Homology and Cohomology with Coefficients I

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Homology of a group

Let G be a group and $\varepsilon : F \to \mathbb{Z}$ a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. We define the homology groups of G by

$$H_*(G) = H_*(F_G).$$

Let R be a ring and M a left R-module. A resolution of M is an exact sequence of R-modules

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

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Let G be a group, $R = \mathbb{Z}G$, and consider $\mathbb{Z}$ as a G-module (with trivial G-action). A projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ is an exact sequence of G-modules

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0,$$

where each F_i is a projective module.

Homology of a group

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$$H_*(G) = H_*(F_G).$$

Let G be a group and M be a G-module. The group of co-invariants of M is

$$M_G = {}^{M}\!/\!_{< gm - m \mid g \in G,\, m \in M >}.$$

The name "co-invariants" comes from the fact that M_G is the largest *quotient* of M on which G acts trivially.

Homology with coefficients

Let G be a group and $\varepsilon : F \to \mathbb{Z}$ a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.
We define the homology groups of G by

$$H_*(G) = H_*(F_G).$$

Let M be a G-module.
Define the homology of G with coefficients in M by

$$H_*(G, M) = H_*(F \otimes_G M).$$

Here $F \otimes_G M$ can be thought of as the complex obtained from F by applying the functor $- \otimes_G M$.

$$\text{If } M = \mathbb{Z} \quad \text{then} \quad F \otimes_G \mathbb{Z} \approx F_G.$$

Homology with coefficients

$$\begin{array}{rcl} G & \rightsquigarrow & \text{group} \\ \varepsilon : F \to \mathbb{Z} & \rightsquigarrow & \text{projective resolution of } \mathbb{Z} \text{ over } \mathbb{Z}G \\ M & \rightsquigarrow & G\text{-module} \end{array}$$

Define the homology of G with coefficients in M by

$$H_*(G, M) = H_*(F \otimes_G M).$$

The complex $F \otimes_G M$ can also be thought of as the tensor product of chain complexes (M is regarded as a chain complex concentrated in dimension 0).

If (C, d) (resp. (C', d')) is a chain complex of right (resp. left) R-modules, then we define their tensor product $C \otimes_R C'$ by

$$(C \otimes_R C')_n = \bigoplus_{p+q=n} C_p \otimes_R C'_q \quad \text{and}$$

$$D(c \otimes c') = dc \otimes c' + (-1)^{\deg c} c \otimes d'c'$$

Homology with coefficients

$$\begin{aligned} G &\quad\rightsquigarrow\quad \text{group} \\ \varepsilon : F \to \mathbb{Z} &\quad\rightsquigarrow\quad \text{projective resolution of } \mathbb{Z} \text{ over } \mathbb{Z}G \\ M &\quad\rightsquigarrow\quad G\text{-module} \end{aligned}$$

Define the homology of G with coefficients in M by

$$H_*(G, M) = H_*(F \otimes_G M).$$

$$\eta : P \to M \quad\rightsquigarrow\quad \text{projective resolution of } M \text{ over } \mathbb{Z}G$$

Define the homology of G with coefficients in M by

$$H_*(G, M) = H_*(F \otimes_G P).$$

These definitions are consistent because η induces a weak equivalence $F \otimes \eta : F \otimes_G P \to F \otimes_G M$.

Homology with coefficients

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These definitions are consistent because η induces a weak equivalence $F \otimes \eta : F \otimes_G P \to F \otimes_G M$.

A chain map $f : C \to C'$ is called a weak equivalence if $H(f) : H(C) \to H(C')$ is an isomorphism.

Easy computation

We apply $- \otimes_G M$ to

$$F_1 \to F_0 \to \mathbb{Z} \to 0$$

and we get

$$F_1 \otimes_G M \to F_0 \otimes_G M \to \mathbb{Z} \otimes_G M \to 0.$$

Then we have

$$H_0(G, M) = \mathbb{Z} \otimes_G M \approx M_G.$$

Tensor product $\otimes_R$

Let R be a ring.

$$\underset{\text{right } R\text{-module}}{M} \otimes_R \underset{\text{left } R\text{-module}}{N}$$

Notation: $M \otimes_{\mathbb{Z}} N = M \otimes N$.

$$M \otimes_R N = {}^{M \otimes N}\big/ < mr \otimes n = m \otimes rn \mid m \in M,\ r \in R,\ n \in N >.$$

If $R = \mathbb{Z}G$, we can avoid having to consider both left and right modules by using the anti-automorphism $g \to g^{-1}$ of G.

