The Homology of a Group II

One-Relator Groups, Functoriality, and the Mayer-Vietoris Sequence for the Homology of Amalgamated Free Products

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May 10, 2021

Some quick reminders: Eilenberg–MacLane spaces

(From Allen Hatcher's *Algebraic Topology*, Section 1.B.)

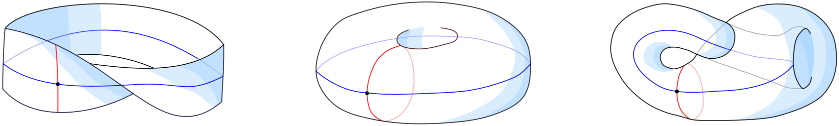
Definition

Let X be a path-connected space whose fundamental group is isomorphic to a given group G and which has a contractible universal covering space. Then X is said to be a $\mathbf{K(G,1)}$**-space**.

- The condition that the universal covering space, be contractible is equivalent to stating that $\pi_i(X) = 0$ for $i > 1$.
- A path connected CW-complex X whose n^{th} homotopy group is isomorphic to G and which is such that $\pi_i(X) = 0$ for $i \neq n$ is called a $\mathbf{K(G,n)}$**-space** (or **complex**) or **Eilenberg–MacLane space**.

Eilenberg–MacLane spaces

So, for instance...



- For the Möbius stripe, M, we have $\pi_1(M) = \mathbb{Z}$, and $\pi_i(M) = 0$ if $i \neq 1$. Therefore, M is a $K(\mathbb{Z}, 1)$-complex.
- For the torus, T, we have $\pi_1(T) = \mathbb{Z}^2$, and $\pi_i(T) = 0$ if $i \neq 1$. Therefore, T is a $K(\mathbb{Z}^2, 1)$-complex.
- For the Klein bottle, K, we have $\pi_1(K) = \langle x, y | x^2 = y^2 \rangle$, and $\pi_i(K) = 0$ if $i \neq 1$. Therefore, K is a $K(\langle x, y | x^2 = y^2 \rangle, 1)$-complex.

The topological definition of the homology of a group

The basic idea behind the notion of the homology of a group can essentially be derived from Whitehead's theorem.

Theorem

If a continuous map $f : X \to Y$ between connected CW-complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence.

- Since homology is homotopy invariant, given a group G, we may construct a space X such that X is a $K(G, 1)$-complex, and then define the homology groups of G to simply be the homology groups of X.
- So, for instance, when we are discussing the homology of $\mathbb{Z}^2$, we are "really" just talking about the homology of the torus.
- This is the **topological definition** of the homology of a group.

Resolutions

- Given a ring R, and a module thereover M, we define the *resolution of M over R* to be an exact sequence (possibly infinite) of R-modules E_i

$$\cdots \to E_n \to E_{n-1} \to \cdots \to E_2 \to E_1 \to E_0 \xrightarrow{\varepsilon} M \to 0.$$

- In the interest of notational economy, this is frequently written in the form $\varepsilon : E \to M$.
- A resolution is *free* if all the modules E_i are free.
- A resolution is *projective* if all the modules E_i are projective.

The algebraic definition of the homology of a group

Brown gives this definition of the homology of a group (p. 35):

Definition

Let G be a group and $\varepsilon : F \to \mathbb{Z}$ a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. We then define the **homology groups of G** by

$$H_i(G) = H_i(F_G).$$

- (We don't need to worry about the choice of free resolution, as they all give rise to the same homology groups. Touched upon by Brown in Sec. II.2, a better (in my view) proof in Hatcher as Lemma 3.1 on p. 194.)
- This is the **algebraic definition** of the homology of a group, and Brown shows that it is in fact equivalent to the topological definition (treated by Dominic Witt last week).

