Groups acting on regular rooted trees

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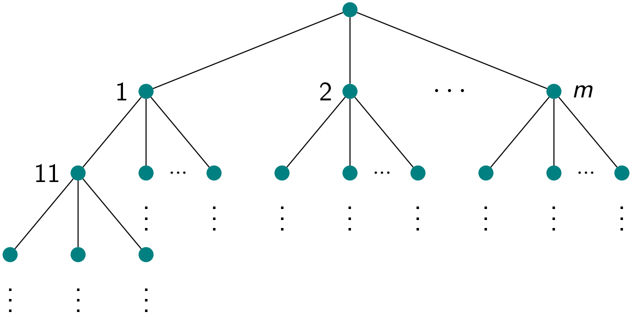
Why are we interested in regular rooted trees?

Groups of automorphisms of regular rooted trees are a rich source of examples in group theory. For example the (first) Grigorchuk group Γ has a number of surprising properties.

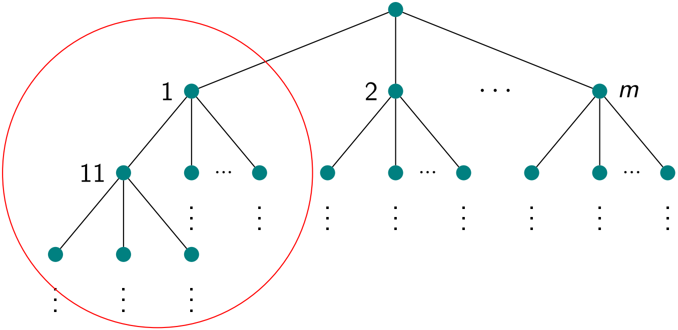
- It has a solvable word problem.
- It is a finitely generated, infinitely torsion group. In fact, every element has order a power of 2. However, it is not finitely presented.
- It has intermediate word growth.
- It is amenable but not elementary amenable.
- It is commensurable its own direct product $\Gamma \times \Gamma$.
- It is just infinite.

What is a regular rooted tree $\mathcal{T}$?



- Is an infinite tree.
- With a fixed vertex, the root.
- Regular: Every vertex has the same number of descendants.

What is a regular rooted tree $\mathcal{T}$?



- The tree is self-similar: Every tree hanging from any vertex is isomorphic to the tree itself.

What is a regular rooted tree $\mathcal{T}$?

Equivalently, let X be an alphabet of m letters. For example, $X = \{1, 2, \dots, m\}$. Denote by X the set of all finite words over the alphabet X, that is

$$X^* = \{x_1 x_2 \cdots x_n \mid n \in \mathbb{N}, x_i \in X\} \cup \{\emptyset\}$$

Then every vertex in the m-adic regular rooted tree can be identified with an element in X as follows:

- The root is identified with the empty word $\emptyset$.
- Each of the m descendants of the root is associated with a unique letter from the alphabet X in an ordered manner.
- Any other vertex of the tree can be then inductively identified with the word vx, where v is the father or predecessor of said vertex, and x indicates which descendant the vertex is.

What is a regular rooted tree $\mathcal{T}$?

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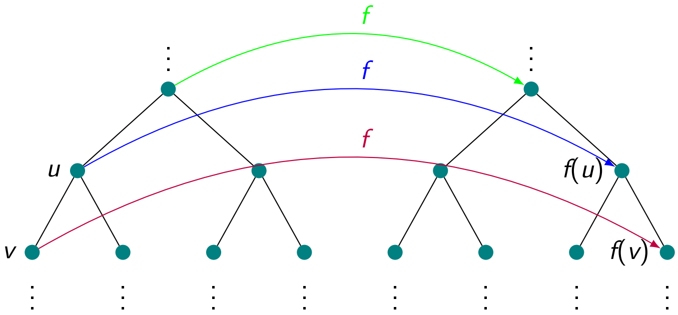
What is an automorphisms of a regular rooted trees?

