The irreducible (complex) representations of various general linear groups

Clotilde Gauthier

Oberseminar SOSE 2026

In this seminar, we are interested in understanding the (complex) representation theory of (finite) general linear groups. We will see that the irreducible representations can be divided in two groups: the principal series and the cuspidal representations. Principal series are obtained as irreducible components of representations induced from representations of specific subgroups, called parabolic subgroups. The cuspidal representations are all the irreducible representations that cannot be obtained in this way. They are much more diffucult to study and come from linear combinations of characters. The main goal of the seminar is to see the machinery involved in the construction of these irreducible representations, and to apply it to small examples to get a better understanding.

The following notations will be used throughout the program.[1]

- For all $n \in \mathbb{N}$, $G_n \coloneqq \mathrm{GL}_n(\mathbb{F}_q)$ and in particular $G \coloneqq G_2 = \mathrm{GL}_2(\mathbb{F}_q)$ and $D \coloneqq \mathrm{SL}_2(\mathbb{F}_q)$
- B the subgroup of upper triangular matrices in $\mathrm{GL}_2(\mathbb{F}_q)$
- N the subgroup of unipotent upper triangular matrices in $\mathrm{GL}_2(\mathbb{F}_q)$
- T the subgroup of diagonal matrices in $\mathrm{GL}_2(\mathbb{F}_q)$ and the matrix $w \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{F}_q)$
- Let $\mathcal{P}$ be the set of all partitions of the integers (including 0). For all partition $\lambda \in \mathcal{P}$, denote $|\lambda|$ the integer of which λ is a partition
- For all partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of $n \in \mathbb{N}$, P_λ is the subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ defined by

$$P_\lambda = \begin{pmatrix} G_{\lambda_1} & * & \cdots & * \\ 0 & G_{\lambda_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & G_{\lambda_r} \end{pmatrix}$$

- $\widehat{G}$ the group of linear characters of a group G (that is group morphisms $G \to \mathbb{C}^\times$)
- $\mathbb{1}_G$ the trivial character of a group G
- Φ_n the set of irreducible polynomials of degree at most n over $\mathbb{F}_q$

[1] You can change them if you like, but please make sure to coordinate with the speakers who need to use objects introduced in your talk

Talk 1: Setting the stage

References: [FH04],[BH06]

Goal: Giving the necessary background of representation theory and describing the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$.

1. Give the definition of *representation*, *subrepresentation*, *irreducible* representation and *morphism* of representation [FH04, p. 3-4].

2. Define the *dimension* of a representation and mention that a representation of a cyclic group is irreducible if and only if it has dimension 1.

3. State Schur's Lemma, that is for any two complex irreducible representations (V, ρ) and (W, π) of a finite group G,
$$\dim\left(\mathrm{Hom}_G(V, W)\right) = \begin{cases} 1 & \text{if } (V, \rho) \simeq (W, \pi) \\ 0 & \text{otherwise} \end{cases}$$
[FH04, Lemma 1.7].

4. State Maschke's Theorem: Any complex representation of a finite group G can be written as a direct sum of irreducible subrepresentations [FH04, Corollary 1.6].

5. As a consequence of the two previous results, we get that any representation can be written as a direct sum of non-equivalent irreducible representations with multiplicity [FH04, Proposition 1.8].

6. Define the *character* χ_V of a representation V, justify that this is a class function and that $\chi_V(1_G) = \dim(V)$ [FH04, p. 13].

7. Define the Herminitan inner product on the class functions of G given by
$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}$$
[FH04, Equation (2.11)].

8. State Frobenius' Theorem, that is, the irreducible characters of G form an orthonormal basis of the class functions for the inner product $\langle \cdot, \cdot \rangle$ [FH04, Proposition 2.30]. We have the following consequences:
   - There are as many irreducible representations of G as conjugacy classes
   - A representation V of G is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$
   - Two representations V and W are equivalent if and only if they have same character

9. Give the orthogonality relations of the irreducible characters [FH04, p. 18] and use it to draw the character table of the symmetric group $\mathfrak{S}_3$.

10. Mention the representation ring, and the notion of *virtual representation* [FH04, p. 22].

11. Give the definition of induced representation. State Frobenius reciprocity, Mackey's induction-restriction formula and give a formula for the value of its character [Ser77, Chapter 7]. [2]

12. Recall that two matrices of $\mathrm{GL}_2(\mathbb{F}_q)$ are conjugated if and only if they have same characteristic and minimal polynomial, describe the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$ and give their respective size [BH06, §5.3].

[2] Mackey's induction-restriction formula is Proposition 22

Talk 2: Principal series of $\mathrm{GL}_2(\mathbb{F}_q)$
References: [BH06],[FH04]
Goal: Defining a first type of irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ that are obtained by induction called *principal series*. We will see that this process produces $\frac{1}{2}(q^2+q)-1$ non-equivalent irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ and we will compute their character.

1. Give an overview of the classification of the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$: In the last talk, we saw that $\mathrm{GL}_2(\mathbb{F}_q)$ has q^2-1 conjugacy classes, hence there are q^2-1 irreducible representations to find. The general idea to get these representations is to induce linear characters of specific subgroups. The first kind of representations, called principal series, will be defined in this talk and are obtained by inducing linear characters of B. The second kind, called cuspidal representations, are obtained as linear combinations in $\mathbb{Z}$ of induced representations and will be studied in the next talk.