Lemma

(Brown's Lemma II.5.1.) *Let*

$$F_n \to \cdots \to F_0 \to \mathbb{Z} \to 0$$

be an exact sequence of $\mathbb{Z}G$*-modules where each* F_i *is projective. Then* $H_i(G) \cong H_i(F_G)$ *for* $i < n$ *and there is an exact sequence*

$$0 \to H_{n+1}(G) \to (H_n(F))_G \to H_n(F_G) \to H_n(G) \to 0.$$

Proof. Extend F to a full projective resolution F^+:

$$\cdots \to F'_{n+2} \to F'_{n+1} \to F_n \to F_{n-1} \to \cdots \to\to F_1 \to F_0 \to \mathbb{Z} \to 0.$$

It is immediately clear that for $i < n$, $H_i(F_G) \cong H_i(F_G^+) = H_i(G)$.

Furthermore, from algebraic topology, we know that the short exact sequence

$$0 \to F_G \to F_G^+ \to F_G^+/F_G \to 0$$

gives rise to a long exact sequence

$$\begin{aligned} \cdots \to & H_{i+1}(F_G^+/F_G) \to H_i(F_G) \to H_i(F_G^+) \to \\ \to & H_i(F_G^+/F_G) \to H_{i-1}(F_G) \to \dots \end{aligned}$$

Since $F_{n+1} = 0$, we may conclude that $H_{n+1}(F_G) = 0$. Further, since $F_n = F_n^+$, we may conclude that $H_n(F_G^+/F_G) = 0$. By definition, $H_i(F_G^+) = H_i(G)$, so we have an exact sequence

$$0 \to H_{n+1}(G) \to H_{n+1}(F_G^+/F_G) \to H_n(F_G) \to H_n(G) \to 0.$$

Hopf's Theorems

Further prying reveals that $H_{n+1}(F_G^+/F_G)$ may be identified with the cokernel of the arrow $(F'_{n+2})_G \to (F'_{n+1})_G$, which, by Brown's II.2.2, may be identified with $(H_n(F))_G$, finishing the proof.

□

Theorem

(The Fundamental Theorem of Homological Algebra.) (FTHA.)

Let

$$\dots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \dots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$$

and

$$\dots \xrightarrow{\partial'} C'_n \xrightarrow{\partial'} C'_{n-1} \xrightarrow{\partial'} \dots \xrightarrow{\partial'} C'_1 \xrightarrow{\partial'} C'_0 \xrightarrow{\partial'} 0$$

be chain complexes and let r be an integer. Let $(f_i : C_i \to C'_i)_{i \leqslant r}$ be a family of maps such that $\partial'_i \circ f_i = f_{i-1} \circ \partial_i$ for $i \leqslant r$.

If C_i is projective for $i > r$ and $H_i(C') = 0$ for $i \leqslant r$, then $(f_i)_{i \leqslant r}$ extends to a chain map $f : C \to C'$, and f is unique up to homotopy. More precisely, any two extensions are homotopic by a homotopy h_i such that $h_i = 0$ for $i \leqslant r$.

Proof.

Due to time constraints, we omit the proof, though it is far from difficult and may be found in Brown's book as Lem. I.7.4 □

It should be noted that usually, the FTHA is stated merely for exact sequences and/or resolutions, losing some of the generality and hence usefulness that can be obtained from the theorem in its full form.

Hopf's Theorems

Proposition

A coving space projection $p : \tilde{X} \to X$ *induces isomorphisms* $p_* : \pi_n(\tilde{X}) \to \pi_n(X)$ *for all* $n \geqslant 2$.

Proof.

Omitted due to time constraints. (See Spanier's *Algebraic Topology* and Hatcher (where it is Prop. 4.1).) $\square$

Basic homotopy theory (see Hatcher, Prop. 1.39), tells us that if $\tilde{X}$ is the universal cover of X, then the group of *deck transformations of* $\tilde{X}$ is isomorphic to $\pi_1(X)$. This implies:

Lemma

Let $\tilde{X}$ *be the universal cover of* X *for which* $G = \pi_1(X)$. *Then* $C_\bullet(\tilde{X})$ *is a complex of free* $\mathbb{Z}G$*-modules augmented over* $\mathbb{Z}$.

Hopf's Theorems

Theorem

(Brown's Theorem II.5.2.) *For any connected CW complex Y, there is a canonical map $\psi : H_\bullet(Y) \to H_\bullet(\pi)$ where $\pi = \pi_1(Y)$. If $\pi_1(Y) = 0$ for $1 < i < n$ (for some $n \geqslant 2$) then ψ is an isomorphism $H_i(Y) \xrightarrow{\sim} H_i(\pi)$ for $i < n$, and the sequence*

$$\pi_n(Y) \xrightarrow{h} H_n(Y) \xrightarrow{\psi} H_n(\pi) \to 0$$

is exact.