Automorphisms of $\mathcal{T}$ (Aut $\mathcal{T}$)

Bijections of the vertices that preserve incidence.



The set of all automorphisms of $\mathcal{T}$, Aut $\mathcal{T}$, is a group with respect to composition between functions.

What is an automorphisms of a regular rooted trees?

Here are some examples of automorphisms of $\mathcal{T}$:

- The identity map, which we will denote by 1.
- Rooted automorphisms: transformations that rigidly permute the subtrees hanging from the first-level vertices, i.e. X, according to some permutation in the symmetric group $\mathrm{Sym}(m)$.

What is an automorphisms of a regular rooted trees?

Keep in mind that, if $|X| = |Y|$, then the regular rooted trees X^* and Y^* are isomorphic. Therefore $\mathrm{Aut}X^* \cong \mathrm{Aut}Y^*$.
Let g be an element in $\mathrm{Aut}\mathcal{T}$. Then:

- the automorphism g sends the root to the root, that is $g(\emptyset) = \emptyset$,
- $g(L_n) = L_n$, in particular if $u \in L_n$, then $g(u) \in L_n$,
- if g fixes a vertex u, it also fixes the path from $\emptyset$ to u.

Labels

Let $g \in \mathrm{Aut}\mathcal{T}$ be an automorphism that sends the vertex $u \in \mathcal{T}$ to another vertex v,in other words $g(u) = v$. Then, by definition, any descendant ux of u must be sent to some descendant vy of v, where $x, y \in X$. Therefore,

$$g(ux) = g(u)\sigma(x) = v\sigma(x) = vy.$$

for some $\sigma \in \mathrm{Sym}(m)$, which corresponds to the rooted automorphism of the tree $\mathcal{T}_v$ that has the vertex v as its root. We σ denote by $g_{(u)}$ and we call it the label of g at the vertex u.

Labels

- The set of all labels of g is called the portrait of g.
- The automorphism $g \in \mathrm{Aut}\mathcal{T}$ is completely determined by its portrait.

Let $u = x_1 \cdots x_{n-1} x_n \in \mathcal{T}$, then

$$g(u) = g(x_1 \cdots x_{n-1}) g_{(x_1 \cdots x_{n-1})}(x_n) = g_{(\emptyset)}(x_1) g_{(x_1)}(x_2) \cdots g_{(x_1 \cdots x_{n-1})}(x_n)$$

Here are some rules for the labels:

- $(f\, g)_u = f_{(u)} g_{(f(u))}$.
- $(f^{-1})_{(u)} = (f_{(f^{-1}(u))})^{-1}$.
- $(f^{g})_{(u)} = (g_{(g^{-1}(u))})^{-1} f_{(g^{-1}(u))} g_{(f(g^{-1}(u))}$.

Section or states

Let $g \in \mathrm{Aut}\mathcal{T}$ be an automorphism. We may consider how g acts on the subtree hanging from $g(u)$, that is, $\mathcal{T}_{g(u)}$, which as we know, is isomorphic to $\mathcal{T}$:
We denote by $g|_u$ the section of g at the vertex u, which is defined by

$$g(uv) = g(u)g|_u(v)$$

where here uv is a descendant of u.
Note that the section of g at the vertex u, which we denote by $g|_u$, is an automorphism that sends v to $g|_u(v)$. In particular, if $g(u) = u$, then $g|_u$ is just the restriction of g to $\mathcal{T}_u$, which is the tree hanging from u.

Sections or states

Here are some rules for the labels:

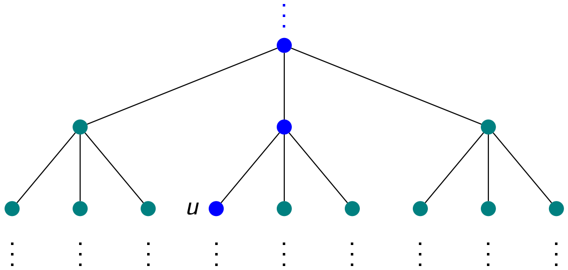
- $(f\,g)|_u = f|_u g|_{f(u)}$.
- $(f^{\,-1})|_u = (f|_{f^{-1}(u)})^{-1}$.
- $(f^{\,g})|_u = (g_{|}g^{-1}(u))^{-1} f|_{g^{-1}(u)} g|_{f(g^{-1}(u))}$.
- If $u = u_1 u_2$, then $f|_u = (f|_{u_1})|_{u_2}$.

