

Arithmetical Similarities

Winter semester 2026/27

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Based on Norbert Klingen, *Arithmetical Similarities: Prime Decomposition and Finite Group Theory*, Clarendon Press, Oxford, 1998.

Non-isomorphic number fields can have the same prime decomposition behavior: the residue degrees above every rational prime can agree, giving identical Dedekind zeta functions. This phenomenon, called arithmetical equivalence, is governed by finite group theory. In a common Galois closure, the corresponding subgroups have equal permutation characters, even though they need not be conjugate. Such Gassmann triples form the central link between arithmetic and group theory in this seminar. Starting with prime decomposition, Frobenius automorphisms and zeta functions, we develop this link through explicit examples, including the point and line actions on the Fano plane. We then introduce local fields and adèle rings, examine what these similarities reveal about class numbers and regulators, and conclude with Sunada’s construction of isospectral manifolds.

NO.	FRIDAY	TALK
1	16 Oct 2026	Prime decomposition and Frobenius
2	23 Oct 2026	From splitting primes to permutation characters
3	30 Oct 2026	Dedekind zeta functions and Chebotarev density
4	06 Nov 2026	Gassmann triples and arithmetical equivalence
5	13 Nov 2026	Gassmann’s example and common invariants
6	20 Nov 2026	An explicit Gassmann pair of degree eight
7	27 Nov 2026	The Fano plane and a first look at local fields
8	04 Dec 2026	Local extensions and adèle rings
9	11 Dec 2026	Local fields: what splitting types fail to determine
-	18 Dec 2026	No talk - pre-Christmas break
10	08 Jan 2027	Adeles and locally isomorphic number fields
11	15 Jan 2027	Class numbers and regulators: what zeta functions miss
-	22 Jan 2027	Reserve slot - flexibility / postponed talk
12	29 Jan 2027	Sunada’s construction: Gassmann triples in geometry
-	05 Feb 2027	Discussion of next semester’s program

Foundations: primes, characters and zeta functions

1 | 16 October 2026

Prime decomposition and Frobenius

Book: I § 1a-b, pp. 3-11 (stop before § 1c); I.1.1-1.10.

Introduce rings of integers, prime ideals, ideal norms, ramification indices and residue degrees. Explain the relation $\sum_i e_i f_i = [K : k]$, then the transitive Galois action on primes, decomposition and inertia groups, and the cyclic quotient D/I . Define Frobenius and its conjugacy class. Use a quadratic field as a running example. State the Dedekind-domain and finiteness results; concentrate proofs on the Galois action and the construction of Frobenius.

Priority: A common vocabulary for participants without algebraic number theory.

75-minute plan: 25 min ideals and e, f ; 35 min Galois theory of primes; 15 min examples.

2 | 23 October 2026

From splitting primes to permutation characters

Book: I § 1c, pp. 11-14; Theorem I.1.12, Corollary I.1.13 and Theorem I.1.14.

Identify the k -embeddings of $K = N^H$ into N with the coset action G/H . Prove that Frobenius cycle lengths are residue degrees. Introduce permutation modules and their characters as fixed-point counts; show how the values $\chi(g^j)$ recover every cycle length. Explain that subgroups H and H' of G are conjugate if and only if the transitive G -sets G/H and G/H' are isomorphic, and that $\chi_H = \chi_{H'}$ if and only if $\mathbb{C}[G/H] \cong_{\mathbb{C}G} \mathbb{C}[G/H']$. Discuss why the latter condition is potentially weaker. Work out the action of S_3 on three letters and relate it to the factorization of $X^3 - X - 3$ at the primes 2, 3, and 7 (the discriminant is -239 , the splitting field has Galois group S_3).

Priority: The finite group theory dictionary used in every later talk.

75-minute plan: 20 min cosets and characters; 30 min splitting/cycle proofs; 25 min examples.

3 | 30 October 2026

Dedekind zeta functions and Chebotarev density

Book: I § 2a-b, pp. 15-20; I.2.1-2.3. Preparation only: III § 1a, p. 81, equation (11).

Additional reference for the arithmetic invariants: Jürgen Neukirch, *Algebraic Number Theory*, Springer, 1999.

Define the Dedekind zeta function and its Euler product. Prove that its Euler factors determine all residue degrees over $\mathbb{Q}$ (I.2.1). Introduce Dirichlet density and state Chebotarev; calculate splitting-type densities in the S_3 example. Deduce from Chebotarev that two number fields with the same splitting types outside a set of primes of density zero have the same splitting type at every unramified prime. Briefly define the ideal class group, class number, signature, roots of unity and regulator, and state the analytic class number formula for later use.