Proof. (Brown.) Let X be the universal cover of Y and let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$. The proof follows from the afore listed results:

- $C_\bullet(X)$ is a complex of free $\mathbb{Z}\pi$-modules over $\mathbb{Z}$.
- Therefore, by the FTHA, we have a chain map $C_\bullet(X) \to F$ well-defined up to homotopy.

Hopf's Theorems

- By the right-exactness of the co-invariant functor (established last week), we obtain a map $C_\bullet(Y) \to F_\pi$, which in turn induces the desired map $\psi : H_n(Y) \to H_n(\pi)$.
- Since $\pi_i(X) \cong \pi_i(Y)$ for $i > 1$, if $\pi_i(Y) = 0$ for $1 < i < n$, then $\pi_i(X) = 0$ for $i < n$. (The case $i = 1$ is covered by the fact that a universal cover of a connect space is simply connected and so has trivial fundamental group.)
- By a result by Hurewicz, this implies that $H_i(X) = 0$ for $i < n$ and the Hurewicz map $h : \pi_n(X) \to H_n(X)$ is an isomorphism.
- Considering how homology arises from chain complexes, we then have a partial free resolution in

$$C_n(X) \to C_{n-1}(X) \to \cdots \to C_1(X) \to C_0(X) \to \mathbb{Z} \to 0.$$

whose n^{th} homology group is the group $Z_n(X)$ of n-cycles of X.

- From hereon, by making use of Lem. 5.1, the fact that $h : \pi_n(X) \to H_n(X)$, and that $\pi_i(X) \cong \pi_i(Y)$ for $i > 1$, we may finally establish that ψ is an isomorphism $H_i(Y) \xrightarrow{\sim} H_i(\pi)$ for $i < n$, and the sequence

$$\pi_n(Y) \xrightarrow{h} H_n(Y) \xrightarrow{\psi} H_n(\pi) \to 0$$

is exact.

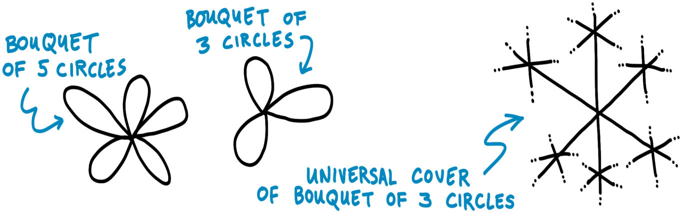
□

Theorem

(Brown's Theorem II.5.3.) *If $G = F/R$ where F is free, then $H_2(G) \cong (R \cap [F,F])/[F,R]$.*

Before we do the proof, we first need to establish a few terms:

- Given a group G with subgroups A, B, by $[A, B]$, we mean the subgroup of G generated by the commutator of A and B.
- By the term a **bouquet of n circles**, we mean a wedge product of n circles, which sort of looking like a flower.

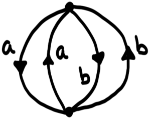


Hopf's Theorems

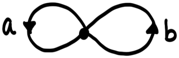
Proof. (Kenneth Brown.)
It is well-known that the fundamental group of a bouquet of n circles is $*_{i=1}^{n}\mathbb{Z}$. Therefore, we may construct the space Y to be a bouquet of as many circles as needed for us to have $\pi_1(Y) = F$.
We may de-
note $F = F(S)$, where S is the set indexing the circles making up Y.

Next, let $\tilde{Y}$ be a covering space of Y such that if $p : \tilde{Y} \to Y$ is the covering projection, then $p_*(\pi_1(\tilde{Y}))$ is isomorphic to the normal subgroup R of $F(S)$.