The stabilizers

We now describe some important subgroups of Aut$\mathcal{T}$. For a vertex $u \in \mathcal{T}$, one can define the vertex stabiliser of u as follows:

$$\mathrm{Stab}(u) = \{g \in \mathrm{Aut}\mathcal{T} \mid g(u) = u\}$$

This is the subgroup of Aut $\mathcal{T}$ consisting on all automorphisms that fix the vertex u.

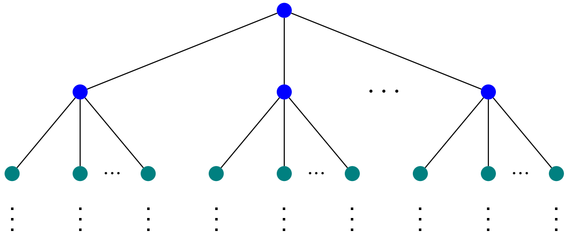


The stabilizers

One can also consider the n-th level stabiliser, which is described as follow:

$$\mathrm{Stab}(n) = \{g \in \mathrm{Aut}\mathcal{T} \mid g(u) = u, \ \forall u \in L_n\}$$

This is the normal subgroup of Aut $\mathcal{T}$ consisting on all automorphisms that fix all vertexes up to level n.



Stabilizers of $G \leq \mathrm{Aut}\ \mathcal{T}$

Let G be a subgroup of Aut $\mathcal{T}$. Then

$$\mathrm{Stab}_G(u) := G \cap \mathrm{Stab}(u)$$

is the subgroup of G consisting on all automorphisms of G that fix the vertex u;

$$\mathrm{Stab}_G(n) := G \cap \mathrm{Stab}(n)$$

is the (normal) subgroup of G consisting on all automorphisms of G that fix all vertexes up to level n.

The ψ maps:

We can define the following maps:

$$\begin{aligned} \psi_u : \mathsf{Stab}(u) &\longrightarrow \mathsf{Aut}\mathcal{T} \\ g &\longmapsto g|_u. \end{aligned}$$

$$\begin{aligned} \psi_n : \mathsf{Stab}(n) &\longrightarrow \mathsf{Aut}\mathcal{T} \times \overset{m^n}{\cdots} \times \mathsf{Aut}\mathcal{T} \\ g &\longmapsto (g|_u)_{u \in L_n}. \end{aligned}$$

Note that $\mathsf{Stab}(n) \cong \mathsf{Aut}\mathcal{T} \times \overset{m^n}{\cdots} \times \mathsf{Aut}\mathcal{T}$.

The ψ maps:

Consider the following subgroup:

$$\begin{aligned} H_n &= \{g \in \mathrm{Aut}\mathcal{T} \mid g_{(u)} = 1 \forall u \in L_{\geq n}\} \\ &= \{g \in \mathrm{Aut}\mathcal{T} \mid g|_u = 1 \forall u \in L_{\geq n}\} \end{aligned}$$

- Rooted automorphism belong to $H_1 \cong \mathrm{Sym}(m)$.

