The Reidemeister zeta function and the Pólya–Carlson dichotomy

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Consider V - **nonsingular projective algebraic variety** of dimension n over a finite field K with q elements.

This variety is defined by homogenous polynomial equations with coefficients in K for $m+1$ variables $x_0, x_1, ..., x_m \in \bar{K}$.

The variety V is invariant under **the Frobenius map** $F : (x_0, x_1, ..., x_m) \to (x_0^q, x_1^q, ..., x_m^q)$.

Hasse and Weil introduced a zeta function which counts the points on V which are fixed under F^n for some $n \geqslant 1$:

$$\zeta(z, V) = \exp\left(\sum_{n=1}^{\infty} \frac{\#Fix(F^n)}{n} z^n\right).$$

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over all primitive periodic orbits γ of F on V.

It is a known result that **the Hasse–Weil zeta function is a rational function**.

Inspired by the Hasse–Weil zeta function of an algebraic variety over a finite field, **Artin and Mazur** defined the zeta function for an arbitrary map $f : X \to X$ of a topological space X:

$$F_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{F(f^n)}{n} z^n\right)$$

where $F(f^n)$ is the number of isolated fixed points of f^n.

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For a dense set of the space of smooth maps of a compact smooth manifold into itself, the series $\sum_{n=1}^{\infty} \frac{F(f^n)}{n} z^n$ has positive radius of convergence. For diffeomorphisms of a smooth compact manifold satisfying Smale Axiom A, the Artin–Mazur zeta function is **rational**.
The Artin–Mazur zeta function was historically the first dynamical zeta function of a discrete dynamical system.

The next dynamical zeta function was defined by Smale - **the Lefschetz zeta function** of a discrete dynamical system:

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where

$$L(f^n) = \sum_{k=0}^{dimX} (-1)^k Tr[f^n_{*k} : H_k(X;\mathbb{Q}) \to H_k(X;\mathbb{Q})]$$

is the Lefschetz number of f^n. The Lefschetz zeta function is rational and is given by the formula:

$$L_f(z) = \prod_{k=0}^{dimX} \det(I - f_{*k} \cdot z)^{(-1)^{k+1}}.$$

Ways to count the periodic points of the map:

- **geometrically** - using **the Artin–Mazur** zeta function,
- **using homology groups** to obtain coefficients in **Lefschetz** zeta function,
- provided by **Nielsen**,
- provided by **Reidemeister**.

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Nielsen and the Reidemeister zeta functions counts periodic points in the presence of a fundamental group and this also force us to consider the Reidemeister zeta function of a group homomorphism.

Let

- X be an arbitrary **topological space**,
- $p : \tilde{X} \to X$ be **universal covering of X** and
- $\tilde{f} : \tilde{X} \to \tilde{X}$ **a lifting of** f.

Two liftings $\tilde{f}$ and $\tilde{f}'$ are called ***conjugate*** if there exists

$$\gamma \in \Gamma \cong \pi_1(X) \text{ such that } \tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}.$$

The subset $p(Fix(\tilde{f})) \subset Fix(f)$ is called ***the fixed point class*** of f determined by the lifting class $[\tilde{f}]$. A fixed point class is called ***essential*** if its topological index is nonzero.

The number of all lifting classes of f (and hence the number of fixed point classes) is called ***the Reidemeister number*** and is denoted by $R(f)$. This is positive integer or infinity.
The number of essential fixed point classes is called ***the Nielsen number*** of f and is denoted by $N(f)$. The Nielsen number is always finite.
$R(f)$ and $N(f)$ are homotopy type invariants. In 1987 A. Fel'shtyn introduced new dynamical zeta functions connected with Nielsen fixed point theory as follows:

$$R_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(f^n)}{n} z^n\right),$$

$$N_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n\right).$$

Let

- G be a **group** and
- $\phi : G \to G$ its **endomorphism**.

Two elements $\alpha, \alpha' \in G$ are called ***ϕ-conjugate*** iff there exists

$$\gamma \in G \text{ such that } \alpha' = \gamma\alpha\phi(\gamma)^{-1}.$$

The number of ϕ-conjugacy classes is called ***the Reidemeister number*** of ϕ and is denoted by $R(\phi)$.

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Let us consider **the Reidemeister zeta function** of ϕ:

$$R_\phi(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(\phi^n)}{n} z^n\right).$$

We assume that $R(\phi^n) < \infty$ for all $n > 0$. We investigate for which groups and endomorphisms the Reidemeister zeta function is rational and when it satisfies a functional equation.

If G is an abelian group, then:

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Fact

Let G be a finite abelian group and let $\phi \in End(G)$. Then

$$R_\phi(z) = \prod_{[\gamma]} \frac{1}{1 - z^{\#[\gamma]}}$$

where the product is taken over periodic orbits of ϕ in G. Moreover,

$$R_\phi\left(\frac{1}{z}\right) = (-1)^p z^q R_\phi(z),$$

where q is the number of periodic elements of ϕ in G and p is the number of periodic orbits of ϕ in G.

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Theorem (A. Fel'shtyn, R.Hill)

Let ϕ be an endomorphism of a finite group G. Then:

$$R_\phi(z) = \prod_{[\langle g \rangle]} \frac{1}{1 - z^{\#[\langle g \rangle]}},$$

where the product is taken over all ϕ-periodic orbits or ordinary conjugacy classes of elements of G. The number $\#[\langle g \rangle]$ is the number of conjugacy classes in the ϕ-orbit of the conjugacy class $\langle g \rangle$. Moreover,

$$R_\phi\left(\frac{1}{z}\right) = (-1)^a z^b R_\phi(z),$$

where a is the number of periodic ϕ-orbits of conjugacy classes of elements of G and b is the number of periodic conjugacy classes of elements of G, i.e. such classes that $\langle \phi^n(g) \rangle = \langle g \rangle$ for some $n > 0$.

