Cohomology of groups: products

Iker de las Heras

Oberseminar

May 2021

Homology and cohomology of a group G

Let G be group, M a G-module and $\varepsilon : F \to \mathbb{Z}$ a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. The homology of G with coefficients in M is defined as

$$H_*(G, M) = H_*(F \otimes_G M).$$

If $M = \mathbb{Z}$, then we write $H_*(G, \mathbb{Z}) = H_*(G)$.

The cohomology of G with coefficients in M is defined as

$$H^*(G, M) = H^*(\mathscr{H}\!om_G(F, M)).$$

If $M = \mathbb{Z}$, then we write $H^*(G, \mathbb{Z}) = H^*(G)$.

The tensor product of a resolution

Let G (resp. G') be a group, and let M (resp. M') be a G-module (resp. G'-module).

Then $M \otimes M'$ is a $G \times G'$-module:

$$(g, g') \cdot (m \otimes m') = (gm \otimes g'm').$$

If M (resp. M') is projective over $\mathbb{Z}G$ (resp. $\mathbb{Z}G'$), then $M \otimes M'$ is projective over $\mathbb{Z}[G \times G']$. Indeed, if N (resp. N') is such that $M \oplus N$ (resp. $M' \oplus N'$) is a free $\mathbb{Z}G$-module (resp. $\mathbb{Z}G'$-module), then

$$M \otimes M' \oplus (M \otimes N' \oplus N \otimes M' \oplus N \oplus N') = (M \oplus N) \otimes (M' \oplus N'),$$

which is a free $\mathbb{Z}[G \times G']$-module.

The tensor product of a resolution

Let $\varepsilon : F \to \mathbb{Z}$ (resp. $\varepsilon' : F' \to \mathbb{Z}$) be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ (resp. $\mathbb{Z}G'$).

Recall that $F \otimes F'$ is the complex defined as

$$(F \otimes F')_n = \bigoplus_{p+q=n} F_p \otimes F'_q,$$

with differential D given by

$$D(f \otimes f') = d(f) \otimes f' + (-1)^{\deg f} f \otimes d'(f')$$

for $f \in F$ and $f' \in F'$.

The complex $F \otimes F'$ is a complex of projective $\mathbb{Z}[G \times G']$-modules.

The tensor product of a resolution

Proposition

If $\varepsilon : F \to \mathbb{Z}$ and $\varepsilon' : F' \to \mathbb{Z}$ are projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$, respectively, then

$$\varepsilon \otimes \varepsilon' : F \otimes F' \to \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$$

is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G \times G']$.

Proof. It remains to show that $\varepsilon \otimes \varepsilon'$ is a weak equivalence.

$\varepsilon : F \to \mathbb{Z}$ and $\varepsilon' : F' \to \mathbb{Z}$ are homotopy equivalences if we ignore the action of G (free resolutions of $\mathbb{Z}$ over $\mathbb{Z}$ are always homotopy equivalences).
Then the same is true for $\varepsilon \otimes \varepsilon'$, so in particular it is a weak equivalence over $\mathbb{Z}[G \times G']$. $\square$

The tensor product of a resolution

Corollary

If $\varepsilon : F \to \mathbb{Z}$ and $\varepsilon' : F' \to \mathbb{Z}$ are projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$, then so is

$$\varepsilon \otimes \varepsilon' : F \otimes F' \to \mathbb{Z},$$

where G acts diagonally on $F \otimes F'$.

Proof. This follows from the previous proposition and by restriction of scalars with respect to the diagonal embedding

$$\begin{aligned} d : G &\to G \times G \\ g &\to (g, g). \end{aligned}$$

□

Homology cross product

Let G, G', M, M', F, F' be as before.