2. Mention the isomorphism $T \simeq B/N$ and state the Bruhat decomposition for $\mathrm{GL}_2(\mathbb{F}_q)$ [BH06, §5.2].

3. In order to study the principal series of G, one needs to first get a better understanding of the characters of N. See that $N \simeq \mathbb{F}_q$, fix ψ a non-trivial linear character of $\mathbb{F}_q$ and describe the characters of N in terms of ψ. Describe the orbits of the action of T on $\widehat{N}$ [BH06, §6.2].

4. Start the study of the irreducible representations of G by its linear characters: Prove that if $q \neq 2$ every linear character of G can be written $\phi \circ \det$ with $\phi \in \widehat{\mathbb{F}_q^\times}$. In particular, G has $q-1$ linear characters [FH04, p. 69].
Mention that for $q=2$, $G \simeq \mathfrak{S}_3$ and as a consequence its linear characters are $\mathbb{1}$ and the signature.

5. Prove that for any irreducible representation π of G, either $\mathrm{Res}_N^G \pi$ contains[3] $\mathbb{1}_N$ or $\mathrm{Res}_N^G \pi$ contains every linear characters of N but $\mathbb{1}_N$ (this is a consequence of [BH06, §6.2][4]).
In the rest of the talk we are only concerned with the first case of this result.

6. Give an explicit description of the linear characters of T and justify that the linear characters of B that are trivial on N are exactly obtained as inflations of linear characters of T.
Prove that the irreducible representations of G that contain $\mathbb{1}_N$ on N are exactly the subrepresentations of the representations of G obtained by inducing linear characters of B that are trivial on N [BH06, §6.3].

7. Fix χ a linear character of B that is trivial on N. We need to understand the irreducible components of $\mathrm{Ind}_B^G \chi$.
Define the character χ^w, give the value of $\langle \mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \chi \rangle$ and see that $\mathrm{Ind}_B^G \chi$ is irreducible if and only if $\chi \neq \chi^w$ on T [BH06, §6.3].

8. Give the example of $\chi = \chi^w = \mathbb{1}_B$ and define the *Steinberg representation St_G* [BH06, §6.3].

9. Justify that in the case $\chi = \chi^w$ on T, there exists $\phi \in \widehat{\mathbb{F}_q^\times}$ such that $\chi = \phi \circ \det$ and justify

$$\mathrm{Ind}_B^G \chi \simeq \phi \circ \det \oplus \phi \, \mathrm{St}_G$$

[FH04, p. 69][5].

[3] that is $\mathbb{1}_N$ is a subrepresentation of $\mathrm{Res}_N^G \pi$
[4] If you need a proper proof, please ask me
[5] If you need more details, please ask me

10. Define *principal series* as the irreducible representations of G that contain the trivial character on N and prove that if $\operatorname{Ind}_B^G \chi$ and $\operatorname{Ind}_B^G \xi$ are irreducible, then they are equivalent if and only if $\chi = \xi$ or $\chi = \xi^w$ [BH06, §6.3].
Enumerate the three different kind of principal series of G:
• Linear characters: $q-1$ • Twisted Steinberg: $q-1$ • Irreducible $\operatorname{Ind}_B^G \chi$: $\frac{(q-1)(q-2)}{2}$
and give their respective dimensions [BH06, §6.3].

Talk 3: Cuspidal representations of $\mathbf{GL_2}(\mathbb{F}_q)$

References: [BH06]

Goal: Defining and studying the remaining irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$, that is to say the ones that are not principal series.

1. Recall what happened in the previous talk: We saw that for all irreducible representation π of G, either $\mathrm{Res}_N^G \pi$ contains $\mathbb{1}_N$ or $\mathrm{Res}_N^G \pi$ contains every linear character of N but $\mathbb{1}_N$. The first case are the *principal series* and they can be found as irreducible components of representations $\mathrm{Ind}_B^G \chi$ where χ is a linear character of B that is trivial on N. We have found $\frac{1}{2}(q^2+q)-1$ principal series and we know that there are q^2-1 irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$. Consequently, there are $\frac{1}{2}(q^2-q)$ irreducible representations left to find.

2. Give the definition of *cuspidal representation*, that is any irreducible representation of G that is not a principal series [BH06, §6.4].

3. The cuspidal representations cannot be built directly with induction, but they can be expressed as linear combination in $\mathbb{Z}$ of induced representations. Let F be a quadratic extension of $\mathbb{F}_q$ and α a primitive element, and consider the following subgroup of G

$$E = \left\{ \begin{pmatrix} a & b\alpha^2 \\ b & a \end{pmatrix} \mid a, b \in \mathbb{F}_q \right\}.$$

   Mention that $E \simeq F^\times$, so is abelian of order q^2-1, and that $Z \coloneqq Z(G)$ is contained in E [BH06, §6.4].

4. Justify that any element of G with irreducible characteristic polynomial is conjugated to an element of E and that in particular, any element of $E \setminus Z$ is conjugated to exactly one other element of $E \setminus Z$ [BH06, §6.4][6].

5. Give the definition of *regular character* of E. Fix θ a regular character of E and ϕ a non-trivial character of N and define the character θ_ψ of ZN. Justify that the representation $\mathrm{Ind}_{ZN}^G \theta_\psi$ is independant of the choice of ψ [BH06, §6.4].