Let us take a closer look at the connection between the Reidemeister zeta function and the Lefschetz zeta function and its consequences. For that purpose, we recall the notion of a unitary dual space of a group, which is just a Pontryagin dual in an Abelian case.

Let G be a locally compact Abelian topological group. We write $\hat{G}$ for the set of continuous homomorphisms

$$\chi : G \to U(1) = \{z \in \mathbb{C} : |z| = 1\}.$$

This is a group with pointwise multiplication. We call $\hat{G}$ ***the Pontryagin dual*** of G.

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A continuous endomorphism $f : G \mapsto G$ gives a rise to a continuous endomorphism $\hat{f} : \hat{G} \mapsto \hat{G}$ defined by

$$\hat{f}(\chi) := \chi \circ f.$$

For a finitely generated Abelian group G we define the finite subgroup G^{finite} to be the subgroup of the torsion elements of G. We denote the quotient $G/G^{finite} =: G^{\infty}$. The group G^{∞} is torsion free. Since the image of any torsion element by a homomorphism has to be a torsion element, the function $\phi : G \to G$ induces maps

$$\phi^{finite} : G^{finite} \to G^{finite} \text{ and } \phi^{\infty} : G^{\infty} \to G^{\infty}.$$

Theorem - Connection with Lefschetz numbers (A. Fel'shtyn)

Let $\phi : G \to G$ be an endomorphism of a finitely generater Abelian group. Then we have the following

$$R(\phi^n) = |L(\hat{\phi}^n)|,$$

where $\hat{\phi}$ is the continuous endomorphism of $\hat{G}$ and $L(\hat{\phi}^n)$ is the Lefschetz number of $\hat{\phi}$ thought as a self-map of the topological space $\hat{G}$. From that it follows

$$R_\phi(z) = L_{\hat{\phi}}(\sigma z)^{(-1)^r},$$

where $\sigma = (-1)^p$ where p is the number of real eigenvalues $\gamma \in Spec\phi^\infty$ such that $\gamma < -1$ and r is the number of real eigenvalues $\gamma \in Spec\phi^\infty$ such that $|\gamma| > 1$. If G is finite this reduces to

$$R(\phi^n) = L(\hat{\phi}^n) \text{ and } R_\phi(z) = L_{\hat{\phi}}(z).$$

Consider finitely generated torsion free nilpotent group Γ and its endomorphism ϕ. To prove the rationality of its Reidemeister zeta function we can admit two different approach.

First result was obtained by A. Fel'shtyn in the 80's and is connected with a topology and Anosovs's theorem.

The second approach is algebraic and provides a direct calculation of the Reidemeister number of ϕ. It was presented by V. Roman'kov in 2011.

Theorem (A. Fel'shtyn)

If Γ is a finitely generated torsion free nilpotent group and $\phi \in End(\Gamma)$, then $R_\phi(z)$ is a rational function.

The following theorem gives an explicit formula for the Reidemeister numbers of Γ. We can easily deduce the rationality from it.

Theorem (V. Roman'kov)

Let N be a finitely generated, nilpotent group. Let

$$N = N_1 \geqslant N_2 \geqslant ... \geqslant N_c \geqslant N_{c+1} = 1$$

be a central series of N, such that all factors N_k/N_{k+1} are torsion-free. If $\phi \in End(N)$ and $\phi(N_k) \subset N_k$ for all k, then

$$R(\phi) = \prod_{k=1}^{c} R(\phi_k),$$

where $R(\phi_k)$ is the induced endomorphism on the factor N_k/N_{k+1}.

In 2019 B. Klopsch and A. Fel'shtyn proposed to study of the coincidence Reidemeister zeta function.

Let Γ be a finitely generated group and $\phi, \psi : \Gamma \to \Gamma$ two endomorphisms. Two elements $\alpha, \alpha' \in \Gamma$ are said to be (ϕ, ψ) - ***conjugate*** if and only if there exists $\gamma \in \Gamma$ with

$$\alpha' = \psi(\gamma)\alpha\phi(\gamma)^{-1}.$$

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$$\alpha' = \psi(\gamma)\alpha\phi(\gamma)^{-1}.$$

The number of (ϕ, ψ)-conjugacy classes is called ***the Reidemeister coincidence number*** of an endomorphisms ϕ and ψ, denoted by $R(\phi, \psi)$. If ψ is the identity map then the (ϕ, id)-conjugacy classes are the ϕ - conjugacy classes in the group Γ and $R(\phi, id) = R(\phi)$.

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The coincidence Reidemeister zeta function is defined as follows

$$R_{\phi,\psi}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(\phi^n, \psi^n)}{n} z^n\right).$$

Whenever we mention the zeta function $R_{\phi,\psi}(z)$, we shall assume that it is well-defined and so $R(\phi^n, \psi^n,) < \infty$ for all $n > 0$.

Lemma

Let $\phi, \psi : \Gamma \to \Gamma$ are two automorphisms. Two elements x, y of Γ are $\psi^{-1}\phi$-conjugate if and only if elements $\psi(x)$ and $\psi(y)$ are (ψ, ϕ)-conjugate. Therefore the Reidemeister number $R(\psi^{-1}\phi)$ is equal to $R(\phi, \psi)$. For commuting automorphisms $\phi, \psi : \Gamma \to \Gamma$ the coicidence Reidemeister zeta function $R_{\phi,\psi}(z)$ is equal to the Reidemeister zeta function $R_{\psi^{-1}\phi}(z)$.

Lemma

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Proof

If x and y are $\psi^{-1}\phi$-conjugate, then there is a $\gamma \in \Gamma$ such that $x = \gamma y \psi^{-1}\phi(\gamma^{-1})$. This implies $\psi(x) = \psi(\gamma)\psi(y)\phi(\gamma^{-1})$. So $\psi(x)$ and $\psi(y)$ are (ϕ, ψ)-conjugate. The converse statement follows if we move in opposite direction in previous implications.