$$\text{right } G\text{-module } M \leftrightsquigarrow \text{left } G\text{-module } M$$

$$mg = g^{-1}m$$

Tensor product $\otimes_G$

$$\begin{array}{ccc} & M \otimes_{\mathbb{Z}G} N & \\ \nearrow & & \nwarrow \\ \text{left } G\text{-module} & & \text{left } G\text{-module} \end{array}$$

Notation: $M \otimes_{\mathbb{Z}G} N = M \otimes_G N$.

$$M \otimes_G N = {}^{M \otimes N}\big/ < g^{-1}m \otimes n = m \otimes gn \mid m \in M,\, g \in G,\, n \in N >.$$

If we replace m by gm, these relations take the form $m \otimes n = gm \otimes gn$, and then

$$M \otimes_G N = (M \otimes N)_G,$$

where G acts "diagonally" on $M \otimes N : g \cdot (m \otimes n) = gm \otimes gn$.

Tensor product $\otimes_G$

In particular, $- \otimes_G -$ is commutative :

$$M \otimes_G N \approx N \otimes_G M.$$

Warning: The passage between left and right modules is convenient, but it can sometimes be confusing, for instance if M naturally admits both a left and a right G-action. In such cases we will revert to the standard notation $M \otimes_{\mathbb{Z}G} N$ if we want to indicate that the tensor product is to be formed with respect to the given right action of G on M.

Cohomology with coefficients

Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and let M be a G-module. Define the cohomology of G with coefficients in M by

$$H^*(G, M) = H^*(\mathcal{Hom}_G(F, M)).$$

Given (C, d) and (C', d') two chain complexes, the "function complex" $\mathcal{Hom}_R(C, C')$ is defined as follows:

- $\mathcal{Hom}_R(C, C')_n = \prod_{q \in \mathbb{Z}} \mathrm{Hom}_R(C_q, C'_{q+n})$ is the set of graded module homomorphisms of degree n;
- $D_n : \mathcal{Hom}_R(C, C')_n \to \mathcal{Hom}_R(C, C')_{n-1}$ is defined by $D_n(f) = d'f + (-1)^{n+1} fd$.

Cohomology with coefficients

Given (C, d) and (C', d') two chain complexes, the "function complex" $\mathcal{H}om_R(C, C')$ is defined as follows:

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In our case:

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

$$\cdots \to 0 \to 0 \to M \to 0 \to 0.$$

So we follow the standard convention in regarding $\mathcal{H}om_G(F, M)$ as a *co*chain complex, with

$$\mathcal{H}om_G(F, M)^n = \mathcal{H}om_G(F, M)_{-n} = \mathrm{Hom}_G(F_n, M).$$

Cohomology with coefficients

$$\cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$$\cdots \to\ 0 \to\ 0 \to\ M \to\ 0 \to\ 0.$$

So we follow the standard convention in regarding $\mathcal{Hom}_G(F, M)$ as a *co*chain complex, with

$$\mathcal{Hom}_G(F, M)^n = \mathcal{Hom}_G(F, M)_{-n} = \mathrm{Hom}_G(F_n, M).$$

So we have

$$\cdots \leftarrow \mathrm{Hom}_G(F_{n+1}, M) \xleftarrow{\delta_n} \mathrm{Hom}_G(F_n, M) \xleftarrow{\delta_{n-1}} \mathrm{Hom}_G(F_{n-1}, M) \leftarrow \cdots$$

where δ is the coboundary operator and for $u \in \mathcal{Hom}_G(F, M)^n$ and $x \in F_{n+1}$, we have

$$(\delta_n u)(x) = (-1)^{n+1} u(\partial_{n+1} x).$$

Easy computation

The sequence

$$F_1 \to F_0 \to \mathbb{Z} \to 0,$$

yields an exact sequence

$$0 \to \mathrm{Hom}_G(\mathbb{Z}, M) \to \mathrm{Hom}_G(F_0, M) \to \mathrm{Hom}_G(F_1, M).$$

Then we have

$$H^0(G, M) = \mathrm{Hom}_G(\mathbb{Z}, M) = M^G,$$

where M^G is the group of invariants, i.e. is the largest submodule of M on which G acts trivially.