6. Define the *virtual representation* $\pi_\theta \coloneqq \mathrm{Ind}_{ZN}^G \theta_\psi - \mathrm{Ind}_E^G \theta$ and prove that it is an (irreducible) cuspidal representation of G of dimension $q-1$ [BH06, §6.4].

7. Prove that two cuspidal representation π_{θ_1} and π_{θ_2} are equivalent if and only if $\theta_2 = \theta_1$ or $\theta_2 = \theta_1^q$ [BH06, §6.4].

8. Prove that all cuspidal representations of G have form π_θ for some θ regular character of E, that is to say there are $\frac{1}{2}(q^2-q)$ such representations [BH06, §6.4].

9. Draw a recap table with the four kinds of irreducible representations of G, their dimension and their amount.

10. Draw the character table of $\mathrm{GL}_2(\mathbb{F}_q)$.

[6] If you need more details, please ask me

Talk 4: What about $\mathbf{SL_2}(\mathbb{F}_q)$?
References: [FH04]
Goal: Understanding the representation theory of $\mathrm{SL}_2(\mathbb{F}_q)$, q odd, under the light of the classfication done for $\mathrm{GL}_2(\mathbb{F}_q)$.

1. Give an overview of the talk: Since $SL_2(\mathbb{F}_q)$ is a subgroup of $\mathrm{GL}_2(\mathbb{F}_q)$, its irreducible representations are all **contained** in the restriction[7] of the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$. The general idea is to restrict all the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ to $\mathrm{SL}_2(\mathbb{F}_q)$ and see which remain irreducible, which become equivalent, and which split into several irreducible subrepresentations.

2. In order to know the amount of irreducible representations we are looking for and be able to use the character formula on $D \coloneqq \mathrm{SL}_2(\mathbb{F}_q)$, one needs to know the conjugacy classes.
Draw a table with a representative for each conjugacy class, the size of the conjugacy classes and the amount of conjugacy classes associated to each type of representatives, and briefly explain how to get it from the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$ [FH04, p. 71].

3. Justify that the restriction of the linear characters of G to D is always $\mathbb{1}_D$ and that the trivial character is the only linear character of $\mathrm{SL}_2(\mathbb{F}_q)$ [FH04, p. 71].

4. Prove (sketch) that $\mathrm{SL}_2(\mathbb{F}_q)$ has $\frac{1}{2}(q-3)$ irreducible representations of dimension $q+1$ of form $\mathrm{Res}_D^G \mathrm{Ind}_B^G \chi$ where χ is a linear character of B that is trivial on N such that $\chi^2 \neq \mathbb{1}$ on $T \cap D$ [FH04, p. 72][8].

5. Prove that the Steinberg representation of G remains irreducible after restriction to D. Deduce from this that all the representations of G of dimension q restrict to the Steinberg representation on D [FH04, p. 72].

6. Prove (sketch) that the group $\mathrm{SL}_2(\mathbb{F}_q)$ has $\frac{1}{2}(q-1)$ irreducible representations of dimension $q-1$ that are obtained as $\mathrm{Res}_D^G \pi_\theta$ where θ is a regular character of E such that $\theta^2 \neq 1$ on $E \cap D$ [FH04, p. 72].

7. So far the restriction of the irreducible representations of G gave q irreducible non-equivalent representations of D. Since there are $q+4$ conjugacy classes in $\mathrm{SL}_2(\mathbb{F}_q)$, there are 4 representations left to find. This is achieved by studying the two representations $\mathrm{Res}_D^G \mathrm{Ind}_B^G \chi_0$ and $\mathrm{Res}_D^G \pi_{\theta_0}$ where $\chi_0^2 = \mathbb{1}$ on $T \cap D$ and $\theta_0^2 = \mathbb{1}$ on $E \cap D$.
Prove that these two representations have length 2, that is there exists W', W'', X' and X'' irreducible representations of D such that $\mathrm{Res}_D^G \mathrm{Ind}_B^G \chi_0 = W' \oplus W''$ and $\pi_{\theta_0} = X' \oplus X''$ [FH04, p. 72].

8. Justify that the representations W', W'', X' and X'' are not equivalent to each other and that $\dim W' = \dim W'' = \frac{1}{2}(q+1)$ and $\dim X' = \dim X'' = \frac{1}{2}(q-1)$ [FH04, p. 72].

Further reading: In order to work out the complete character table of $\mathrm{SL}_2(\mathbb{F}_q)$ there are only the characters of the four representations W', W'', X' and X'' left to compute. This uses the representation theory of subgroups of index 2 applied to the subgroup $H = \{g \in G \mid \exists\, \alpha \in \mathbb{F}_q^\times,\ \det g = \alpha^2\}$. In particular, this gives that the representations W' and W'' (resp. X' and X'') are conjugated representations of $\mathrm{SL}_2(\mathbb{F}_q)$.

[7] Note that the restriction of an irreducible representation may not remain irreducible.
[8] If you need help to work out the proof, please ask me

Talk 5: Conjugacy classes of $\mathbf{GL}_n(\mathbb{F}_q)$
References: [Gre55]
Goal: Describing the conjugacy classes of $\mathrm{GL}_n(\mathbb{F}_q)$ in order to be able to give explicit values to the characters we will build in the next talks.