The following theorem generalize the result obtained by V. Roman'kov. It also follows from B. Klopsch and A. Fel'shtyn result for torsion free finite rank nilpotent group of the nilpotency class c.

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Theorem (M. Ziętek [2020])

Let N be a finitely generated, torsion–free, nilpotent group and

$$N = \sqrt[G]{\gamma_1(N)} \geqslant \sqrt[G]{\gamma_2(N)} \geqslant \ldots \geqslant \sqrt[G]{\gamma_c(N)} \geqslant \sqrt[G]{\gamma_{c+1}(N)} = 1$$

be an adapted lower central series of N. If $\phi, \psi \in End(N)$, then

$$R(\phi,\psi) = \prod_{k=1}^{c} R(\phi_k,\psi_k),$$

where ϕ_k, ψ_k are induced endomorphisms on the factor $\sqrt[N]{\gamma_k(N)}/\sqrt[N]{\gamma_{k+1}(N)}$. Moreover, if ϕ, ψ are commuting, the coincidence Reidemeister zeta function of group N, $R_{\phi,\psi}(z)$ is rational.

Before we go to the proof let us provide some necessary notions and theorems.

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Let G be a group. For a subgroup $H \leqslant G$, we define ***the isolator*** $\sqrt[G]{H}$ of H in G as:

$$\sqrt[G]{H} = \{g \in G \mid g^n \in H \text{ for some } n \in \mathbb{N}\}.$$

Lemma 1

Let G be a group. Then

i. for all $k \in \mathbb{N}$, $\sqrt[G]{\gamma_k(G)}$ is a fully characteristic subgroup of G.

ii. for all $k \in \mathbb{N}$, $G/\sqrt[G]{\gamma_k(G)}$ torsion-free.

iii. for all $k, l \in \mathbb{N}$, $[\sqrt[G]{\gamma_k(G)}, \sqrt[G]{\gamma_l(G)}] \leq \sqrt[G]{\gamma_{k+l}(G)}$.

iv. for all $k, l \in \mathbb{N}$ such that $k \geqslant l$ if $N := \sqrt[G]{\gamma_l(G)}$, then

$$\sqrt[G/N]{\gamma_k(G/N)} = \sqrt[G]{\gamma_l(G)}/N$$

.

We define ***the adapted lower central series*** of a group G as

$$G = \sqrt[G]{\gamma_1(G)} \geqslant \sqrt[G]{\gamma_2(G)} \geqslant \dots \sqrt[G]{\gamma_k(G)} \geqslant \dots,$$

where $\gamma_k(G)$ is the k-th commutator of G.

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where $\gamma_k(G)$ is the k-th commutator of G.

The adapted lower central series will terminate if and only if G is a torsion-free, nilpotent group.Moreover, all factors $\sqrt[G]{\gamma_k(G)}/\sqrt[G]{\gamma_{k+1}(G)}$ are torsion-free.

Let N be a normal subgroup of a group G and $\phi, \psi \in$End(G) with $\phi(N) \subseteq N, \psi(N) \subseteq N$.

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We denote the restriction of ϕ to N by $\phi|_N$, ψ to N by $\psi|_N$ and the induced endomorphisms on the quotient G/N by ϕ', ψ' respectively.

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Then we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \xrightarrow{i} & G & \xrightarrow{p} & G/N & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi|_N, \psi|_N} & & \downarrow{\scriptstyle \phi, \psi} & & \downarrow{\scriptstyle \phi', \psi'} & & \\ 1 & \longrightarrow & N & \xrightarrow{i} & G & \xrightarrow{p} & G/N & \longrightarrow & 1 \end{array} \qquad (1)$$

Note that both i and p induce functions $\hat{i}, \hat{p}$ on the set of Reidemeister classes so that the sequence

$$\mathfrak{R}[\phi|_N, \psi|_N] \xrightarrow{\hat{i}} \mathfrak{R}[\phi, \psi] \xrightarrow{\hat{p}} \mathfrak{R}[\phi', \psi'] \longrightarrow 0$$

is exact, i.e. $\hat{p}$ is surjective and $\hat{p}^{-1}[1] = im(\hat{i})$, where 1 is the identity element of G/N.

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Lemma 2

If $R(\phi|_N) < \infty, R(\phi') < \infty$, $R(\psi|_N) < \infty, R(\psi') < \infty$ and $N \subseteq Z(G)$, then $R(\phi, \psi) \leqslant R(\phi|_N, \psi|_N) R(\phi', \psi')$.

Lemma 3

Let G be a group such that $G \cong \mathbb{Z}^n$, ϕ, ψ commuting endomorphisms of G, $\lambda_1, ...\lambda_n$ - eigenvalues of ϕ and $\mu_1, \mu_2, ..., \mu_n$ - eigenvalues of ψ. Then

$$R(\phi, \psi) = \#Coker(\psi - \phi) = |\det(\psi - \phi)| = \prod_{i=1}^{n} |\mu_i - \lambda_i|,$$

If we denote $R(\phi^j, \psi^j)$ as the Reidemeister number of j-th iteration of ϕ and ψ, then:

$$R(\phi^j, \psi^j) = |\det(\psi^j - \phi^j)| = \prod_{i=1}^{n} |\mu_i^j - \lambda_i^j|.$$

Let us also recall the well known fact:

Lemma 4

$R_\phi(z)$ is a rational function if and only if there extists a finite set of $\alpha_i, \beta_j \in \mathbb{C}$ such that

$$R(\phi^n) = \sum_j \beta_j^n - \sum_i \alpha_i^n.$$

Proof

First of all, for the sake of a clarity let us denote $\sqrt[N]{\gamma_k(N)}$ as M_k.

