WORD GROWTH IN GROUPS

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The goal of this seminar is to investigate the "word growth" of finitely generated infinite groups. The idea is simple: fix a finite generating set S of a group G and count the number $\gamma_S^G(n)$ of distinct elements of G that can be (minimally) represented as words of length n in the generators; γ_S^G is called *a word growth function* of G. The study of word growth first arose in a geometric setting, but this seminar will be mostly about algebraic aspects. We will follow the book "How groups grow" by A. Mann [4]. It is easy to see that the word growth function γ_S of a finitely generated infinite group has to be between "polynomial" and "exponential". Moreover being polynomial and exponential turns out to be a property of the group and not of the generating set, so we can talk about groups with polynomial and with exponential growth. The idea is to deduce structural information about the group from the asymptotic behaviour of its word growth function.

The first part of the seminar is devoted to the proof of Gromov's Theorem: a finitely generated group has polynomial growth if and only if it has a finite index subgroup which is nilpotent.

In the second part we will deal with groups of *intermediate* growth, i.e. groups which are neither polynomial nor exponential. The first example of such a group was the *Grigorchuk group*. We will go through the basic properties of this group and see its relevance with respect to other properties related to word growth, such as amenability.

Finally there will be some talks about *uniformly exponential word growth* and *generating functions* associated to word growth functions.

IMPORTANT: There are currently 16 available talks in this programme, but only (at most) 15 available weeks. Note that Talks 12*, 13*, 15* and 16* are marked with a star and are *optional*. We should choose at most 3 of them. Also if you have suggestions for other topics in word growth, please let me know and I can try to include them in the programme. Finally, if you have any questions about the programme or the preparation of your talk please let me know.

BLOCK A: GROUPS OF POLYNOMIAL GROWTH

TALK 1: Introduction. *Chapter 1 of [4].*

This talk introduces the word growth function of a finitely generated group.

Follow Chapter 1. Mention briefly the relation with the solvability of the word problem. Sketch the plan of the Seminar (see historical paragraph at the end of Chapter 1).

TALK 2: Basics of Group Theory. *Chapter 2 of [4].*

Here we review the group-theoretic concepts that are necessary for later talks.

Go quickly through Section 2.1. Cover Sections 2.2 and 2.4 in more detail. Make sure to cover at least the needed prerequisites for Talk 3 (e.g. the series δ_i) and Proposition 2.5.

TALK 3: Growth of nilpotent groups. *Chapter 4 of [4].*

In this talk we prove the easy implication of Gromov's theorem: a finitely generated virtually-nilpotent group has polynomial growth.

Prove Theorem 4.1, Theorem 4.2, Lemma 4.3 and Corollary 4.4. If there is time, survey some results from Chapter 3 and/or Proposition 4.8.

TALK 4: Growth of solvable groups. *Chapter 5 of [4].*

This chapter contains a proof of a strong version of Gromov's theorem for solvable groups: a finitely generated solvable group of subexponential growth is virtually-nilpotent.

Recall the needed facts about solvable and polycyclic groups from Section 2.4 of Chapter 2. Then prove Theorem 5.1, Corollary 5.2, Theorem 5.3 and Corollary 5.4.

TALK 5: Ultralimits and Asymptotic cones (Part 1). *Chapter 7 of [4].*

The next talks have a "more geometric flavour", we introduce the notion of asymptotic cone of a finitely generated group and we prove some first properties.

Cover Chapter 7 until the end of the proof of Theorem 7.4.

TALK 6: Dimension and Asymptotic cones (Part 2). *Chapter 6/7/8 of [4].*
We continue the study of asymptotic cones started in the previous talk. Here we look at the dimension of the cone of a finitely generated group of polynomial growth.

Introduce the concepts of dimension of a topological space and Hausdorff dimension of a metric space (Chapter 6 from page 65 till end of the Chapter). Prove Proposition 8.1, 8.2 and Theorem 8.3.

TALK 7: Gromov's Theorem. *Chapter 6/8 of [4].*
The aim of this talk is to finish the proof of Gromov's theorem. Here we will put together all results from the previous three talks. To conclude the proof, it is also necessary to recall a few deep theorems about linear groups, such as the Tits's alternative and the solution to Hilbert's Fifth Problem.

Mention the missing prerequisites from Chapter 6 (pages 63 and 64). Go through the proof of Theorem 7.5 and Corollary 8.4. Finish with the proof of Corollary 8.5, Theorem 8.6 and Corollary 8.7.

TALK 8: Gromov's Theorem for infinitely generated groups. *Chapter 9 of [4].*
In this talk we generalise Gromov's theorem to infinitely generated groups in which every finitely generated subgroup has polynomial growth.

Go over the results in Chapter 9. Of course it would be nice to see as much as possible, but feel free to decide which proofs and/or results should be skipped.

Start of the GRK 2240.

BLOCK B: GROUPS OF INTERMEDIATE GROWTH

TALK 9: Grigorchuk's group. *Chapter 10 of [4].*
In this talk we start to investigate the first known example of a group of intermediate growth: the Grigorchuk group.