1. Motivate the talk: In order to understand the representation theory of a finite group, one needs to know its conjugacy classes as this provides with the number of irreducible representations and the value of their characters (that are class functions). In the case of $\mathrm{GL}_n(\mathbb{F}_q)$ we describe the conjugacy classes in terms of partitions and irreducible polynomials over $\mathbb{F}_q$.

2. Explain that two matrices in $\mathrm{GL}_n(\mathbb{F}_q)$ are conjugate if and only if they are similar over $\mathbb{F}_q$ and introduce the matrices $U(f)$, $U_m(f)$ and $U_\lambda(f)$. See that if $\chi_A = f_1^{k_1} \cdots f_N^{k_N}$ is the characteristic polynomial of a matrix $A \in \mathrm{GL}_n(\mathbb{F}_q)$, then there exists a sequence $(\nu_1, \dots, \nu_N)$ of partitions of $k_1, \dots, k_N$ respectively such that

$$A \sim \mathrm{diag}\left(U_{\nu_1}(f_1), \dots, U_{\nu_N}(f_N)\right).$$

   [Gre55, p. 405-406]

3. We now aim at giving a more general description of the conjugacy classes. Fix c a conjugacy class of $\mathrm{GL}_n(\mathbb{F}_q)$ and define the function $\nu_c : \Phi_n \to \mathcal{P}$ which to any irreducible polynomial f associates its partition (possibly 0) in c. Prove that for all function $\Phi_n \to \mathcal{P}$ there exists a conjugacy class c of $\mathrm{GL}_n(\mathbb{F}_q)$ such that $\nu = \nu_c$ if and only if

$$\sum_{f \in \Phi_n} |\nu(f)| \deg(f) = n$$

   [Gre55, Lemma 1.1].

4. From now on, χ_c denotes the characteristic polynomial of the matrices in c, and we call *eigenvalues of c* the roots of χ_c[9]. Give the definitions of *principal conjugacy class*, *conjugacy class of principal type ρ* and *primary conjugacy class*. Define the notion of primary classes of same *type* [Gre55, p. 407].

5. We want to generalise the notion of *type*. Define the partition $\rho_c(\nu)$. This then gives a function $\rho_c : \mathcal{P} \setminus \{0\} \to \mathcal{P}$ that is the *type* of c. Two conjugacy classes c_1 and c_2 have *same type* if the functions ρ_{c_1} and ρ_{c_2} are equal [Gre55, p.407].

6. Prove that any function $\rho : \mathcal{P} \setminus \{0\} \to \mathcal{P}$ describes a type of conjugacy class if

$$\sum_{\nu \in \mathcal{P} \setminus \{0\}} |\rho(\nu)| |\nu| = n$$

   and explain that the number $t(n)$ of types of conjugacy classes does not depend on q [Gre55, p. 408].

7. Give the generating functions for the number of types and the number of conjugacy classes of $\mathrm{GL}_n(\mathbb{F}_q)$ [Gre55, p. 408].

8. As an example, describe the conjugagcy classes of $\mathrm{GL}_3(\mathbb{F}_q)$.

Bonus depending on time: Study the size of the conjugacy classes of $\mathrm{GL}_n(\mathbb{F}_q)$ [Gre55, p. 409].

[9] Those are called latent roots in the article

Talk 6: Parabolic induction

References: [Gre55]

Goal: Defining parabolic subgroups and parabolic induction and understanding the formula for the value of the character of such a representation.

1. Explain the concept of parabolic induction: Given any partition λ of n, one can consider the *parabolic subgroup* P_λ associated to λ and create representations of $\mathrm{GL}_n(\mathbb{F}_q)$ by inducing representations of P_λ. The particular form of P_λ provides that its representations can be seen as tensor product of representations of some $\mathrm{GL}_m(\mathbb{F}_q)$ for $m < n$.
   In a nutshell, parabolic induction boils down to creating representations of $\mathrm{GL}_n(F_q)$ by inducing tensor products of representations of smaller $\mathrm{GL}_m(\mathbb{F}_q)$.

2. Fix $\lambda = (\lambda_1, \dots, \lambda_r)$ a partition of n and define the *parabolic subgroup*

$$P_\lambda = \begin{pmatrix} G_{\lambda_1} & * & \cdots & * \\ 0 & G_{\lambda_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & G_{\lambda_r} \end{pmatrix}$$

   [Gre55, p. 403].

3. For all $i \in [\![1, r]\!]$, fix α_i a character of G_{λ_i} and consider the character $\alpha_1 \otimes \cdots \otimes \alpha_r$ of P_λ. The representation obtained by inducing this character to G_n is called *parabolic induction*

$$\alpha_1 \circ \cdots \circ \alpha_r = \mathrm{Ind}_{P_\lambda}^{G_n} \alpha_1 \otimes \cdots \otimes \alpha_r$$

   [Gre55, p. 403].

4. We now want to express the value of the characters obtained by parabolic induction. Recall the formula of the character of an induced representation [Ser77, p. 55].