We will prove this by induction on the length of the central series. If $c = 1$, the result follows trivially.

Let $c > 1$ and assume the theorem holds for a central series of length $c-1$. Let $\phi, \psi \in End(N)$, then $\phi(M_c) \subseteq M_c, \psi(M_c) \subseteq M_c$ and hence we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M_c & \xrightarrow{i} & M & \xrightarrow{p} & M/M_c & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi_c,\psi_c} & & \downarrow{\scriptstyle \phi,\psi} & & \downarrow{\scriptstyle \phi',\psi'} & & \\ 1 & \longrightarrow & M_c & \xrightarrow{i} & M & \xrightarrow{p} & M/M_c & \longrightarrow & 1 \end{array},$$

where ϕ_c, ψ_c are induced automorphisms on the M_c/M_{c+1}.

The quotient M/M_c is a finitely generated, nilpotent group with a central series

$$M/M_c = M_1/M_c \geqslant M_2/M_c \geqslant ... \geqslant M_{c-1}/M_c \geqslant M_c/M_c = 1$$

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of length $c-1$.

By the third isomorphism theorem, every factor of this series is of the form

$$(M_k/M_c)/(M_{k+1}/M_c) \cong M_k/M_{k+1},$$

hence it is also torsion-free.

Moreover, because of this natural isomorphism we know that for every induced pair of endomorphisms ϕ_k, ψ_k on $(M_k/M_c)/(M_{k+1}/M_c)$ it is true that

$$R(\phi'_k, \psi'_k) = R(\phi_k, \psi_k).$$

Let us assume, that $R(\phi', \psi') < \infty$ and $R(\phi_c, \psi_c) < \infty$.

Moreover, let

$$[g_1 M_c]_{\phi', \psi'}, \ldots, [g_n M_c]_{\phi', \psi'}$$

be the (ϕ', ψ')-Reidemeister classes and

$$[c_1]_{\phi_c, \psi_c}, \ldots, [c_m]_{\phi_c, \psi_c}$$

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- the (ϕ_c, ψ_c)-Reidemeister classes.

Since $M_c \subseteq Z(N)$, by Lemma 2 we obtain that

$$R(\phi, \psi) \leqslant R(\phi_c, \psi_c) R(\phi', \psi').$$

To prove the opposite inequality it suffices to prove that every class $[c_i g_j]_{\phi,\psi}$ represents a different (ϕ, ψ)-Reidemeister class. Then we obtain

$$R(\phi, \psi) = R(\phi_c, \psi_c)R(\phi', \psi')$$

and then the theorem follows from the induction hypothesis.

Suppose, that there exists some $h \in N$ such that

$$c_i g_j = \psi(h) c_a g_b \phi(h)^{-1}.$$

Then by taking the projection to M/M_c we find that

$$g_j M_c = p(c_i g_j) = p(\psi(h) c_a g_b \phi(h)^{-1}) = \psi'(hM_c)(g_b M_c)\phi'(hM_c)^{-1},$$

hence $[g_j M_c]_{\phi',\psi'} = [g_b M_c]_{\phi',\psi'}$.

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hence $[g_j M_c]_{\phi',\psi'} = [g_b M_c]_{\phi',\psi'}$.

Now assume that

$$c_i g_j = \psi(h) c_a g_j \phi(h)^{-1}.$$

If $h \in M_c$, then

$$c_i g_j = \psi(h) c_a \phi(h)^{-1} g_j$$

and consequently $[c_i]_{\phi_c,\psi_c} = [c_a]_{\phi_c,\psi_c}$, so let us assume that $h \notin M_c$ and that M_k is the smallest group in the central series which contains h.

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Then

$$c_i g_j = \psi(h) c_a g_j \phi(h)^{-1} \Leftrightarrow$$

$$c_i c_a^{-1} = g_j^{-1} \psi(h) g_j \phi(h)^{-1}$$

and therefore

$$\begin{aligned} c_i c_a^{-1} M_{k+1} &= g_j^{-1} \psi(h) g_j \phi(h)^{-1} M_{k+1} \\ &= [g_j, \psi(h)^{-1}](\psi(h)\phi(h)^{-1}) M_{k+1} \end{aligned}$$

Since $M_k/M_{k+1} \subseteq Z\left(M/M_{k+1}\right)$, then $[g_j, h^{-1}] \in M_{k+1}$.
As $c_i c_a^{-1} \in M_c \subseteq M_{k+1}$, we obtain

$$(\phi_k)(hM_{k+1}) = (\psi_k)hM_{k+1}.$$

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That means that $\mathrm{Coin}(\phi'_k, \psi'_k) \neq \{1\}$, which implies that $R(\phi', \psi') = \infty$ and this contradicts assumption.

Now we will prove the rationality of $R_{\phi,\psi}(z)$.

Let us denote the factors M_k/M_{k+1} as G_k. G_k is finitely generated, abelian and torsion free for all k, hence $G_k \cong \mathbb{Z}^{r_k}$ for some $r_k \in \mathbb{N}$.

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Let ϕ_k, ψ_k are represented by integer matrices $A, B \in M_{r_k}(\mathbb{Z})$ associated to them respectively.

Let $a_{1,k}, a_{2,k}, \ldots, a_{r_k,k}$ be the eigenvalues of A and $b_{1,k}, b_{2,k}, \ldots, b_{r_k,k}$ be the eigenvalues of B.