For instance,



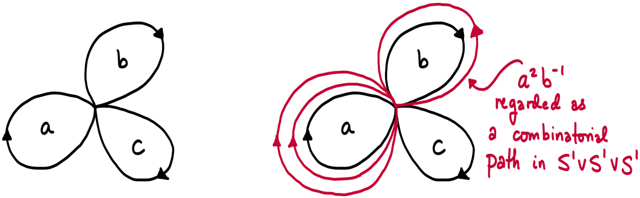
is the regular cover of



corresponding to the normal subgroup $\langle a^2, b^2, ab\rangle \subset \langle a, b\rangle = \mathbb{Z} * \mathbb{Z}$.

Then, picking a basepoint $\tilde{v} \in \tilde{Y}$ lying over the vertex $v \in Y$, we may identify $G = F(S)/R$ with the group of deck transformations of $\tilde{Y}$.

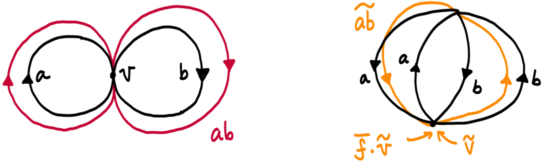
Given an element $f \in F$, we may identify regard f as a **combinatorial path** in F as per the following scheme. By means of example, let $F = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = \langle a, b, c \rangle$, so that Y is a bouquet of three circles. Then:



Hopf's Theorems

By $\tilde{f}$ denoting the lifting of f to $\tilde{Y}$ starting at $\tilde{v}$. The path $\tilde{f}$ then ends at the vertex $\overline{f} \cdot \tilde{v}$ where $\overline{f}$ is the image of f in G.

Letting $F(S) = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$ and $R = \langle a^2, b^2, ab \rangle$,



Following Brown further, the complex $C_\bullet(\tilde{Y})$ is then a complex of free $\mathbb{Z}G$-modules, giving us a partial resolution in

$$C_1(\tilde{Y}) \to C_0(\tilde{Y}) \to \mathbb{Z} \to 0.$$

Lem. II.5.1 then says $H_2(G) \cong \ker\{(H_1(\tilde{Y}))_G \to H_1(Y)\}$.

Brown next notes that $H_1(\tilde{Y}) \cong (\pi_1(\tilde{Y}))_{ab} \cong R_{ab}$ (where *ab* stands for *abelianized*), and makes the claim that $H_1(\tilde{Y})$ and $\cong R_{ab}$ are in fact isomorphic as G-modules.

Specifically, the G-action $G \times R_{ab} \to R_{ab}$ is given as $\overline{f} \cdot r := frf^{-1}$.

With the machinery sketched out above, this is surprisingly easy to prove. The morphism $R_{ab} \to H_1(\tilde{Y})$ is induced by the map $d : R \to H_1(\tilde{Y})$ which merely lifts $r \in R$ to its associated path $\overline{r}$ in $\tilde{Y}$. Since these are always closed and brings you back to where you started, we have

$$\begin{aligned} d(\overline{f} \cdot r) &= d(frf^{-1}) \\ &= \overline{f}\,\overline{fr}\,\overline{f^{-1}} \end{aligned}$$

Hopf's Theorems

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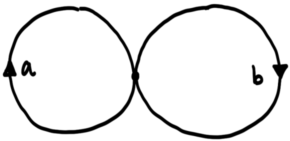
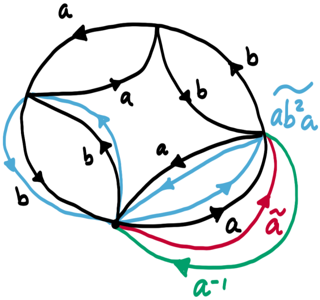
$$
\begin{aligned}
d(\overline{f} \cdot r) &= d(frf^{-1}) \\
&= \overline{f}\,\overline{fr}\,\overline{f^{-1}}
\end{aligned}
$$

The matter in question is probably best illustrated by an illustration. Let Y be a bouquet of 2 circles, and let $\tilde{Y}$ be the cover associated with $R = \langle a^2, b^2, (ab)^2, (ba)^2, ab^2a \rangle$, and consider the action of $a \in \langle a, b \rangle = \mathbb{Z} * \mathbb{Z}$ on $ab^2a \in R$.