5. Explain how for all $A \in G_n$, representatives s of $P_\lambda \backslash G_n$ such that $sAs^{-1} \in P_\lambda$ is in 1-1 correspondance with the chains

$$V = V^{(0)} \supset V^{(1)} \supset \cdots \supset V^{(r)} = 0$$

   of submodules of V_A such that for all $i \in [\![1, r]\!]$, $V^{(i-1)}/V^{(i)} \simeq V_{(sAs^{-1})_{ii}}$ [Gre55, p. 410].

6. State Theorem 2 in [Gre55, p. 410]. State Lemma 2.7 in [Gre55, p. 413] - this gives the degree of the representations obtained through parabolic induction - and state Lemma 2.8 in [Gre55, p. 413] which gives the value of the characters on the principal classes.

7. In order to get a better grasp on what is going on, give the exemple of the character of the representation

$$\chi \circ \phi = \mathrm{Ind}_{P_{(2\,1)}}^{G_3} \chi \otimes \phi$$

   of $\mathrm{GL}_3(\mathbb{F}_q)$, where χ is an arbitrary caracter of $\mathrm{GL}_2(\mathbb{F}_q)$ and ϕ is an arbitrary character of $\mathbb{F}_q^\times$.

Talk 7: Uniform functions

References: [Gre55]

Goal: Define uniform functions, see their basic properties and compute a few examples.

1. Give a motivation for this talk: In his paper, Green defines a specific type of class function on $\mathrm{GL}_n(\mathbb{F}_q)$ and introduce then the concept of *uniform function* in order to better understand them. Indeed, uniform functions are defined as a linear combination of *principal parts* which makes them easier to work out. Green also proves that knowing the scalar product of uniform functions boils down to computing the scalar products of the principal parts.
Since the combinatorics in the article is too involved for the purpose of a seminar, we skip the considerations leading to the construction of the irreducible characters and instead we use the definition of uniform function to define a type of class function that will appear to be exactly the irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$[10].

2. Start by defining the polynomials Q^λ_ρ [Gre55, p. 420].

3. Define the terms *ρ-variables*, *substitution* and *mode of substitution* [Gre55, p. 421]. Give the definition of *ρ-function* and give an exemple[11][Gre55, p. 422-423].

4. Give the definition of *uniform class function* [Gre55, p. 423] and present the following example: Let θ be a generator of $\mathbb{F}_{q^{n!}}$ and λ be a partition of 3. Define the following ρ-functions for the partitions of 3:
$$U^\lambda_{(1,1,1)}(\xi_{11},\xi_{12},\xi_{13}) = \chi^\lambda_{(1,1,1)}\theta^k(\xi_{11})\theta^k(\xi_{12})\theta^k(\xi_{13})$$
$$U^\lambda_{(2,1)}(\xi_{11},\xi_{21}) = \chi^\lambda_{(2,1)}\theta^k(\xi_{11})\theta^k(\xi_{21}^{1+q})$$
$$U^\lambda_{(3)}(\xi_{31}) = \chi^\lambda_{(3)}\theta^k(\xi_{31}^{1+q+q^2}),$$
where χ^λ_ρ is the value of the character of the symmetric group $\mathfrak{S}_3$ associated to λ on the class parametrized by ρ. Explain how to get the uniform function $U^\lambda = (U^\lambda_\rho)$ and draw a table of its values on the conjugacy calsses of $\mathrm{GL}_3(\mathbb{F}_q)$[12].

5. State Theorem 6 and 7 on [Gre55, p. 424] and state Theorem 11 [Gre55, p. 431]. These results are used to prove the irreducibility of the characters that will be defined in the next talk, but their proofs are too long for this seminar.

[10]Magic!

[11]You can introduce the notation $T_{s,e}(k:\xi)$ as it will be used later. It first appears on page 417.

[12]You can use the beamer to show the table if you think it would take too much time to draw

Talk 8: Irreducible characters of $\mathbf{GL}_n(\mathbb{F}_q)$
References: [Gre55]
Goal: Defining the irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ and using the results to compute the character table of $\mathrm{GL}_3(\mathbb{F}_q)$.

1. Motivate the talk: In the previous talk, we studied the concept of *uniform function*. In this talk, we define a specific uniform functions that will turn out to be exactly the irreducible characters we are looking for. Once again, the proof of this is too long to be presented but it comes form the properties of uniform function and some representation theory results such as Brauer's characterization of characters.

2. Define the functions $T_{s,e}(k:\xi)$ [Gre55, p. 417].
   Note: they already appeared in an example in the previous talk.

3. Define the notion of *s-simplex* [Gre55, p. 438-439] and see that there are as many simplexes of degree s as irreducible polynomials of degree s over $\mathbb{F}_q$ [Gre55, Lemma 7.7].

4. State Theorem 12 [Gre55, p. 439]. Recall the last example of the previous talk and see that the uniform function U^λ is actually the *primary irreducible character* (g^λ).

5. State Theorem 13 [Gre55, p. 439] that gives the definition of all irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ as parabolic induction of primary irreducible characters.

6. Using the table of parabolic induction computed in Talk 6 and the table of values of U^λ computed in Talk 7, draw the character table of $\mathrm{GL}_3(\mathbb{F}_q)$.

Talk 9: The irreducible representations of $\mathbf{GL_2}(\mathbb{Q}_p)$: An overview
References: [BH06]
Goal: Stating the classification of the irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and understanding what needs to be changed compared to the classification we saw for $\mathrm{GL}_2(\mathbb{F}_q)$.