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Then, by Lemma 3

$$R(\phi_k^j, \psi_k^j) = \#Coker(\phi_k^j - \psi_k^j) = |\det(\phi_k^j - \psi_k^j)| = |a_{1,k}^j - b_{1,k}^j| \dots |a_{r_k,k}^j - b_{r_k,k}^j|.$$

Let us define $c^j_{i,k} := \max\{|a^j_{i,k}|, |b^j_{i,k}|\}$ and $d^j_{i,k} := \frac{a^j_{i,k} b^j_{i,k}}{c_{i,k}}$. Then

$$R(\phi^j_k, \psi^j_k) = (c^j_{1,k} - d^j_{1,k})\ldots(c^j_{r_k,k} - d^j_{r_k,k}).$$

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$$R(\phi^j_k, \psi^j_k) = (c^j_{1,k} - d^j_{1,k})\ldots(c^j_{r_k,k} - d^j_{r_k,k}).$$

Then there exists a finite set $\delta_{m,k}, \epsilon_{n,k} \in \mathbb{C}$ such that

$$R(\phi^j_k, \psi^j_k) = \sum_n \epsilon^j_{n,k} - \sum_m \delta^j_{m,k} \text{ for all } k = 1, \ldots, c.$$

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Then there exists a finite set $\delta_{m,k}, \epsilon_{n,k} \in \mathbb{C}$ such that

$$R(\phi^j_k, \psi^j_k) = \sum_n \epsilon^j_{n,k} - \sum_m \delta^j_{m,k} \text{ for all } k = 1, \ldots, c.$$

Since $R(\phi^j, \psi^j) = \prod_{k=1}^c R(\phi^j_k, \psi^j_k)$, then there exists a finite set $\alpha_a, \beta_b \in \mathbb{C}$, such that

$$R(\phi^j, \psi^j) = \sum_b \beta^j - \sum_a \alpha^j,$$

which finishes the proof by Lemma 4. □

We have proven, that for **finitely generated abelian groups** the Reidemeister zeta function is **rational**. For infinitely generated abelian groups, situation is completely different.

For further studies of the Reidemeister zeta function we apply tools from commutative algebra, algebraic number theory and valuation theory.

Let G be an abelian subgroup of $\mathbb{Q}^d, d \geqslant 1$ and $\phi : G \to G$ - a endomorphism of G.
Then G has the structure of a $\mathbb{Z}[t]$-module, given by the formula $\phi(g) = tg$, where $t \in \mathbb{Q}$. We can extend this multiplication in a natural way to polynomials, i.e. or $g \in G, f = \sum_{j \in \mathbb{Z}} c_j t^j \in R = \mathbb{Z}[t]$ we set:

$$fg = \sum_{j \in \mathbb{Z}} c_j t^j g = \sum_{j \in \mathbb{Z}} c_j \phi^j(g),$$

where all but finitely many $c_n \in \mathbb{Z}$ are equal to zero.

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where all but finitely many $c_n \in \mathbb{Z}$ are equal to zero.

Remark

For abelian case (both finitely and infinitely generated) we have:

$$R(\phi^n) = \#Coker(1-\phi^n) = |G/(1-\phi^n)G|.$$

Lemma 1

Let G be an R-module. If either $R(\phi^j)$ or $|G/(t^j - 1)G|$ is finite, then these quantities are equal.

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Lemma 2

Let $L \subset G$ be R-module and $r \in R$.

i. $\left|\frac{G}{rG}\right| = \left|\frac{G/L}{r(G/L)}\right| \left|\frac{L}{L \cap rG}\right|$.

ii. If G/L is finite and the map $g \mapsto rg$ is a monomorphism of G then $|G/rG| = |L/rL|$.

Lemma 3

Let G be an R-module for which $Ass(G)$ consists of finitely many non-trivial principal ideals and suppose $m(\mathfrak{p}) = dim_{\mathbb{K}(\mathfrak{p})} G_{\mathfrak{p}} < \infty$ for each $\mathfrak{p} \in Ass(G)$. If $r \in R$ is such that the map $g \mapsto rg$ is a monomorphism of G, then G/rG is finite.

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Lemma 4

Let $R(z) = \sum_{n=1}^{\infty} R(\phi^n)z^n$. If $R_\phi(z)$ is rational then $R(z)$ is rational. If $R_\phi(z)$ has analytic continuation beyond its circle of convergence, then so does $R(z)$. In particular, the existence of a natural boundary at the circle of convergence for $R(z)$ implies the existence of natural boundary for $R_\phi(z)$.

The crucial tool for our work is following theorem.

Pólya–Carlson Theorem

A power series with integer coefficients and radius of convergence 1 is either rational or has the unit circle as a natural boundary.

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Lemma 5

Let S be a finite set of places of algebraic number fields and for each $v \in S$, let ξ_v be a non-unit root in the appropriate number field such that $|\xi_v|_v = 1$. Then the function

$$F(z) = \sum_{n=1}^{\infty} f(n) z^n,$$

where $f(n) = \prod_{v \in S} |\xi_v^n - 1|_v$ for $n \geqslant 1$, has the unit circle as a natural boundary.

Let $\phi : G \to G$ be an automorphism of countable abelian group G that it is a subgroup of $\mathbb{Q}^d$, where $d \geqslant 1$. Suppose that the group G as $R = \mathbb{Z}[t]$-module satisfies the following conditions:

(1) the set of associated primes $Ass(G)$ is finite and constists entirely of non-zero principal ideals of R,

(2) the map $g \mapsto (t^j - 1)g$ is a monomorphism of G for all $j \in \mathbb{N}$ (equivalently, $t^j - 1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in Ass(G)$ and all $j \in \mathbb{N}$,

(3) for each $\mathfrak{p} \in Ass(G), m(\mathfrak{p}) = dim_{\mathbb{K}(\mathfrak{p})} G_{\mathfrak{p}} < \infty$, where $\mathbb{K}(\mathfrak{p})$ denotes the field of fractions of $R/\mathfrak{p}$ and $G_{\mathfrak{p}} = G \otimes_R \mathbb{K}(\mathfrak{p})$ is the localization of the module G at $\mathfrak{p}$.

The following results are based to R.Miles' and T.Ward's work. They studied properties of the Artin–Mazur zeta function on dual spaces (namely, solenoids).