There is an isomorphism

$$(F \otimes_G M) \otimes (F' \otimes_{G'} M') \to (F \otimes F') \otimes_{G \times G'} (M \otimes M'),$$

given by

$$(x \otimes m) \otimes (x' \otimes m') \mapsto (x \otimes x') \otimes (m \otimes m').$$

(This follows from the fact that both are surjective images of

$$F \otimes M \otimes F' \otimes M',$$

and the kernel of both surjections coincide.)

Homology cross product

If $z \in F \otimes_G M$ and $z' \in F' \otimes_{G'} M'$, we write $z \times z'$ for the image of $z \otimes z'$, and note that

$$\partial(z \times z') = \partial(z) \times z' + (-1)^{\deg z} z \times \partial(z').$$

Hence:

- The image of two cycles is a cycle.
- Its homology class only depends on the classes of the given cycles.

We have an induce product

$$\times : H_p(G, M) \otimes H_q(G', M') \to H_{p+q}(G \times G', M \otimes M')$$

called the *homology cross-product* and still denoted $z \times z'$, where in this case $z \in H_p(G, M)$ and $z' \in H_q(G', M')$.

Cohomology cross product

Similarly, we have a map

$$\mathscr{Hom}_G(F, M) \otimes \mathscr{Hom}_G(F', M') \to \mathscr{Hom}_{G\times G'}(F \otimes F', M \otimes M'),$$

where for $u \in \mathscr{Hom}_G(F, M)$ and $u' \in \mathscr{Hom}_{G'}(F', M')$, we define $u \times u'$ by

$$(u \times u')(x \otimes x') = (-1)^{\deg u' \deg x} u(x) \otimes u'(x').$$

As before, this map induces a product

$$\times : H^p(G, M) \otimes H^q(G', M') \to H^{p+q}(G \times G', M \otimes M')$$

called the *cohomology cross product*.

Cup product

Recall:

Let $\alpha : G \to G'$ be a group homomorphism.
Let $f : M' \to M$ such that $f(\alpha(g)m') = gf(m')$ for every $g \in G, m \in M$.
Let $\tau : F \to F'$ such that $\tau(gx) = \alpha(g)\tau(x)$ for every $x \in F$.

Then, there is a map

$$\mathscr{Hom}(\tau, f) : \mathscr{Hom}_{G'}(F', M') \to \mathscr{Hom}_G(F, M),$$

which induces

$$(\alpha, f)^* : H^*(G', M') \to H^*(G, M),$$

so that H^* is a contravariant functor. If $M = M'$ and $f = \mathrm{Id}_M$, then we simply write α^*.

Cup product

Let M and N be two G-modules.

If $u \in H^p(G, M)$ and $v \in H^q(G, N)$, then

$$u \times v \in H^{p+q}(G \times G, M \otimes N).$$

We define the *cup product* of u and v as

$$u \cup v = d^*(u \times v) \in H^{p+q}(G, M \otimes N),$$

where d is the diagonal map $d : G \to G \otimes G$

Properties of the cup product

Dimension 0:

The cup product in dimension 0

$$H^0(G, M) \otimes H^0(G, N) \to H^0(G, M \otimes N)$$

is the map

$$M^G \otimes N^G \to (M \otimes N)^G$$

induced from the inclusions

$$M^G \hookrightarrow M \quad \text{and} \quad N^G \hookrightarrow N.$$

Properties of the cup product

Naturality with respect to coefficient homomorphism:

Let $f : M \to M'$ and $g : N \to N'$ be two G-module maps and $u \in H^*(G, M)$ and $v \in H^*(G, N)$. Then,

$$(f \otimes g)_*(u \cup v) = f_*(u) \cup g_*(v) \in H^*(G, M' \otimes N'),$$

where $f_* \in H^*(G, f)$, $g_* \in H^*(G, h)$ and $(f \otimes g)_* \in H^*(G, f \otimes g)$.

Properties of the cup product

Compatibility with δ:

Let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of G-modules, and let N be a G-module such that the sequence

$$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$$

is exact. Then, $\delta(u \cup v) = \delta(u) \cup v$ for any $u \in H^p(G, M'')$ and $v \in H^q(G, N)$.