1. Motivate the talk: The group we are now studying is no longer finite, which prevents from using character theory to study the representations. In order to still have some structure, one can get topology involve and restrict the study to *smooth representations*. The outcome of the study for $\mathrm{GL}_2(\mathbb{Q}_p)$ is that principal series behave in a similar way as the finite case whereas cuspidal representations are a lot harder to find.
   Note: Instead of $\mathbb{Q}_p$ you can use F a non-archimedean local field.

2. Give the definition of *smooth representation* and explain *smooth induction*.

3. Define the *Jacquet module* at N and sketch the proof of Proposition 9.1 [BH06, p. 62]. This results enable to separate principal series and cuspidal representations as in the finite case.

4. Mention the results of [BH06, §9.2] to explain the difference between $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathrm{GL}_2(\mathbb{F}_q)$.

5. State the classification theorem for the principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$.

6. The full classification being very involved, we will only give a way to build cuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Define compact induction [BH06, §2.5]. The cusidal representations will be constructed using compact induction from open subgroups.

7. Define the notion of *an element intertwining representations* [BH06, §11.1].

8. State Theorem 11.4 [BH06, p. 80] and give the example [BH06, §11.5]

A Various character tables for $n = 2$ and $n = 3$

Let α, β and γ be three distinct elements of $\mathbb{F}_q^\times$, let $\varepsilon \in \mathbb{F}_{q^2}$ be the root of an irreducible polynomial of degree 2 over $\mathbb{F}_q$ and let $\mu \in \mathbb{F}_{q^3}$ be the root of an irreducible polynomial of degree 3 over $\mathbb{F}_q$. Let θ be a character of $\mathbb{F}_{q^{n!}}^\times$ of order $q^{n!} - 1$. We denote g_1, g_1' and g_1'' three 1-simplexes with k, k' and k'' as root respectively, we denote g_2 a 2-simplex with l as a root and g_3 a 3-simplex with m as a root.

A.1 Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

$\mathrm{GL}_2(\mathbb{F}_q)$	$(X-\alpha)^{(1,1)}$	$(X-\alpha)^{(2)}$	$(X-\alpha)^{(1)}(X-\beta)^{(1)}$	$(X^2-\gamma^2)^{(1)}$
$\left(g_1^{(2)}\right)$	$\theta^k(\alpha^2)$	$\theta^k(\alpha^2)$	$\theta^k(\alpha\beta)$	$\theta^k(\gamma^{1+q})$
$\left(g_2^{(1)}\right)$	$(q-1)\theta^l(\alpha)$	$-\theta^l(\alpha)$	0	$-(\theta^l(\gamma)+\theta^{lq}(\gamma))$
$\left(g_1^{(1,1)}\right)$	$q\theta^k(\alpha^2)$	0	$\theta^k(\alpha\beta)$	$\theta^k(\gamma^{1+q})$
$\left(g_1^{(1)}\right) \circ \left({g'_1}^{(1)}\right)$	$(q+1)\theta^{k+k'}(\alpha)$	$\theta^{k+k'}(\alpha)$	$\theta^k(\alpha)\theta^{k'}(\beta)+\theta^k(\beta)\theta^{k'}(\alpha)$	0

Table 1: Character table of $\mathrm{GL}_2(\mathbb{F}_q)$

The notations of Talks 2 and 3, give

$$\left(g_1^{(2)}\right) = \theta^k \circ \det, \quad \left(g_2^{(1)}\right) = \pi_{\theta^l}, \quad \left(g_1^{(1,1)}\right) = \theta^k \circ \mathrm{St}_G \quad \text{and} \quad \left(g_1^{(1)}\right) \circ \left({g'_1}^{(1)}\right) = \mathrm{Ind}_B^G \theta^k \otimes \theta^{k'}.$$

A.2 Parabolic induction from $P_{(2,1)}$ to $\mathrm{GL}_3(\mathbb{F}_q)$

Let π be a character of $\mathrm{GL}_2(\mathbb{F}_q)$ and let ϕ be a character of $\mathbb{F}_q^\times$.

$\mathrm{GL}_3(\mathbb{F}_q)$	$(X-\alpha)^{(1,1,1)}$	$(X-\alpha)^{(2,1)}$	$(X-\alpha)^{(3)}$	$(X-\alpha)^{(1,1)}(X-\beta)^{(1)}$
$\pi \circ \phi$	$(q^2+q+1)\pi\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\phi(\alpha)$	$\pi\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\phi(\alpha)+q\pi\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}\phi(\alpha)$	$\pi\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}\phi(\alpha)$	$(q+1)\pi\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\phi(\alpha)+\pi\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\phi(\beta)$

$\mathrm{GL}_3(\mathbb{F}_q)$	$(X-\alpha)^{(2)}(X-\beta)^{(1)}$	$(X-\alpha)^{(1)}(X-\beta)^{(1)}(X-\gamma)^{(1)}$	$(X-\alpha)^{(1)}(X^2-\varepsilon^2)^{(1)}$	$(X^3-\mu^3)^{(1)}$
$\pi \circ \phi$	$\pi\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\phi(\alpha)+\pi\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}\phi(\beta)$	$\pi\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\phi(\gamma)+\pi\begin{pmatrix}\alpha & 0\\ 0 & \gamma\end{pmatrix}\phi(\beta)+\pi\begin{pmatrix}\beta & 0\\ 0 & \gamma\end{pmatrix}\phi(\alpha)$	$\pi\begin{pmatrix}0 & \varepsilon^2\\ 1 & 0\end{pmatrix}\phi(\alpha)$	0