Theorem (Fel'shtyn, Ziętek [2019])

Let $\phi : G \to G$ be an automorphism of countable abelian group G that it is a subgroup of $\mathbb{Q}^d$, where $d \geqslant 1$. Suppose that the group G as $R = \mathbb{Z}[t]$-module satisfies the above (1)-(3) conditions. Then there exist algebraic number fields $\mathbb{K}_1, ..., \mathbb{K}_n$, sets of finite places $P_i \subset \mathcal{P}(\mathbb{K}_i)$ and elements $\xi_i \in \mathbb{K}_i$, no one of which is a root of unity for $i = 1, ..., n$, such that:

$$R(\phi^j) = \prod_{i=1}^{n} \prod_{v \in P_i} |\xi_i^j - 1|_v^{-1} = \prod_{i=1}^{n} |\xi_i^j - 1|_{P_i} = \prod_{i=1}^{n} |\xi_i^j - 1|_{P_i^\infty \cup S_i}$$

for all $j \in \mathbb{N}$. Moreover, suppose that the above product only involves finitely many places and that $|\xi_i|_v \neq 1$ for all v in the set of infinite places P_i^∞ of $\mathbb{K}_i$ and $S_i = \mathcal{P}(\mathbb{K}_i) \backslash P_i$ for all $i = 1, ..., n$. Then the Reidemeister zeta function $R_\phi(z)$ is either rational or has a natural boundary at its circle of convergence and the latter occurs if and only if $|\xi_i|_v = 1$ for some $v \in S_i, 1 \leqslant n$.

Main steps:

(1) Make a chain of submodules of G.

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Sketch of the proof:

- Consider abelian group G as $\mathbb{Z}[t]$-module, we have:

$$R(\phi^j) = |Coker(\phi^j - Id_G)| = |G/(\xi^j - 1)G| \\ = |G/(t^j - 1)G)|.$$

- The multiplicative set $U = \bigcap_{\mathfrak{p} \in Ass(G)} R \backslash \mathfrak{p}$ has $U \cap ann(a) = \emptyset$ for all non-zero $a \in G$, so the natural map $G \to U^{-1}G$ is a monomorphism. Then there exists a chain of submodules atural map $G \to U^{-1}G$ is a monomorphism. Then there exists a chain of submodules

$$\{0\} = L_0 \subset L_1 \subset ... \subset L_n = G,$$

where $L_i = U^{-1}G_i \cap G$.

Step:

(2) Apply Lemma 2 to this submodule recursively to obtain product.

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- $N_i = L_I/L_{i-1}$ may be considered as a fractional ideal of $E_i = R/\mathfrak{p}_i$. Using Lemma 2(i)

$$\left|\frac{L_i}{(t^j-1)L_i}\right| = \left|\frac{N_i}{(t^j-1)N_i}\right|\left|\frac{L_{i-1}}{(t^j-1)L_{i-1}}\right|,$$

where $1 \leqslant i \leqslant n$.

- Successively applying this formula to each of the modules $L_i, 1 \leqslant i \leqslant n$,gives $|L_n/(t^j-1)L_n| = \prod_{i=1}^{n} |N_i/(t^j-1)N_i|$.

Step:

(3) Treat ach term of the product similar to G by localizing it at an associated prime ideal which leads us to a corresponding place and make a chain.

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- Consider now an individual term $|N_i/(t^j - 1)N_i|$. Since E_i is a finitely generated domain, the integral closure D_i of E_i, in K_i is a finitely generated Dedekind E_i-module. We may consider $I_i = D_i \otimes_{E_i} N_i$ as a fractional ideal of D_i.From Lemma 2 and Lemma 3 we obtain that $|N_i/(\xi_i^j - 1)N_i| = |I_i/(\xi_i^j - 1)I_i|$.

Step:

(4) Repeat step (2) to obtain product formula for each term and consequently for $R_\phi(z)$ through all corresponding places.

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- By considering $I_i/(\xi_i^j - 1)I_i$, as a D_i-module, localizing at each of its associated primes, repeating above staps and applying the Artin product formula we obtain

$$|I_i/(\xi_i^j - 1)I_i| = \prod_{\mathfrak{m}\in P_i} |D_i/\mathfrak{m}|^{v_{\mathfrak{m}}(\xi_i^j - 1)}$$

$$= \prod_{\mathfrak{m}\in P_i} |\xi_i^j - 1|_{\mathfrak{m}}^{-1} = \prod_{\mathfrak{m}\in P_i} |\xi_i^j - 1|_{P_i^\infty \cup S_i},$$

where P_i^∞ denotes set of infinite places of $\mathbb{K}_i$ and $S_i = \mathcal{P}(\mathbb{K}_i)\backslash P_i$.

Step:

(5) Split places into finite and infinite ones and then decompose the Reidemeister numbers into two factors f and g.

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Lemma 6

$R_\phi(z)$ is a rational function if and only if there exist a finite set of complex numbers α_m, β_n such that $R(\phi^j) = \sum_n \beta_n^j - \sum_m \alpha_m^j$ for every $j > 0$.

- Let

$$S_i^* = \{v \in S_i : |\xi_i|_v \neq 1\},$$
$$S_i^{**} = \{v \in S_i : |\xi_i|_v > 1\}$$

and

$$f(j) = \prod_{i=1}^{n} |\xi_i^j - 1|_{S_i \setminus S_i^*}, g(j) = \prod_{i=1}^{n} |\xi_i^j - 1|_{P_i^\infty \cup S_i^*}.$$

So $R(\phi^j) = f(j)g(j)$.

Step:

(6) Show that factor f is the reason for irrationality and natural boundary for $R_\phi(z)$.