Priority: Chebotarev and the class number formula are inputs; their analytic proofs are omitted.

75-minute plan: 25 min zeta functions; 30 min density and examples; 20 min arithmetic invariants.

The equivalence theorem and explicit triples

4 | 06 November 2026

Gassmann triples and arithmetical equivalence

Book: III § 1a, pp. 75-79; Definition III.1.1, Theorem III.1.2 and Theorem III.1.3(i)-(v). For the ramified-prime step: Appendix A, using I § 1b, pp. 7-11, and I.1.13, pp. 12-14.

Work over the ground field $\mathbb{Q}$ and define arithmetical equivalence. Prove the equivalence between identical splitting types almost everywhere, equal permutation characters, and equal cardinalities of the intersections of U and U' with every conjugacy class of G (III.1.3). Call such (G, U, U') a Gassmann triple. For ramified primes, restrict the permutation modules to decomposition groups and take inertia invariants; recover residue degrees from Frobenius on inertia orbits, using Appendix A. This replaces the Artin L -function step on p. 78. Deduce equality of splitting types at every prime and, by the Euler product, of Dedekind zeta functions. Use I.2.1 for the converse. Leave the common arithmetic invariants to Talk 5.

Priority: Prove III.1.3(i)-(v) over $\mathbb{Q}$ and its equivalence with equality of Dedekind zeta functions; omit (vi).

75-minute plan: 15 min definitions and theorem; 30 min Chebotarev and characters; 30 min ramified primes.

5 | 13 November 2026

Gassmann's example and common invariants

Book: III § 1b, p. 85, Example III.1.7 (omit the subsequent Kronecker-equivalence remark); III § 1a, pp. 79-81, Theorem III.1.4a-j. Omit III.1.4k and III.1.5.

Work out Gassmann's two Klein four-subgroups in S_6 : compare their intersections with conjugacy classes, prove non-conjugacy by their fixed letters, and obtain degree-180 coset actions. State the existence of Galois realizations with group S_6 . Derive equal degrees, Galois closures, maximal Galois subfields, roots of unity and signatures for arithmetically equivalent fields. For maximal Galois subfields, use equality of the normal closures of H and H' in G ; for signatures, use complex conjugation. State discriminant equality and equality of the sets of ramified primes. Use the analytic class number formula to deduce $h_K R_K = h_{K'} R_{K'}$.

Priority: Explain which invariants agree despite non-isomorphism.

75-minute plan: 30 min Gassmann's example; 30 min common invariants; 15 min the class number formula.

6 | 20 November 2026

An explicit Gassmann pair of degree eight

Book: III § 1b, pp. 86-87, Example III.1.9; use the direct affine-group proof on p. 87 and omit the Kronecker-equivalence discussion. Preview: III.1.10, pp. 87-88.

Construct the common splitting field of $\mathbb{Q}(\sqrt[8]{3})$ and $\mathbb{Q}(\sqrt[8]{48})$. Prove that $X^8 - 3$ remains irreducible over $\mathbb{Q}(\zeta_8)$, using Eisenstein at a prime above 3, and identify its Galois group with the affine group of $\mathbb{Z}/8\mathbb{Z}$. Verify that the two fixed subgroups are $H = \{\pm x, \pm 3x\}$ and $H' = \{\pm x, \pm 3x + 4\}$. Match their elements by conjugacy class and prove that the subgroups are not conjugate, using the absence of a common fixed point for H' . Apply Talk 4 to obtain equal zeta functions for non-isomorphic fields. Explain the same construction for suitable radicands a , including $a = 97$ for Talk 9, and preview the degree-seven example.

Priority: Give a complete explicit construction, including the Galois group and the fixed fields.

75-minute plan: 25 min splitting field and Galois group; 35 min the subgroup pair; 15 min consequences and variants.

The Fano plane, local fields and adeles

7 | 27 November 2026

The Fano plane and a first look at local fields

Book: III § 1b, Example III.1.10, pp. 87-88. Local foundations: VI § 2a, p. 235, including the local-degree formula (3); supplement with Lang, Algebraic Number Theory, Ch. II § 1 (Klingen [67]).

Draw the Fano plane and construct its point and line stabilizers in $\mathrm{GL}_3(\mathbb{F}_2)$. Prove non-conjugacy and equality of permutation characters by comparing fixed vectors and fixed covectors. Present the degree-seven fields defined by $X^7 - 7X + 3$ and $X^7 + 14X^4 - 42X^2 - 21X + 9$; take their stated Galois realization as input. Then introduce p -adic absolute values, the completions $\mathbb{Q}_p$ and their valuation rings $\mathbb{Z}_p$, and completions of number fields at prime ideals. Define uniformizers and residue fields, state the local degree formula ef , and relate its ramification index and residue degree to those from Talk 1.