Properties of the cup product

In other words, the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^p(G, M'') & \xrightarrow{\delta} & H^{p+1}(G, M') & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle -\cup v} & & \downarrow{\scriptstyle -\cup v} & & \\ \cdots & \longrightarrow & H^{p+q}(G, M'' \otimes N) & \xrightarrow{\delta} & H^{p+q+1}(G, M' \otimes N) & \longrightarrow & \cdots. \end{array}$$

Proof. Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(G, M') & \longrightarrow & C^*(G, M) & \longrightarrow & C^*(G, M'') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle -\cup v} & & \downarrow{\scriptstyle -\cup v} & & \downarrow{\scriptstyle -\cup v} & & \\ 0 & \longrightarrow & C^*(G, M' \otimes N) & \longrightarrow & C^*(G, M \otimes H) & \longrightarrow & C^*(G, M'' \otimes N) & \longrightarrow & 0, \end{array}$$

where v and $\cup$ are taken in the (boundary of) the cochain level.

Properties of the cup product

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(G,M') & \longrightarrow & C^*(G,M) & \longrightarrow & C^*(G,M'') & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle -\cup v} & & \Big\downarrow{\scriptstyle -\cup v} & & \Big\downarrow{\scriptstyle -\cup v} & & \\ 0 & \longrightarrow & C^*(G,M'\otimes N) & \longrightarrow & C^*(G,M\otimes N) & \longrightarrow & C^*(G,M''\otimes N) & \longrightarrow & 0, \end{array}$$

where v and $\cup$ are taken in the cochain level.

If δ is the coboundary operator in $C^*(G,-)$, we have

$$\delta(a\cup b) = \delta a \cup b + (-1)^{\deg a} a \cup \delta b.$$

If b is a cocycle, it reduces to $\delta a \cup b$, so the diagram is commutative.

The result follows from the naturality of connecting homomorphisms with respect to maps of short exact sequences. □

Similarly, let

$$0 \to N' \to N \to N'' \to 0$$

be a short exact sequence of G-modules such that the sequence

$$0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$$

is exact. Then,

$$\delta(u \cup v) = (-1)^p u \cup \delta v,$$

for any $u \in H^p(G, M)$ and $v \in H^q(G, N'')$.

Properties of the cup product

Existence of identity:

For the element $1 \in H^0(G, \mathbb{Z}) = \mathbb{Z}$, we have

$$1 \cup u = u = u \cup 1$$

for all $u \in H^*(G, M)$ (here $\mathbb{Z} \otimes M = M = M \otimes \mathbb{Z}$).

Proof. Note that $1 \in H^0(G, \mathbb{Z})$ is represented by the augmentation map ε.

Moreover, $F \otimes \varepsilon$ and $\varepsilon \otimes F$ are maps of resolutions $F \otimes F \to F$, so they induce the "identity map" in the cohomology.

The result follows then from the definition. □

Properties of the cup product

Associativity:

For $i = 1, 2, 3$, let $u_i \in H^*(G, M_i)$. Then,

$$u_1 \cup (u_2 \cup u_3) = (u_1 \cup u_2) \cup u_3$$

in $H^*(G, M_1 \otimes M_2 \otimes M_3)$.

Actually, associativity also hold in the cochain level

$$\mathscr{H}om(F \otimes F \otimes F, M_1 \otimes M_2 \otimes M_3).$$

Properties of the cup product

Commutativity:

For any $u \in H^p(G, M)$ and $v \in H^q(G, N)$, we have

$$u \cup v = (-1)^{pq} t_*(v \cup u),$$

where $t : N \otimes M \to M \otimes N$ is the canonical map and $t_* = H^*(G, t)$.