Table 2: Parabolic induction for $P_{(2,1)}$

A.3 Characters of $\mathrm{GL}_3(\mathbb{F}_q)$ obtained through uniform functions

$\mathrm{GL}_3(\mathbb{F}_q)$	$(X-\alpha)^{(1,1,1)}$	$(X-\alpha)^{(2,1)}$	$(X-\alpha)^{(3)}$	$(X-\alpha)^{(1,1)}(X-\beta)^{(1)}$
$\left(g_1^{(1,1,1)}\right)$	$q^3\theta^k(\alpha^3)$	0	0	$q\theta^k(\alpha^2\beta)$
$\left(g_1^{(2,1)}\right)$	$(q^2+q)\theta^k(\alpha^3)$	$q\theta^k(\alpha^3)$	0	$(q+1)\theta^k(\alpha^2\beta)$
$\left(g_1^{(3)}\right)$	$\theta^k(\alpha^3)$	$\theta^k(\alpha^3)$	$\theta^k(\alpha^3)$	$\theta^k(\alpha^2\beta)$
$\left(g_3^{(1)}\right)$	$(q^2-1)(q-1)\theta^m(\alpha)$	$(1-q)\theta^m(\alpha)$	$\theta^m(\alpha)$	0

$\mathrm{GL}_3(\mathbb{F}_q)$	$(X-\alpha)^{(2)}(X-\beta)^{(1)}$	$(X-\alpha)^{(1)}(X-\beta)^{(1)}(X-\gamma)^{(1)}$	$(X-\alpha)^{(1)}(X^2-\varepsilon^2)^{(1)}$	$(X^3-\mu^3)^{(1)}$
$\left(g_1^{(1,1,1)}\right)$	0	$\theta^k(\alpha\beta\gamma)$	$-\theta^k(\alpha)\theta^k(\varepsilon^{1+q})$	$\theta^k(\mu^{1+q+q^2})$
$\left(g_1^{(2,1)}\right)$	$\theta^k(\alpha^2\beta)$	$2\theta^k(\alpha\beta\gamma)$	0	$-\theta^k(\mu^{1+q+q^2})$
$\left(g_1^{(3)}\right)$	$\theta^k(\alpha^2\beta)$	$\theta^k(\alpha\beta\gamma)$	$\theta^k(\alpha)\theta^k(\varepsilon^{1+q})$	$\theta^k(\mu^{1+q+q^2})$
$\left(g_3^{(1)}\right)$	0	0	0	$\theta^m(\mu)+\theta^{mq}(\mu)+\theta^{mq^2}(\mu)$

Table 3: Characters defined by uniform functions

A.4 Character table of $\mathrm{GL}_3(\mathbb{F}_q)$

$\mathrm{GL}_3(\mathbb{F}_q)$	$(X-\alpha)^{(1,1,1)}$	$(X-\alpha)^{(2,1)}$	$(X-\alpha)^{(3)}$	$(X-\alpha)^{(1,1)}(X-\beta)^{(1)}$
$\left(g_1^{(3)}\right)$	$\theta^k(\alpha^3)$	$\theta^k(\alpha^3)$	$\theta^k(\alpha^3)$	$\theta^k(\alpha^2\beta)$
$\left(g_1^{(2,1)}\right)$	$(q^2+q)\theta^k(\alpha^3)$	$q\theta^k(\alpha^3)$	0	$(q+1)\theta^k(\alpha^2\beta)$
$\left(g_1^{(1,1,1)}\right)$	$q^3\theta^k(\alpha^3)$	0	0	$q\theta^k(\alpha^2\beta)$
$\left(g_1^{(2)}\right)\circ\left(g_1'^{(1)}\right)$	$(q^2+q+1)\theta^{2k+k'}(\alpha)$	$(q+1)\theta^{2k+k'}(\alpha)$	$\theta^{2k+k'}(\alpha)$	$(q+1)\theta^{k+k'}(\alpha)\theta^k(\beta)$ $+\theta^{2k}(\alpha)\theta^{k'}(\beta)$
$\left(g_1^{(1,1)}\right)\circ\left(g_1'^{(1)}\right)$	$(q^2+q+1)q\theta^{2k+k'}(\alpha)$	$q\theta^{2k+k'}(\alpha)$	0	$(q+1)\theta^{k+k'}(\alpha)\theta^k(\beta)$ $+q\theta^{2k}(\alpha)\theta^{k'}(\beta)$
$\left(g_1^{(1)}\right)\circ\left(g_1'^{(1)}\right)\circ\left(g_1''^{(1)}\right)$	$(q^2+q+1)(q+$ $1)\theta^{k+k'+k''}(\alpha)$	$(2q+1)\theta^{k+k'+k''}(\alpha)$	$\theta^{k+k'+k''}(\alpha)$	$(q+1)\Big[\theta^{k+k''}(\alpha)\theta^{k'}(\beta)$ $+\theta^{k'+k''}(\alpha)\theta^k(\beta)$ $+\theta^{k+k'}(\alpha)\theta^{k''}(\beta)\Big]$
$\left(g_2^{(1)}\right)\circ\left(g_1^{(1)}\right)$	$(q^3-1)\theta^{k+l}(\alpha)$	$-\theta^{k+l}(\alpha)$	$-\theta^{k+l}(\alpha)$	$(q-1)\theta^l(\alpha)\theta^k(\beta)$
$\left(g_3^{(1)}\right)$	$(q^2-1)(q-1)\theta^m(\alpha)$	$(1-q)\theta^m(\alpha)$	$\theta^m(\alpha)$	0