Tor

There is an obvious generalization of $H_*(G,-)$ called $\mathrm{Tor}_*^G(-,-)$, obtained by removing the restriction that F be a resolution of the particular module $\mathbb{Z}$.

$$\begin{array}{rcl} G & \rightsquigarrow & \text{group} \\ M \text{ and } N & \rightsquigarrow & G\text{-modules} \\ F \to M & \rightsquigarrow & \text{projective resolution of } M \text{ over } \mathbb{Z}G \\ P \to N & \rightsquigarrow & \text{projective resolution of } N \text{ over } \mathbb{Z}G \end{array}$$

$$\mathrm{Tor}_*^G(M,N) = H_*(F \otimes_G N) = H_*(F \otimes_G P) = H_*(M \otimes_G P).$$

We recover $H_*(G,-)$ as $\mathrm{Tor}_*^G(\mathbb{Z},-)$.

Ext

Similarly, there is an obvious generalization of $H^*(G,-)$ called $\mathrm{Ext}_*^G(-,-)$, obtained by removing the restriction that F be a resolution of the particular module $\mathbb{Z}$.

$$\begin{array}{rcl} G & \rightsquigarrow & \text{group} \\ M \text{ and } N & \rightsquigarrow & G\text{-modules} \\ F \to M & \rightsquigarrow & \text{projective resolution of } M \text{ over } \mathbb{Z}G \\ P \to N & \rightsquigarrow & \text{projective resolution of } N \text{ over } \mathbb{Z}G \end{array}$$

$$\mathrm{Ext}_*^G(M,N) = H^*(\mathcal{H}om_G(F,N)).$$

We recover $H^*(G,-)$ as $\mathrm{Ext}_*^G(\mathbb{Z},-)$.

Two results about Tor and Ext

Proposition 1

Let $\varepsilon : F \to M$ and $\eta : P \to N$ be resolutions, not necessarily projective. If either F or P is a complex of flat modules, then

$$\mathrm{Tor}_*^G(M, N) = H_*(F \otimes_G P).$$

Recall that an R-module F is flat if the functor $- \otimes_R F$ is exact.

Two results about Tor and Ext

Proposition 2

Let M and N be G-modules. If M is $\mathbb{Z}$-torsion-free then

$$\mathrm{Tor}_*^G(M, N) \approx H_*(G, M \otimes N),$$

where G acts diagonally on $M \otimes N$. If M is $\mathbb{Z}$-free, then

$$\mathrm{Ext}_*^G(M, N) \approx H^*(G, \mathrm{Hom}(M, N)),$$

where G acts diagonally on $\mathrm{Hom}(M, N)$.

Restriction of scalars

Let $\alpha : R \to S$ be a ring homomorphism and M be a S-module. Let $\cdot_S$ be an action of S on M. Then we have an action of R on M defined as follows

$$r \cdot_R m := \alpha(r) \cdot_S m,$$

where $m \in M$, $s \in S$ and $r \in R$.

In this way we obtain a functor called restriction of scalars

$$S\text{-Mod} \longrightarrow R\text{-Mod}.$$

Now we want to study two constructions which go in the opposite direction, from R-modules to S- modules.

Extension of scalars

Let $\alpha : R \to S$ be a ring homomorphism.

We regard S as a right R-module by $s \cdot r := s\alpha(r)$.
For any (left) R-module M consider the tensor product

$$S \otimes_R M.$$

Since the natural left action of S on itself commutes with the right action of R on S, we can make $S \otimes_R M$ a (left) S-module by setting

$$s \cdot (s' \otimes m) := ss' \otimes m.$$

This S-module is said to be obtained form M by extension of scalars from R to S.

Universal mapping property I

There is a natural map $i : M \to S \otimes_R M$ given by $i(m) = 1 \otimes m$.Since $1 \otimes rm = \alpha(r) \otimes m = \alpha(r) \cdot (1 \otimes m)$ for $r \in R$, we have

$$i(rm) = \alpha(r) i(m).$$

The map i is an R-module map ($S \otimes_R M$ is regarded as an R-module by restriction of scalars).

Given an S-module N and an R-module map $f : M \to N$, there is a unique S-module map $g : S \otimes_R M \to N$ such that $gi = f$

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "f"'] & S \otimes_R M \arrow[dl, dashed, "g"] \\ N & \end{tikzcd}$$

Extension of scalars

Thus we have

$$\mathrm{Hom}_S(S \otimes_R M, N) \approx \mathrm{Hom}_R(M, N),$$

showing that the extension of scalars functor

$$(R\text{-modules}) \to (S\text{-modules})$$

is left adjoint to the restriction of scalars functor

$$(S\text{-modules}) \to (R\text{-modules}).$$

Co-Extension of scalars

Let $\alpha : R \to S$ be a ring homomorphism.