Priority: Prove the Fano example using elementary linear algebra; introduce local fields through examples.

75-minute plan: 40 min Fano plane and degree-seven fields; 35 min completions, uniformizers, residue fields and local degrees.

8 | 04 December 2026

Local extensions and adele rings

Book: VI § 2a-b, pp. 235-237: unramified extensions and the definition of adeles before VI.2.3. Background: Lang, Algebraic Number Theory, Ch. II §§ 1,4 (Klingen [67]); Neukirch, as in Talk 3.

Build on the local foundations from Talk 7. State uniqueness of unramified extensions of a prescribed degree, Hensel's lemma in its valuation form, and the Eisenstein criterion for total ramification. Prepare Talk 9 explicitly: use Hensel at $x = 5$ for $X^8 - 97$ to show that 97 has an eighth root in $\mathbb{Q}_2$. Factor $X^8 - 1$ into cyclotomic factors and $X^8 - 16$ into $X^2 \pm 2$ and $X^2 \pm 2X + 2$; use shifts where needed to make the cyclotomic factors Eisenstein. Then define the adèle ring as a restricted product, including the real and complex places. Explain its basic open sets, the compact-open valuation rings, and the diagonal embedding of the number field.

Priority: Take the local-extension theorems as inputs; work out the calculations and illustrate adeles with $\mathbb{Q}$.

75-minute plan: 15 min local-extension theorems; 30 min Hensel and the factorizations; 30 min adeles.

9 | 11 December 2026

Local fields: what splitting types fail to determine

Book: VI § 2a, pp. 234-236; VI.2.1-2.2 and the example on p. 235. Recall I § 1b, pp. 7-10.

Work over $\mathbb{Q}$. Prove VI.2.1: arithmetical equivalence matches residue fields at every prime, and matches the quotient rings modulo p at all but finitely many primes. Using the uniqueness of unramified local extensions from Talk 8, show that this is equivalent to matching completions over $\mathbb{Q}_p$ for almost every p . Define local isomorphism at every finite prime. For $K = \mathbb{Q}(\sqrt[8]{97})$ and $K' = \mathbb{Q}(\sqrt[8]{16 \cdot 97})$, explain how factorization over $\mathbb{Q}_2$ gives the completions above 2. Scale using the eighth root from Talk 8 and deduce ramification indices $(1, 1, 2, 4)$ and $(2, 2, 2, 2)$, although all four residue degrees are one in both fields. Discuss why equal zeta functions and discriminants do not determine the completions.

Priority: Make the difference between residue fields and completions concrete; use the calculations already prepared.

75-minute plan: 20 min residue fields and quotient rings; 25 min local isomorphism; 30 min the degree eight counterexample.

Adeles, class groups and spectral geometry

10 | 08 January 2027

Adeles and locally isomorphic number fields

Book: VI § 2b, pp. 236-240; VI.2.3b, the closed-maximal-ideal part of VI.2.4, and the topological parts of VI.2.5. State VI.2.6 for degree eight; VI.2.7 leads to Talk 11.

Recall the restricted product from Talk 8 and prove that number fields are locally isomorphic over $\mathbb{Q}$ exactly when their adèle rings are isomorphic as topological rings. Identify the closed maximal ideals as kernels of coordinate projections and recover the completions as quotients; use the density of the elements with finite support for the key step. Explain how equal signatures supply the archimedean factors in the converse. State Komatsu's congruence criterion for the degree eight pairs from Talk 6 and compare $a = 97$ with $a = -33$. The latter gives $\mathbb{Q}(\sqrt[8]{-33})$ and $\mathbb{Q}(\sqrt[8]{-528})$ with isomorphic adèles. Explain that such an isomorphism need not carry one diagonally embedded number field onto the other.

Priority: Use the topological proof over $\mathbb{Q}$; omit the principal-ideal characterization and the general-base refinements.

75-minute plan: 15 min adèle recap; 30 min recovering completions; 20 min the equivalence theorem; 10 min examples.

11 | 15 January 2027

Class numbers and regulators: what zeta functions miss

Book: III.1.4j, p. 81; IV.2.2, p. 151, with IV.1.2c, p. 134 (consequence stated without proof); IV.3.1, p. 164; VI.2.7, p. 240. Additional reference: de Smit-Perlis [119], pp. 213-215.