$\mathrm{GL}_3(\mathbb{F}_q)$	$(X-\alpha)^{(2)}(X-\beta)^{(1)}$	$(X-\alpha)^{(1)}(X-\beta)^{(1)}(X-\gamma)^{(1)}$	$(X-\alpha)^{(1)}(X^2-\varepsilon^2)^{(1)}$	$(X^3-\mu^3)^{(1)}$
$\left(g_1^{(3)}\right)$	$\theta^k(\alpha^2\beta)$	$\theta^k(\alpha\beta\gamma)$	$\theta^k(\alpha)\theta^k(\varepsilon^{1+q})$	$\theta^k(\mu^{1+q+q^2})$
$\left(g_1^{(2,1)}\right)$	$\theta^k(\alpha^2\beta)$	$2\theta^k(\alpha\beta\gamma)$	0	$-\theta^k(\mu^{1+q+q^2})$
$\left(g_1^{(1,1,1)}\right)$	0	$\theta^k(\alpha\beta\gamma)$	$-\theta^k(\alpha)\theta^k(\varepsilon^{1+q})$	$\theta^k(\mu^{1+q+q^2})$
$\left(g_1^{(2)}\right)\circ\left(g_1'^{(1)}\right)$	$\theta^{k+k'}(\alpha)\theta^k(\beta)$ $+\theta^{2k}(\alpha)\theta^{k'}(\beta)$	$\theta^k(\alpha\beta)\theta^{k'}(\gamma)+\theta^k(\alpha\gamma)\theta^{k'}(\beta)$ $+\theta^k(\beta\gamma)\theta^{k'}(\alpha)$	$\theta^k(\varepsilon^{1+q})\theta^{k'}(\alpha)$	0
$\left(g_1^{(1,1)}\right)\circ\left(g_1'^{(1)}\right)$	$\theta^{k+k'}(\alpha)\theta^k(\beta)$	$\theta^k(\alpha\beta)\theta^{k'}(\gamma)+\theta^k(\alpha\gamma)\theta^{k'}(\beta)$ $+\theta^k(\beta\gamma)\theta^{k'}(\alpha)$	$-\theta^k(\varepsilon^{1+q})\theta^{k'}(\alpha)$	0
$\left(g_1^{(1)}\right)\circ\left(g_1'^{(1)}\right)\circ\left(g_1''^{(1)}\right)$	$\theta^{k+k''}(\alpha)\theta^{k'}(\beta)$ $+\theta^{k'+k''}(\alpha)\theta^k(\beta)$ $+\theta^{k+k'}(\alpha)\theta^{k''}(\beta)$	$\theta^k(\alpha)\theta^{k'}(\beta)\theta^{k''}(\gamma)+\theta^k(\beta)\theta^{k'}(\alpha)\theta^{k''}(\gamma)$ $+\theta^k(\beta)\theta^{k'}(\gamma)\theta^{k''}(\alpha)+\theta^k(\gamma)\theta^{k'}(\beta)\theta^{k''}(\alpha)$ $+\theta^k(\gamma)\theta^{k'}(\alpha)\theta^{k''}(\beta)+\theta^k(\alpha)\theta^{k'}(\gamma)\theta^{k''}(\beta)$	0	0
$\left(g_2^{(1)}\right)\circ\left(g_1^{(1)}\right)$	$-\theta^l(\alpha)\theta^k(\beta)$	0	$-(\theta^l(\varepsilon)+\theta^{lq}(\varepsilon))\theta^k(\beta)$	0
$\left(g_3^{(1)}\right)$	0	0	0	$\theta^m(\mu)+\theta^{mq}(\mu)+$ $\theta^{mq^2}(\mu)$

Table 4: Character table of $\mathrm{GL}_3(\mathbb{F}_q)$

References

[BH06] C. J. Bushnell and G. Henniart. *The local Langlands conjecture for* GL(2). Vol. 335. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, 2006. URL: https://doi.org/10.1007/3-540-31511-X.

[FH04] William Fulton and Joe Harris. *Representation theory*. Vol. 129. Graduate Texts in Mathematics. A first course, Readings in Mathematics. Springer-Verlag, New York, 2004, pp. xvi+551. URL: https://doi.org/10.1007/978-1-4612-0979-9.

[Gre55] J. A Green. "The characters of the finite linear groups". In: *Trans. Amer. Math. Soc.* 80 (1955), pp. 402–447. URL: https://doi.org/10.2307/1992997.

[Ser77] Jean-Pierre Serre. *Linear representations of finite groups*. Vol. Vol. 42. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1977, pp. x+170. URL: https://link.springer.com/book/10.1007/978-1-4684-9458-7.

If you have any question or remark, feel free to come to my office (03.47) or send me an email: clotilde.gauthier@hhu.de