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- By the ultrametric property $g(j) = \prod_{i=1}^{n} |\xi_i|^j_{S_i^{**}} |\xi_i^j - 1|_{P_i^\infty}$. We can expand the product over infinite places using an appropriate symmetric polynomial to obtain an expression of the form similar to the one in the Lemma 6: $g(j) = \sum_{I \in \mathcal{I}} d_I w_I^j$, where $\mathcal{I}$ in a finite indexing set, $d_I = \pm 1$ and $w_I \in \mathbb{C}$.
- Thanks to this formula, we can represent the Reidemeister zeta function as follows

$$R_\phi(z) = \exp\left(\sum_{I \in \mathcal{I}} d_I \sum_{j=1}^{\infty} \frac{f(j)(w_I z)^j}{j} \right).$$

Step:

(6) Show that factor f is the reason for irrationality and natural boundary for $R_\phi(z)$.

- If $S_i \backslash S_i^* = \emptyset$ for all $i = 1, ..., n$, then $f(j) \equiv 1$, and it follows that the Reidemeister zeta function $R_\phi(z)$ is a rational function.
- Now suppose that $S_i \backslash S_i^* \neq \emptyset$ for some i. Thanks to Lemma 4, we need only exhibit a natural boundary at the circle of convergence for $\sum_{I\in\mathcal{I}} d_I \sum_{j=1}^{\infty} f(j)(w_I z)^j$ to obtain the one for $R_\phi(z)$. By using the Cauchy-Hadamard theorem, we have that the series $\sum\limits_{j=1}^{\infty} f(j)(w_I z)^j$ has radius of convergence $|w_I|^{-1}$.

Step:

(6) Show that factor f is the reason for irrationality and natural boundary for $R_\phi(z)$.

- Since $|\xi_i|_{S_i^{**}}$ for all $v \in P_i^\infty$, $i = 1, ..., n$, there is a dominant term w_J in the expansion of $R_\phi(z)$, for which

$$|w_J| = \prod_{i=1}^{n} |\xi_i|_{S_i^{**}} \prod_{v \in P_i^\infty} \max\{|\xi_i|_v, 1\}$$

$$= \prod_{i=1}^{n} \prod_{v \in P_i^\infty \cup P(\mathbb{K}_j)} \max\{|\xi_i|_v, 1\}$$

and $|w_J| > |w_I|$ for all $I \neq J$.

Step:

(6) Show that factor f is the reason for irrationality and natural boundary for $R_\phi(z)$.

- The circle of convergence $|z| = |w_J|^{-1}$ is a natural boundary for $\sum_{j=1}^{\infty} f(j)(w_I z)^j$, because this is the case precisely when $\sum_{j=1}^{\infty} f(j)z^j$ has the unit circle as a natural boundary and this has already been dealt with by Lemma 5.

□

This section is dedicated to present the generalization of the Pólya–Carlson dychotomy to the coincidence Reidemeister zeta function.

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Let $\phi, \psi : G \to G$ be endomorphisms of a countable Abelian group G that is a subgroup of $\mathbb{Q}^d$, where $d \geqslant 1$. Let $R = \mathbb{Z}[t_\phi, t_\psi]$ be a polynomial ring. Then the Abelian group G naturally carries the structure of a R-module over the ring $R = \mathbb{Z}[t_\phi, t_\psi]$ where multiplication by t_ϕ and t_ψ correspond to application of the endomorphisms ϕ and ψ: $t_\phi g = \phi(g), t_\psi g = \psi(g)$ and extending this in a natural way to polynomials.

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Let $\phi, \psi : G \to G$ be endomorphisms of a countable Abelian group G that is a subgroup of $\mathbb{Q}^d$, where $d \geqslant 1$. Let $R = \mathbb{Z}[t_\phi, t_\psi]$ be a polynomial ring. Then the Abelian group G naturally carries the structure of a R-module over the ring $R = \mathbb{Z}[t_\phi, t_\psi]$ where multiplication by t_ϕ and t_ψ correspond to application of the endomorphisms ϕ and ψ: $t_\phi g = \phi(g), t_\psi g = \psi(g)$ and extending this in a natural way to polynomials.

That is, for $g \in G$ and $f = \sum_{(n_1,n_2)\in\mathbb{Z}^2} c_{(n_1,n_2)} t_\phi^{n_1} \cdot t_\psi^{n_2} \in R = \mathbb{Z}[t_\phi, t_\psi]$ set

$$fg = \sum_{(n_1,n_2)\in\mathbb{Z}^2} c_{(n_1,n_2)} t_\phi^{n_1} \cdot t_\psi^{n_2} g = \sum_{(n_1,n_2)\in\mathbb{Z}^2} c_{(n_1,n_2)} \phi^{n_1}\psi^{n_2}(g),$$

where all but finitely many $c_{(n_1,n_2)} \in \mathbb{Z}^2$ are zero.

Let $\phi, \psi : G \to G$ be automorphisms of a countable Abelian group G that is a subgroup of $\mathbb{Q}^d$, where $d \geqslant 1$. Suppose that the group G as $R = \mathbb{Z}[t_\phi, t_\psi]$- module satisfies the following conditions:

(1) the set of associated primes $Ass(G)$ is finite and consists entirely of non-zero principal ideals of the polynomial ring $R = \mathbb{Z}[t_\phi, t_\psi]$,

(2) the map $g \to (t_\phi^j - t_\psi^j)g$ is a monomorphism of G for all $j \in \mathbb{N}$ (equivalently, $t_\phi^j - t_\psi^j \notin \mathfrak{p}$ for all $\mathfrak{p} \in Ass(G)$ and all $j \in \mathbb{N}$),

(3) for each $\mathfrak{p} \in Ass(G)$, $m(\mathfrak{p}) = \dim_{\mathbb{K}(\mathfrak{p})} G_\mathfrak{p} < \infty$.

The following theorem also follows from approach from B. Klopsch and A. Fel'shtyn via profinite completion.