Recall ideal classes and describe the regulator using the logarithmic embedding of units, stating Dirichlet's unit theorem. Illustrate with $\mathbb{Q}(\sqrt{2})$, then revisit the equality of the products hR from Talk 5. Take the following theorem as a black box: for arithmetically equivalent fields in a common Galois extension with group G , $\text{Cl}(K)_p \cong \text{Cl}(K')_p$ whenever $p \nmid |G|$. For the degree eight family, conclude that the odd parts agree and the class-number quotient is a power of two. Present the de Smit-Perlis pair $\mathbb{Q}(\sqrt[8]{-15})$, $\mathbb{Q}(\sqrt[8]{-240})$. Take their computation $h_K = 2h_{K'}$ as input and deduce $R_{K'} = 2R_K$. Finally use VI.2.7 to show that even isomorphic adèle rings do not determine the class number.

Priority: Explain the arithmetic meaning of the examples; accept the comparison theorem and the published computations.

75-minute plan: 25 min units and regulators; 20 min the class-group theorem and degree eight family; 30 min counterexamples.

12 | 29 January 2027

Sunada's construction: Gassmann triples in geometry

Book: VI § 5a-b, pp. 250-254; VI.5.1-5.5, with the main emphasis on VI.5.5.

Additional references: Sunada, *Riemannian coverings and isospectral manifolds*, Ann. Math. 121 (1985), 169-186 (Klingen [123]); Bérard, Bourbaki exposé 705 (1988/89), pp. 127-154 (Klingen [9]).

Explain the dictionary between field extensions and finite Riemannian coverings, and between prime splitting and lifts of closed geodesics. State the length-spectrum result. Prove Sunada's Laplace-isospectrality theorem by viewing each eigenspace on a common finite regular cover as a G -module and comparing H - and H' -invariants. Reuse the S_6 or Fano-plane triple to explain the construction of isospectral examples. Discuss separately the additional geometric argument needed for non-isometry: non-conjugacy alone does not prove it for an arbitrary metric.

Priority: The same equality of permutation characters controls arithmetic and spectral data.

75-minute plan: 20 min coverings and length spectra; 35 min Sunada proof; 20 min examples and non-isometry.

Ramified primes and permutation modules

This sketch is for Talk 4, with ground field $\mathbb{Q}$. It proves (iv) $\Rightarrow$ (ii) in Klingen's Theorem III.1.3 directly, replacing the Artin L -function step on p. 78. Use I.1.6–1.9, pp. 7–10, for decomposition and inertia groups, and I.1.13, pp. 12–14, for recovering cycle lengths from characters.

Let $N|\mathbb{Q}$ be a common finite Galois extension containing K and K' . Following Klingen, put

$$G = G(N|\mathbb{Q}), \quad U = G(N|K), \quad U' = G(N|K'),$$

and assume $1_U^G = 1_{U'}^G$. Fix a rational prime p , write $\mathfrak{p} = p\mathbb{Z}$, and choose a prime $\mathfrak{Q}$ of N above $\mathfrak{p}$. Abbreviate Klingen's decomposition and inertia groups by

$$D = D_{N|\mathbb{Q}}(\mathfrak{Q}) = D(\mathfrak{Q}|\mathfrak{p}), \quad I = I_{N|\mathbb{Q}}(\mathfrak{Q}) = I(\mathfrak{Q}|\mathfrak{p}).$$

Choose a Frobenius automorphism $\sigma \in D$ as in I.1.9b; its image generates the cyclic group D/I .

1. Read residue degrees from inertia orbits. The D -orbits on G/U correspond to the primes $\mathfrak{P}$ of K above $\mathfrak{p}$. The orbit corresponding to $\mathfrak{P}$ consists of $f(\mathfrak{P}|\mathfrak{p})$ inertia orbits, which D/I permutes transitively. Thus the cycle lengths of σ on $I \backslash (G/U)$ are precisely the residue degrees above p , including when p ramifies. The same holds for U' .

2. Pass from equal characters to inertia invariants. The equality $1_U^G = 1_{U'}^G$, gives an isomorphism of $\mathbb{C}G$ -modules $\mathbb{C}[G/U] \cong \mathbb{C}[G/U']$. Restricting to D and taking I -invariants yields

$$\mathbb{C}[G/U]^I \cong \mathbb{C}[G/U']^I \quad \text{as } \mathbb{C}[D/I]\text{-modules.}$$

These invariant spaces have bases given by the sums over individual I -orbits. Consequently, the permutation characters of D/I on $I \backslash (G/U)$ and $I \backslash (G/U')$ agree.

3. Recover splitting types and zeta functions. By I.1.13, the character values on all powers of σ determine its cycle lengths. Step 2 therefore gives identical cycle types, and Step 1 gives identical residue degrees in K and K' above every rational prime. Their ordinary Dedekind zeta Euler factors are therefore equal, so $\zeta_K = \zeta_{K'}$. For the converse, use I.2.1.