Hopf's Theorems

The G-action functioning as it should, we finally obtain a commutative diagram

$$\begin{array}{ccccc} (H_1(\tilde{Y}))_G & \cong & (R_{ab})_G & = & R/[F,R] \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ H_1(Y) & \cong & F_{ab} & = & F/[F,F] \end{array}$$

From which he is finally able to draw the desired conclusion, namely, that

$$H_2(G) \cong \ker R/[F,R] \to F/[F,F] = (R \cap [F,F])/[F,R].$$

$\square$

The abelianization of R in the above proof when considered as a G-module is called the *relation module*, and within the topological framework just discussed, you can draw some interesting conclusions from the form that R takes.
Let $G = \langle S; r_1, r_2, r_3, \ldots \rangle = F(S)/R$, where R is the normal closure in $F(S)$ of elements $r_1, r_2, r_3, \ldots$

Exercise

(Ex. 5.2(a).) *The relation module R_{ab} is generated by the images of $r_1, r_2, r_3, \ldots$*

Next, define the 2-complex associated to the presentation of G as the two dimensional CW-complex

$$\left(\bigvee_{s \in S} S^1_s\right) \bigcup_{r_1} e^2_1 \bigcup_{r_2} e^2_2 \bigcup_{r_3} e^2_3 \cup \ldots$$

...in the above, the 2-cell e_i^2 is attached to $\bigvee_{s\in S} S_s^1$ by the map $S^1 \to \bigvee_{s\in S} S_s^1$ corresponding to the path r_1 assumes in Y.

Exercise

(Ex. 5.2(b).) *The elements $r_1, r_2, r_3, \ldots$ generate R_{ab} freely if and only if the 2-complex associated to the given presentation of G is a $K(G,1)$-complex.*

Let $G = \langle S; r\rangle$, where r is an arbitrarily chosen element (making G a one-relator group). We can then write $r = u^n$ for some u where $n \leqslant 1$ is maximal.

Lyndon and Schupp showed that the image t of u has then order exactly n, and that if by C we denote the cyclical group of order n generated by t, then the surjection $\mathbb{Z}[G/C] \to R_{ab}$ is an isomorphism.

Exercise 5.2(c) challenges the reader to
come up with a rather elaborate geometric interpretation of this fact.

Full expositions of the answers to this question can be found in the official solutions manual to Kenneth Brown's book, compiled by **Christopher A. Gerig**.

Functoriality

This section is basically only here to say that just as in the "ordinary case", homology is a (covariant) functor from the category of topological spaces to the category of abelian abelian groups, so in our case, homology is a (covariant) functor from the category of groups to the category of abelian groups.

This follows from the FTHA.

The chain complex $C_\bullet(G)$ is, in the words of Kenneth Brown, "clearly" functorial in G. Personally, I'd like to add a note that $C_\bullet(G)$ is of course just the chain complex F_G where F is the standard resolution of G.

Consequently, given a group morphism $\alpha : G \to G'$, we can

Functoriality

By means of the FTHA, it can then be shown that there exists an augmentation preserving G-chain map $\tau : F \to F'$ (well-defined up to homotopy), which induces a map $F_G \to F'_G$ (also well-defined up to homotopy), which finally, then in turn, induces a well-defined map

$$\alpha_* : H_*(G) \to H_*(G).$$

The Mayer-Vietoris Sequence

The Mayer-Vietoris sequence is a tool in algebraic topology that has long spooked me by virtue of how counter-intuitive it can be.

Algebraically, little stands out about Mayer-Vietoris. Consider what Hatcher has to say (p. 149):

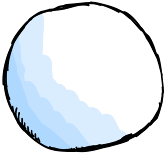
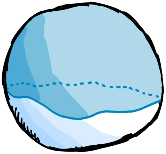
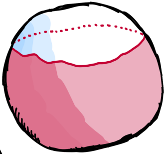
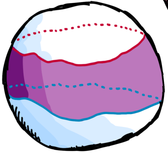
Definition

Let X be a space with pairs of subspaces $A, B \subset X$ such that X is the union of the interiors of A and B. Then there exists an exact sequence of the form

$$\ldots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to \\ \to H_{n-1}(A \cap B) \to H_{n-1}(A) \oplus H_{n-1}(B) \to \ldots$$

Nothing too out of the ordinary, is it? But consider then the geometric implications!

