over $\mathbb{F}_q \supseteq \mathbb{F}_p$.

Let $q = p^f$ be a prime power. Let

$$f(t) \in \mathbb{F}_p[t] \subseteq \mathbb{F}_q[t]$$

be a non-constant monic polynomial which is irreducible over $\mathbb{F}_q \supseteq \mathbb{F}_p$. Using the second theorem, we can show the following.

Proposition

For $R = \mathbb{F}_q[t, t^{-1}, f(t)^{-1}]$, the groups

$$\{B_n^+(R), \mathbb{P}B_n^+(R), \mathbb{A}\mathrm{ff}^+(R) \mid n \in \mathbb{N}_{\geq 2}\},$$

have R_∞.

Thank you!