The ring structure of $H^*(G,\mathbb{Z})$, G cyclic

Let $G = \langle t \rangle$ be cyclic. If $|G| = n$, then

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a projective resolution, where $N = 1+t+t^2+\dots t^{n-1}$.

We have $\mathrm{Hom}_G(\mathbb{Z}G,\mathbb{Z}) \cong \mathbb{Z}$, so

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow \cdots$$

$$H^p(G) = \begin{cases} G & \text{if } n = 2,4,6,\dots \\ 0 & \text{if } n \text{ odd} \\ \mathbb{Z} & \text{if } n = 0 . \end{cases}$$

Theorem. Let G be a finite group which acts freely on S^{2k-1} (for example, $k=1$ and G cyclic). For any G-module M (for example $\mathbb{Z}$) there exists an iterated coboundary map

$$d : H^i(G,M) \to H^{i+2k}(G,M)$$

which is an isomorphism for $i>0$ and an epimorphism for $i=0$. □

For our G, $k=1$ and $M=\mathbb{Z}$. Then, for every $i \geq 0$,

$$d : H^i(G) \to H^{i+2}(G)$$

is an epim. (iso. if $i>0$). Since d is an iterated cobounday,

$$d(u \cup v) = d(u) \cup v .$$

In particular, $d(v) = d(1 \cup v) = d(1) \cup v$ for any $v \in H^*(G)$. Write $d(1) = \alpha \in H^2(G)$, so that $d(v) = \alpha \cup v$.

Since d is an epimorphism, α is a generator of $H^2(G)$. Similarly,

$$d(\alpha) = d(1 \cup \alpha) = d(1) \cup \alpha = \alpha \cup \alpha = \alpha^2,$$

so α^2 is a generator of $H^4(G)$.

In general, α^m is a generator for $H^{2m}(G)$, and

$$H^*(G) \cong \mathbb{Z}[\alpha]/(n\alpha) .$$

The integral cohomology of $\mathrm{Sym}(3) = S_3$

We know that $S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by conjugation on $\mathbb{Z}_3$. Hence,

$$H^*(S_3) = H^*(S_3)_{(2)} \oplus H^*(S_3)_{(3)} \underset{\mathbb{Z}_3 \text{ normal}}{=} H^*(S_3)_{(2)} \oplus H^*(\mathbb{Z}_3)^{\mathbb{Z}_2} .$$

$H^n(S_3)_{(2)} \cong \mathbb{Z}_2$ for n even. (we will not prove it)

What is $H^*(\mathbb{Z}_3)^{\mathbb{Z}_2}$?

How does $\mathbb{Z}_2$ act on $H^*(\mathbb{Z}_3)$?

If $\mathbb{Z}_3 = \{0,1,2\}$, then the generator of $\mathbb{Z}_2$ acts via

$$0 \to 0 = 2\cdot 0, \quad 1 \to 2 = 2 \cdot 1, \quad 2 \to 1 = 2\cdot 2,$$

multipication by 2 (write $m(2) : \mathbb{Z}_3 \to \mathbb{Z}_3$)

It can be seen (not easy) that $m(2)^*$ acts also by multiplication by 2 in $H^2(\mathbb{Z}_3)$.

Recall that the cohomology ring is $H^*(\mathbb{Z}_3) = \mathbb{Z}[\alpha]/n\alpha$, where α is a generator of $H^2(G)$ and α^n a generator of $H^{2n}(G)$. Then,

$$\begin{aligned} m(2)^* \cdot \alpha^n &= m(2)^*(\alpha \cup \overset{n}{\cdots} \cup \alpha) \\ &= m(2)^*\alpha \cup \overset{n}{\cdots} \cup m(2)^*\alpha \\ &= 2\alpha \cup \overset{n}{\cdots} \cup 2\alpha = 2^n \alpha^n , \end{aligned}$$

so that $m(2)^*$ is multiplication by 2^n in the $2n^{th}$ level.

Now, $2^n \equiv 1 \pmod 3$ for n even and $2^n \equiv 2 \pmod 3$ for n odd.

Therefore,

$$H^n(\mathbb{Z}_3)^{\mathbb{Z}_2} = \begin{cases} \mathbb{Z}_3 & \text{if } n \equiv 0 \pmod 4 \\ 0 & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

Then,

$$H^n(S_3) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}_6 & \text{if } n \equiv 0 \pmod 4,\ n \neq 0 \\ \mathbb{Z}_2 & \text{if } n \equiv 2 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$