The Mayer-Vietoris Sequence

Consider the sphere, S^2, , and consider the following two subspaces, $N =$ , "the north", and $S =$ . It can plainly then be seen that their intersection, , is homeomorphic to the cylinder, C.

Mayer-Vietoris then gives us

$$\cdots \longrightarrow \underset{\overset{\shortparallel}{0}}{H_2(N)} \oplus \underset{\overset{\shortparallel}{0}}{H_2(S)} \to H_2(S^2) \longrightarrow \underset{\overset{\shortparallel}{\mathbb{Z}}}{H_1(C)} \longrightarrow \underset{\overset{\shortparallel}{0}}{H_1(N)} \oplus \underset{\overset{\shortparallel}{0}}{H_1(S)} \longrightarrow \cdots$$

So $H_2(S^2) \cong H_1(C) = \mathbb{Z}$, and we can deduce properties about surfaces on a sphere from the properties about curves on a cylinder!

But I digress...

It is actually not too difficult to translate the notion of the Mayer-Vietoris sequence of a space into a notion of a **Mayer-Vietoris sequence of a group**, or rather, a collection of groups.

We can conceive of a notion of "intersections of groups" and "unions of groups" by means of the notion of the **pushout square**, or as Brown calls it, the **amalgamated sum**.

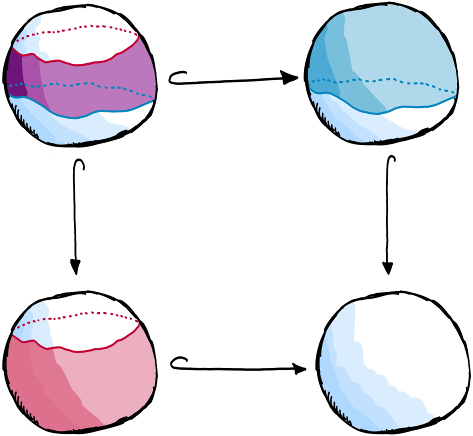
Definition

(Brown.) Let A, G_1, G_2 be groups and let α_1 and α_2 be group morphisms $A \to G_1$ and $A \to G_2$. The **amalgamated sum** of G_1 and G_2 is then a group G fitting into a commutative square

$$\begin{tikzcd} A \arrow[r, "\alpha_2"] \arrow[d, "\alpha_1"'] & G_2 \arrow[d, "\beta_2"] \\ G_1 \arrow[r, "\beta_1"] & G \end{tikzcd}$$

with the following *universal mapping property*: Given a group H and group morphisms $\gamma_i : G_1 \to H$ $(i = 1, 2)$ with $\gamma_1\alpha_1 = \gamma_2\alpha_2$, there is a unique map $\varphi : G \to H$ such that $\varphi \circ \beta_i = \gamma_i$. We then write $G = G_1 *_A G_2$.

In the context of spaces, pushout squares, or amalgamated sums are defined similarly, and for our purposes, it suffices with a quick illustration to get the gist:



With reference to the discussion at the outset about the topological interpretation of the homology of a group, everything turns out to follow by means of this wonderful theorem of Whitehead's:

Theorem

(WHITEHEAD, AS QUOTED BY BROWN.) *Any amalgamation diagram [as on the slide before last] with α_1 and α_2 injective can be realized by a diagram*

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, hook] & X_2 \arrow[d, hook] \\ X_1 \arrow[r, hook] & X \end{tikzcd}$$

of $K(\pi, 1)$-complexes such that $X = X_1 \cup X_2$ and $Y = X_1 \cap X_2$.

The proof is not particularly difficult, and is broken up into a succession of neat little lemmas by Brown. Most neat of all is this corollary that follows from Whitehead's theorem (marked as II.7.7 in Brown's book):

Corollary

Given $G = G_1 *_A G_2$ where $A \hookrightarrow G_1$ and $A \hookrightarrow G_2$, there is a Mayer Vietoris sequence

$$\cdots \to H_n(A) \to H_n(G_1) \oplus H_n(G_2) \to H_n(G) \to H_{n-1}(A) \to \dots$$

...and with that, we are done!