Then

$$\mathrm{Aut}\mathcal{T} = \mathrm{Stab}(n) \rtimes H_n,$$

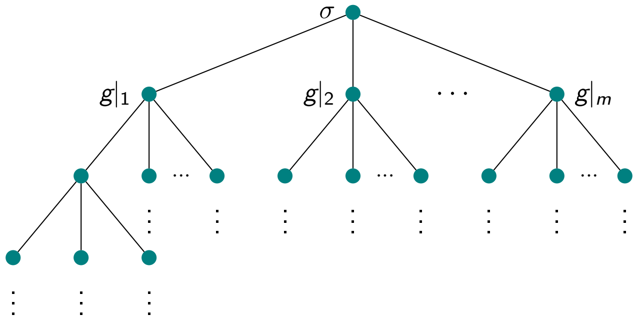
and therefore the map ψ_n can be extended to

$$\begin{aligned} \psi_n : \mathrm{Aut}\mathcal{T} = \mathrm{Stab}(n) \rtimes H_n &\longrightarrow (\mathrm{Aut}\mathcal{T} \times \overset{m^n}{\cdots} \times \mathrm{Aut}\mathcal{T}) \rtimes H_n \\ gh &\longmapsto (g|_u)_{u \in L_n} h. \end{aligned}$$

The ψ maps:

We are particularly interested in the case $n = 1$:

$$\psi = \psi_1 : \mathrm{Aut}\mathcal{T} \longrightarrow (\mathrm{Aut}\ \mathcal{T} \times \overset{m}{\cdots} \times \mathrm{Aut}\ \mathcal{T}) \rtimes \mathrm{Sym}(m)$$
$$g \longmapsto (g|_1, \ldots, g|_m)\sigma.$$



THE ψ MAPS:

Let $\psi(f) = (f_1, \ldots, f_m)\sigma$ and $\psi(g) = (g_1, \ldots, g_m)\tau$ be two automorphisms of $\mathcal{T}$. Then

$$\begin{aligned}\psi(fg) &= (f_1, \ldots, f_m)\sigma(g_1, \ldots, g_m)\tau \\ &= (f_1, \ldots, f_m)(g_1, \ldots, g_m)^{\sigma^{-1}}\sigma\tau \\ &= (f_1, \ldots, f_m)(g_{\sigma(1)}, \ldots, g_{\sigma(m)})\sigma\tau \\ &= (f_1 g_{\sigma(1)}, \ldots, f_m g_{\sigma(m)})\sigma\tau\end{aligned}$$

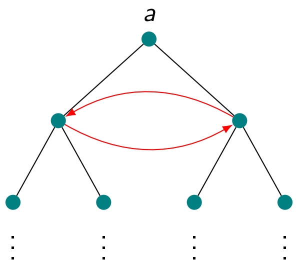
In particular, if $\psi(f) = (f_1, \ldots, f_m) \in \psi(\mathrm{Stab}(1))$ and σ is a rooted automorphism:

$$\psi(f^{\sigma}) = (f_{\sigma^{-1}(1)}, \ldots, f_{\sigma^{-1}(m)})$$

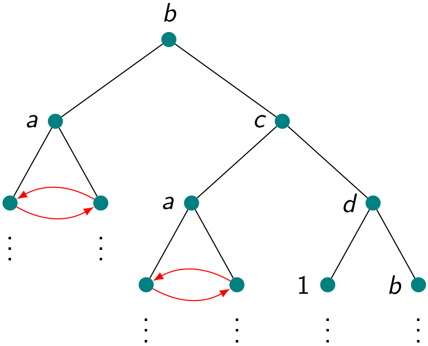
The (first) Grigorchuk group

The (first) Grigorchuk is generated by three automorphisms of the 2-adic tree

$$\psi(a) = (1, 1)(1\ 2)$$
$$\psi(b) = (a, c)$$
$$\psi(c) = (a, d)$$
$$\psi(d) = (1, b)$$



The (first) Grigorchuk group



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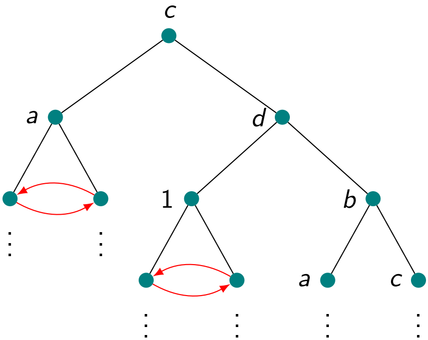

