DIMENSION AND RANDOMNESS IN GROUPS ACTING ON ROOTED TREES

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In these last few weeks of the seminar we will look at (selected parts of) a paper of Abert and Virag [1]. There are two main strands of results:

(1) average order of a random element in the Sylow p-subgroup $W_n(p)$ of the symmetric group and
(2) study of possible Hausdorff dimensions of closed subgroups of $\mathrm{Aut}(T_d)$.

1. General set-up

Let $T = T(d)$ denote the infinite rooted d-ary tree and let $H \subseteq \mathrm{Sym}(d)$ be a permutation group. Let $W(H)$ denote the infinite iterated wreath product of H acting on T with respect to H. For example, $W(\mathrm{Sym}(d))$ is the full automorphism group of $T(d)$. Let $W_n(H)$ denote the n-fold wreath product of H, acting on T_n, the d-ary tree of depth n. The case $H = C_p$, the cyclic group of order p, is of particular interest. The pro-p group $W(p) = W(C_p)$ obtained this way is called the group of p-adic automorphisms. The group $W_n(p)$ is called the symmetric p-group of depth n, as it can also be obtained as the Sylow p-subgroup of the symmetric group $\mathrm{Sym}(p^n)$.

1.1. Haar measure.

The group $W(p)$ is a pro-p group (in particular a compact topological group) and its Haar measure μ becomes a probability measure on $W(p)$. For a *closed* subset X of $W(p)$, one can take the following expression as a definition of μ:

$$\mu(X) = \inf_{N \lhd_o G} |XN : N|,$$

so it is the limit of the counting measures on the finite quotients.

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1.2. **Hausdorff dimension.** For $l \in \mathbb{N}$, let $W^{[l]}$ be the subgroup of $W(p)$ that stabilises all vertices of the tree up to level l. For a closed subgroup $G \leq \mathrm{Aut}(T)$, define the *density sequence* of G as

$$\gamma_l(G) = \frac{\log |G/G \cap W^{[l]}|}{\log |W/W^{[l]}|}$$

and the *Hausdorff dimension* of G to be

$$\dim_{\mathrm{H}}(G) = \liminf_{l \to \infty} \gamma_l(G).$$

2. Orders of ramdom elements

TALK 1. *Galton-Watson random trees and random automorphisms.*

This first talk deals with random elements in $W_n(p)$. Refresh notions of Haar measure. Give ideas about Section 2 (skip 2.12).

3. Hausdorff dimensions of subgroups

TALK 2. *Word maps.*

Go through Section 3 and Section 4 (skip maybe half of proof of 3.10 and the proof of 4.4). Emphasis on Corollary 3.8 and Corollary 4.6.

IMPORTANT: we will use Sections 5 and 6 as black boxes.

TALK 3. *Small subgroups.*

State the definitions at the beginning of Section 7 about Hausdorff dimension. Cover the results in Section 8.

TALK 4. *Large subgroups.*

Cover Section 9. You might have to recall some notions from Section 8 (like n-samples).

References

[1] Abèrt, Miklòs ; Virág, Bálint. Dimension and randomness in groups acting on rooted trees. *J. Amer. Math. Soc.* 18 (2005), no. 1, 157–192. Available here.