Proof. Let $\tau : F \otimes F \to F \otimes F$ be the chain automorphism such that

$$\tau(x \otimes y) = (-1)^{\deg x \deg y} y \otimes x.$$

Properties of the cup product

We have a commutative diagram

$$\begin{array}{ccc} \mathscr{H}om_G(F,M)\otimes\mathscr{H}om_G(F,N) & \xrightarrow{\cup} & \mathscr{H}om_G(F\otimes F, M\otimes N) \\ \downarrow & & \downarrow{\scriptstyle \mathscr{H}om_G(\tau,t)} \\ \mathscr{H}om_G(F,N)\otimes\mathscr{H}om_G(F,M) & \xrightarrow{\cup} & \mathscr{H}om_G(F\otimes F, N\otimes M) \end{array}$$

where the vertical arrow on the left is given by

$$u\otimes v \mapsto (-1)^{\deg u \deg v} v\otimes u.$$

Since τ is an augmentation-preserving chain map, it induces the identity in the cohomology level.

Then $\mathscr{H}om_G(\tau, t)$ induces t_* in the cohomology level. □

Properties of the cup product

These properties show that $H^*(G, \mathbb{Z})$ is an anti-commutative graded ring, and that every $H^*(G, M)$ is a graded $H^*(G, \mathbb{Z})$-module.

See example of $\mathbb{Z}_n$.

Naturality with respect to group homomorphism:

Let $\alpha : H \to G$ be a group homomorphism. Then,

$$\alpha^*(u \cup v) = \alpha^*(u) \cup \alpha^*(v)$$

for any $u \in H^*(G, M)$ and $v \in H^*(G, N)$.

Properties of the cup product

Transfer formula:

Let $H \leq G$ be a subgroup of finite index. For any $u \in H^*(G, M)$ and $v \in H^*(G, N)$, we have

$$\mathrm{cor}_H^G(\mathrm{res}_H^G(u) \cup v) = u \cup \mathrm{cor}_H^G v.$$

Cap product

Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Then, there is a map

$$\gamma : \mathscr{H}\!om_G(F, M) \otimes ((F \otimes F) \otimes_G N) \to F \otimes_G (M \otimes N)$$

given by

$$u \otimes (x \otimes y \otimes n) \mapsto (-1)^{\deg u \deg x} x \otimes u(y) \otimes n.$$

If $u \in \mathscr{H}\!om_G(F, M)^p = \mathscr{H}\!om_G(F, M)_{-p}$ and $z \in ((F \otimes F)_q \otimes_G N)$, then we define the *cap product* of u and z as

$$u \cap z = \gamma(u \otimes z) \in F_{q-p} \otimes_G (M \otimes N).$$

Cap product

This induces a product, still called *cap product* and denoted by $\cap$:

$$H^p(G,M) \otimes H_q(G,N) \to H_{q-p}(G, M \otimes N).$$

This product is adjoint to the cup product in the following sense:

Consider the map

$$\mathscr{H}om_G(F,M) \otimes (F \otimes_G N) \to M \otimes_G N$$

given by

$$u \otimes (x \otimes n) \mapsto u(x) \otimes n.$$

For $u \in \mathscr{H}om_G(F,M)$ and $z \in F \otimes_G N$, denote $\langle u, z \rangle$ for the image of this map.

Cap product

This map is a chain map, since $\langle \delta u, z \rangle + (-1)^{\deg u} \langle u, \partial z \rangle = 0$.

Hence, there is an induced map

$$H^p(G, M) \otimes H_p(G, N) \to M \otimes_G N,$$

still denoted $\langle , \rangle$. It follows from the definition that for $u \in H^p(G, M_1)$, $v \in H^q(G, M_2)$ and $z \in H_{p+q}(G, M_3)$ the following adjunction formula holds:

$$\langle u \cup v, z \rangle = \langle u, v \cup z \rangle.$$

Lemma

For any $v \in H^q(G, M)$ and $z \in H_q(G, N)$, we have

$$v \cap z = \langle v, z \rangle \in H_0(G, M \otimes N) = M \otimes_G N.$$