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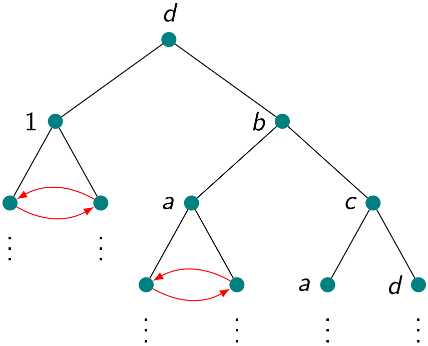

The (first) Grigorchuk group

Exercise

Let $\Gamma = \langle a, b, c, d \rangle$ be the (first) Grigorchuk group.

- What are the labels of a, b, c, d at the vertices 1, 2 and 22?
- What are the sections of a, b, c, d at the vertices $\emptyset$, 1, 2 and 21?
- Which is the order of a, b, c and d?
- Show that $bc = cd = d$ and therefore $\langle b, c, d \rangle = C_2 \times C_2$.

The (first) Grigorchuk group

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- What are the sections of a, b, c, d at the vertices $\emptyset$, 1, 2 and 21?
- Which is the order of a, b, c and d?
- Show that $bc = cd = d$ and therefore $\langle b, c, d \rangle = C_2 \times C_2$.

As a consequence, every nontrivial word in Γ can be written as an alternating product of a and $\{b, c, d\}$.

The (first) Grigorchuk group

For any finitely generated group G with finite symmetric generating set S define

$$\ell_S = \ell : G \longrightarrow \mathbb{N}$$

via $\ell(g) = \min\{k \mid g = s_1 \dots s_k \text{ with } s_i \in S\}$.

Lemma

Let $g \in \Gamma$ and suppose that $\psi(g) = (g_1, g_2)\sigma$. Then

$$\ell(g_i) \leq \left\lfloor \frac{\ell(g) + 1}{2} \right\rfloor$$

for $i \in \{1, 2\}$.

The (first) Grigorchuk group

Exercise

Prove:

- $\psi(aba) = \psi(b^a) = (c, a)$,
- $\psi(aca) = \psi(c^a) = (d, a)$,
- $\psi(ada) = \psi(d^a) = (b, 1)$,
- $\psi(abac) = \psi(b^a c) = (c, a)(a, d) = (ca, ad)$, note that $\ell(ca) = \ell(ad) = 2$.

Complete the proof of the lemma.

The (first) Grigorchuk group

Lemma

Γ has a solvable world problem.

Proof.

Let w be a word in the generating set of Γ written as an alternating product of a with elements of $\{b, c, d\}$. Does w have an odd number of a's?

- If yes, then $w \neq 1$ acts non-trivially at the root.
- If not, then the action at the root is trivial and so w decomposes as (w_1, w_2), with w_1 and w_2 having length less than the length of w. Repeat the steps now with w_1 and w_2.

$\square$

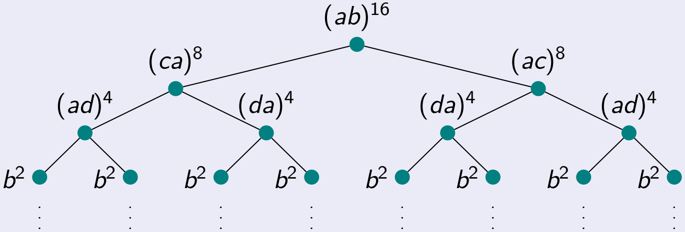
The (first) Grigorchuk group

Lemma

Γ is a torsion group. In fact, each element of Γ has order a power of 2.

Proof.

The general proof is a case by case analysis for which we direct the reader to Topics in geometric group theory by Pierre de la Harpe. For individual words, one can find the order using the length reduction as below:



Self-similar:

Let G be a subgroup of $\mathrm{Aut}\mathcal{T}$. We say that G is self-similar if $g|_u \in G$ for all $g \in G$ and $u \in \mathcal{T}$.

- The (first) Grigorchuk group is self-similar.