Cover Chapter 10 until the beginning of page 97. Alternatively, the same results are presented in [3] (here the Grigorchuk group is represented as a group of automorphisms of a rooted tree).

TALK 10: Amenability and intermediate growth. *Chapter 12 of [4].*
A group is amenable if it admits a non-trivial finite, finitely additive, translation-invariant measure. It is easy to see that amenability is closed under subgroups (S), quotients (Q), extensions (E) and direct limits (D) and finite and abelian groups are amenable. The minimal class of groups that is SQED-closed and contains finite and abelian groups is called the class of "elementary amenable groups". A long standing question asked whether there are amenable groups that are not elementary amenable. In this talk we prove that the Grigorchuk group is amenable but not elementary amenable.

Cover Chapter 12, Section 12.1.

BLOCK C: Uniformly Exponential Growth

For groups of polynomial growth there is a natural notion of "minimal growth speed" (i.e. the degree of the polynomial, see Theorem 4.2 of [4]). In the same spirit, for a group of exponential growth G with a generating set S, we can define the *growth rate* of G w.r.t. S as

$$\omega_S(G) := \lim_{n\to\infty} \sqrt[n]{\gamma_S(n)}$$

and the *minimal growth rate* as $\omega(G) = \inf_S \omega_S(G)$ where the inf is taken over all finite generating sets S of G. We say that G has *uniformly exponential growth* if $\omega(G) > 1$. It turns out that in many classes of groups exponential growth implies uniformly exponential growth, for instance in solvable groups (Talk 11), elementary amenable groups (Talk 12*) and linear groups (Talk 13*). On the other hand, Wilson managed to construct a finitely generated group of non-uniformly exponential growth (Talk 14). We will try to survey a few results in this area.

TALK 11: Uniformly exponential growth in solvable groups. *Chapter 5 of [4] and [5].*
We prove that solvable groups of superpolynomial growth have uniformly exponential growth. This is done by reducing the general case to polycyclic groups.

Follow [5]. The reduction to polycyclic groups can also be found in Chapter 5, Section 5.2 of [4]. Feel free to skip the more technical details.

TALK 12*: Uniform exponential growth in elementary amenable groups. *[6].*

This talk is devoted to prove that every elementary amenable group of superpolynomial growth has uniformly exponential growth.

Many of the technical lemmas and ideas have been laid out in Talk 12. Follow [6]. Skip the geometrical results.

TALK 13*: Uniform exponential growth in other classes of groups, a survey. *[2].*

Here we survey other classes of groups of uniformly exponential growth.

Cover Section 2, Section 3 and Section 5 of [2]. Feel free to browse the references and expand on some of the proofs.

TALK 14: A group of non-uniformly exponential growth. *[1].*

In this talk we look at an example by Bartholdi. The original example was constructed by Wilson in [8].

Go over [1]. It would be nice to also have a comparison between this example and the groups in [8] or [9].

TALK 15*: Another proof of Gromov's theorem. *[7].*

Recently Kleiner obtained a more direct proof of Gromov's theorem using ideas from harmonic analysis.

The main ideas of the proof are outlined in [7]. We would be happy to see the main ideas of the proof without too many technical details.

BLOCK D: Generating functions

The *generating function* of a finitely generated group G w.r.t. a finite generating set S is the complex function

$$A_{G,S}(z) = \sum_{n\in\mathbb{N}} \gamma_S^G(n) z^n, \quad z \in \mathbb{C}.$$

This function depends heavily on the chosen generating set: for some it might be a rational function and for some others a transcendental one, so some generating sets are "nicer" than others.

TALK 16*: A survey on generating functions. *Chapter 14/15 of [4].*

Survey some of the results of Section 14.3 and Chapter 15 of [4].

References

[1] L. Bartholdi. A Wilson Group of Non-Uniformly Exponential Growth.
`https://arxiv.org/abs/math/0210471`

[2] P. De la Harpe. Uniform Growth in Groups of Exponential Growth . *Geom. Dedicata*, 95(1):1–17, 2002. Available here.

[3] R. Grigorchuk and I. Pak. Groups of Intermediate Growth: an Introduction for Beginners.
`https://arxiv.org/abs/math/0607384`.

[4] A. Mann. How groups grow. LMS Lecture Notes Series 395, Cambridge University Press, 2012.

[5] D.V. Osin. The entropy of solvable groups .
`http://www.sci.ccny.cuny.edu/~osin/papers/entsol.pdf`

[6] D.V. Osin. Algebraic entropy of elementary amenable groups.
`https://arxiv.org/pdf/math/0404075.pdf`

[7] T. Tao. A proof of Gromovs theorem.
`https://terrytao.wordpress.com/2010/02/18/a-proof-of-gromovs-theorem/`

[8] J.S. Wilson. On exponential growth and uniformly exponential growth for groups. *Invent. math.* 155:287–303, 2004. Available here.

[9] J.S. Wilson. Further groups that do not have uniformly exponential growth. *J. of Algebra*, 279(1):292–301, 2004.