Theorem (Fel'shtyn, Ziętek [2020])

Let $\phi, \psi : G \to G$ be commuting automorphisms of a countable abelian group G that is a subgroup of $\mathbb{Q}^d$, $d \geqslant 1$. Suppose that the group G as $R = \mathbb{Z}[t_\phi, t_\psi]$- module satisfies the above (1)-(3) conditions. Then there exist algebraic number fields $\mathbb{K}_1, \ldots, \mathbb{K}_n$, sets of finite places $P_i \subset \mathcal{P}(\mathbb{K}_i)$, $S_i = \mathcal{P}(\mathbb{K}_i) \setminus P_i$ and elements $\xi_i, \eta_i \in \mathbb{K}_i$, $\xi_i^j \neq \eta_i^j$ for $i = 1, \ldots, n$, such that

$$R(\phi^j, \psi^j) = \prod_{i=1}^{n} \prod_{v \in P_i} |\xi_i^j - \eta_i^j|_v^{-1} = \prod_{i=1}^{n} |\xi_i^j - \eta_i^j|_{P_i}^{-1} = \prod_{i=1}^{n} |\xi_i^j - \eta_i^j|_{P_i^\infty \cup S_i}$$

for all $j \in \mathbb{N}$. Suppose that the last product only involves finitely many places and that $|\xi_i|_v \neq |\eta_i|_v$ for all v in the set of infinite places P_i^∞ of $\mathbb{K}_i$ and all $i = 1, \ldots, n$.

Then $R_{\phi,\psi}(z)$ is either rational function or has a natural boundary at its circle of convergence, and the latter occurs if and only if $|\xi_i|_v = |\eta_i|_v$ for some $v \in S_i$, $1 \leq i \leq n$.

The proof of this result is analogous to the case when only one automorphim occurs. However some technical changes for the corresponding lemmas are required. Since the idea of the proof remains the same, we will omit it and instead we will provide an example for the irrational coincidence Reidemeister zeta function and compute its natural boundary.

Example

Let us consider two endomorphisms $\phi : g \to 6g$ and $\psi : g \to 3g$ on infinitely generated abelian group $\mathbb{Z}[\frac{1}{3}]$.

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Then we can express the coincidence Reidemeister numbers as follows:

$$R(\phi^n, \psi^n) = |Coker(\phi^n - \psi^n)| = |6^n - 3^n| \cdot |6^n - 3^n|_3 = |2^n - 1| \cdot |2^n - 1|_3,$$

hence

$$R_{\varphi,\psi}(z) = \exp\left(\sum_{n=1}^{\infty} \frac{|2^n - 1| \cdot |2^n - 1|_3}{n} z^n\right).$$

Now we follow the method and the calculations of Everest, Stangoe and Ward for the Artin–Mazur zeta function of the dual compact abelian group endomorphism.

First of all we compute the numbers $R(\phi^n, \psi^n)$.

Let $\xi(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}|2^n - 1| \cdot |2^n - 1|_3$ so the coincidence Reidemeister zeta function $R_{\varphi,\psi}(z) = \exp(\xi(z))$.

Let $\xi(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} |2^n - 1| \cdot |2^n - 1|_3$ so the coincidence Reidemeister zeta function $R_{\varphi,\psi}(z) = \exp(\xi(z))$.

Now

$$
\begin{aligned}
\xi(z) &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}(2^{2n+1} - 1) + \sum_{n=1}^{\infty} \frac{z^{2n}}{2n}(2^{2n} - 1)|2^{2n} - 1|_3 \\
&= \log\left(\frac{1-z}{1-2z}\right) - \frac{1}{2}\log\left(\frac{1-z^2}{1-4z^2}\right) + \sum_{n=1}^{\infty} \frac{z^{2n}}{2n}(2^{2n} - 1)|2^{2n} - 1|_3 \\
&= \log\left(\frac{1-z}{1-2z}\right) - \frac{1}{2}\log\left(\frac{1-z^2}{1-4z^2}\right) + 3\sum_{n=1}^{\infty} \frac{z^{2n}}{n}(4^{n} - 1)|4^{n} - 1|_3 \\
&= \log\left(\frac{1-z}{1-2z}\right) - \frac{1}{2}\log\left(\frac{1-z^2}{1-4z^2}\right) + \sum_{n=1}^{\infty} \frac{z^{2n}}{n}(4^{n} - 1)|n|_3.
\end{aligned}
$$

Finally, we can show that

$$|R_{\phi,\psi}(z)| = \left|\frac{1-z}{1-2z}\right| \left|\frac{1-(2z)^2}{1-z^2}\right|^{1/2} \left|\frac{1-z^2}{1-(2z)^2}\right|^{1/6} \prod_{j=1}^{\infty} \left|\frac{1-(2z)^{2\times 3^j}}{1-z^{2\times 3^j}}\right|^{1/3\times 9^j}.$$

if follows that the series defining the coicidence Reidemeister zeta function has zeroes at all points of the form $\frac{1}{2}e^{2\pi ij/3^r}$, $r \geqslant 1$, so $|z| = \frac{1}{2}$ is a natural boundary for $R_{\phi,\psi}(z)$.

What's next?

So far we were mainly interested with subgroups of $\mathbb{Q}^d$, where $d \geqslant 1$, which decribes an abelian torsion free group.

Currently we are working on transfer the dichotomy result to the case when the group G is infinitely generated **torsion Abelian** group. Despite such a change, the method of proving remains similar. The biggest difference is that we have to consider t-adic place and $\mathbb{F}_p(t)$ field instead of $\mathbb{Q}$ and we give a further support to the Pólya–Carlson dichotomy conjecture.

That leads us to the following **open problem**:
If we consider an arbitrary abelian group G - does the dichotomy result remain true?

Thank you for your attention!