We regard S as a left R-module by $r \cdot s := \alpha(r)s$.
For any (left) R-module M consider the abelian group

$$\mathrm{Hom}_R(S, M).$$

Since the natural right action of S on itself commutes with the left action of R on S, we can make $\mathrm{Hom}_R(S, M)$ a (left) S-module by setting

$$(sf)(s') := f(s's).$$

This S-module is said to be obtained form M by co-extension of scalars from R to S.

Universal mapping property II

There is a natural map $\pi : \mathrm{Hom}_R(S, M) \to M$ given by $\pi(f) = f(1)$. Notice that for for $r \in R$ we have

$$\pi(\alpha(r)f) = (\alpha(r)f)(1) = f(\alpha(r)) = rf(1) = r\pi(f).$$

The map π is an R-module map ($\mathrm{Hom}_R(S, M)$ is regarded as an R-module by restriction of scalars).

Given an S-module N and an R-module map $f : N \to M$, there is a unique S-module map $g : N \to \mathrm{Hom}_R(S, M)$ such that $\pi g = f$

$$\begin{tikzcd}
 & \mathrm{Hom}_R(S, M) \arrow[d, "\pi"] \\
N \arrow[ur, dashed, "g"] \arrow[r, "f"'] & M
\end{tikzcd}$$

Co-Extension of scalars

Thus we have

$$\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) \approx \mathrm{Hom}_R(N, M),$$

showing that the extension of scalars functor

$$(R\text{-modules}) \to (S\text{-modules})$$

is right adjoint to the restriction of scalars functor

$$(S\text{-modules}) \to (R\text{-modules}).$$

Induction and Co-Induction

We apply the previous constructions to ring homomorphisms of the form

$$\mathbb{Z}H \hookrightarrow \mathbb{Z}G,$$

where $H \subset G$.

In this case extension of scalars (resp. co-extension of scalars)is called induction (resp. co-induction) form H to G.
Consider an H-module M, we write

$$\mathbb{Z}G \otimes_{\mathbb{Z}H} M = \mathrm{Ind}_H^G M$$

$$\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) = \mathrm{Coind}_H^G M.$$

Some results

Proposition A1

The G-module $\operatorname{Ind}_H^G M$ contains M as an H-submodule and

$$\operatorname{Ind}_H^G M = \bigoplus_{g \in G/H} gM.$$

Note that the summand gM of $\operatorname{Ind}_H^G M$ is closed under the action of gHg^{-1}, hence gM is a gHg^{-1}-module.

Some results

Proposition A2

Let M be an H-module, then

$$\mathrm{Coind}_H^G M = \prod_{g \in G/H} gM.$$

Proposition B

- Let N be a G-module. Then

$$\operatorname{Ind}_H^G \operatorname{Res}_H^G N \approx \mathbb{Z}[G/H] \otimes N,$$

where G acts diagonally on the tensor product.

- Let H and K be subgroups of G and let E be a set of representatives for the double cosets KgH. For any H-module M, there is a K-isomorphism

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G M \approx \bigoplus_{g \in E} \operatorname{Ind}_{K \cap gHg^{-1}}^K \operatorname{Res}_{K \cap gHg^{-1}}^{gHg^{-1}} gM.$$

In partcular, if $H \triangleleft G$, then there is an H-isomorphism.

$$\operatorname{Res}_H^G \operatorname{Ind}_H^G M \approx \bigoplus_{g \in G/H} gM.$$

Some results

Proposition C1

Let N be a G-module whose underlying abelian group is the direct sum $\bigoplus_{i \in I} M_i$. Assume that there is a transitive right action of G on I such that $gM_i = M_{gi}$ for all $i \in I$ and $g \in G$. Let M be the one of the M_i and let $H \subset G$ be the isotropy group of i. Then M is an H-module and

$$N \approx \operatorname{Ind}_H^G M.$$

Some results

Proposition C2

Let N be a G-module which, as an abelian group, admits a direct product decomposition $(\pi_i : N \to M_i)_{i \in I}$. Assume that there is a transitive right action of G on I such that $\pi_i g \; \pi_{ig}$ for all $i \in I$ and $g \in G$. Let $\pi : N \to M$ be the one of the π_i and let $H \subset G$ be the isotropy group of i. Then M inherits an H-module structure from N and

$$N \approx \mathrm{Coind}_H^G M.$$

Final result

Proposition E

If $(G : H) < \infty$, then $\operatorname{Ind}_H^G M \approx \operatorname{Coind}_H^G M$.