Note that if G is self-similar, then we can define the group homomorphism

$$\begin{aligned} \psi_u : \mathrm{Stab}_G(u) &\longrightarrow G \\ g &\longmapsto g|_u, \end{aligned}$$

and the injective group homomorphism

$$\begin{aligned} \psi_n : \mathrm{Stab}_G(n) &\longrightarrow G \times \overset{m^n}{\cdots} \times G \\ g &\longmapsto (g|_u)_{u \in L_u}. \end{aligned}$$

Fractal and level-transitive:

Let G be a self-similar subgroup of Aut$\mathcal{T}$. Then we say that G is fractal if $\psi_u(\mathrm{Stab}_G(u)) = G$ for all $u \in \mathcal{T}$.

- The (first) Grigorchuk group is fractal.

Let G be a subgroup of Aut$\mathcal{T}$. We say that G is level transitive or spherically transitive if it acts transitively on every level. In other words, if for every $n \in \mathbb{N}$, and for any u and v in L_n, there exists some $g \in G$ such that $g(u) = v$.

- The (first) Grigorchuk group is level transitive.

(Weakly) regular branch:

Let G be a fractal level transitive subgroup of $\mathrm{Aut}\mathcal{T}$ and let K be a non-trivial subgroup of $\mathrm{Stab}_G(1)$. We say that G is weakly regular branch over K if

$$K \times \overset{m}{\cdots} \times K \leq \psi(K).$$

If furthermore K has finite index in G, we say that G is regular branch over K.

- The Grigorchuk group is regular branch over $K = \langle (ab)^2, (bd^a)^2, (b^a d)^2 \rangle$.

The (first) Grigorchuk group

Exercise

Prove that Γ is regular branch over $K = \langle (ab)^2, (bd^a)^2, (b^a d)^2 \rangle$.

- Let us denote: $x = (ab)^2, y = (bd^a)^2, z = (b^a d)^2$. Then

$$\psi(x) = (ca, ca) \qquad \psi(y) = (x, 1)\psi(z) = (1, x).$$

- Consider $[y, x]$:

$$\psi([y, x]) = ([x, ca], [1, ac]) = ((b^a d)^2, 1) = (z, 1)$$

Similarly, by taking appropriate products of x, y and z, we see that $\psi(K)$ contains the elements $\{(y, 1), (1, y), (z, 1), (1, z)\}$. This proves that $\psi(K) \geq K \times K$, and therefore Γ is regular branch over $K = \langle (ab)^2, (bd^a)^2 (b^a d)^2 \rangle$.

The rigid stabilizers:

Let G be a subgroup of Aut$\mathcal{T}$, the rigid vertex stabilizer of a vertex u in G, denoted by $\mathrm{Rst}_G(u)$, is the subgroup of G that consists of those automorphisms of $\mathcal{T}$ that fix all vertices not having u as a prefix.
In other words, an automorphism g is in the rigid vertex stabilizer of u if $g \in G$ and all labels of g outside $\mathcal{T}_u$ are equal to 1.
The rigid stabilizer of the n-th level is defined as:

$$\mathrm{Rst}_G(n) = \langle \mathrm{Rst}_G(u) \mid u \in L_n \rangle.$$

Equivalently, the n-th rigid stabiliser, is the largest subgroup of $\mathrm{Stab}_G(n)$ such that

$$\psi_n(\mathrm{Rst}_G(n)) = H_1 \times \overset{m^n}{\cdots} \times H_{m^n}$$

for some $H_i \leq G$.

(Weakly) branch:

Let G be a fractal level transitive subgroup of $\mathrm{Aut}\mathcal{T}$. We say that G is weakly branch if $\mathrm{Rst}_G(n)$ is non-trivial for all $n \in \mathbb{N}$.
If furthermore $\mathrm{Rst}_G(n)$ has finite index in G for all $n \in \mathbb{N}$, we say that G is branch.

- Note that if G is weakly regular branch (regular branch) then it is weakly branch (branch).