Advanced Seminar on Group Theory - SoSe 2026

Hyperbolic Groups

The main idea of geometric group theory is that groups can be thought of and studied as geometric objects.
One way to do this is by introducing a metric structure on groups via word metrics on their Cayley graphs, which then allows us to study the large scale geometry of groups with respect to this metric structure.

In particular, one can introduce a notion of negative curvature to large scale geometry via slim triangles, which can be applied to Cayley graphs.
This then leads to the definition of hyperbolic groups as those finitely generated groups with negative curvature.

The study of hyperbolic groups will be the main focus of our Seminar.
In the first half (Talks 1-4) we will introduce all the basic notions required to define hyperbolic groups. In the second half of the seminar (Talks 5-9) we will look at some interesting properties and applications, including the Rips construction and the solvability of the word problem for hyperbolic groups.

We will mainly follow the reference [1].
If you have any questions about the program or you require further details regarding the content of the talks please let me know.
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Talk 1: Cayley graphs and (quasi-)isometries *(Chapter 3 and 5 of [1])*
Introduce Cayley graphs and show how they can be equipped with a metric structure:
Recall the definition of a Cayley graph (Definition 3.2.1) and give some examples (see Example 3.2.2). Give the definitions of isometry and quasi-isometry (Definition 5.1.2 and Definition 5.1.6) as well as some examples. State Proposition 5.1.11. Give the definition of the metric on a graph (Definition 5.2.1) and the word metric on a group (Definition 5.2.3). Show some examples. Prove Proposition 5.2.5 (instead of bilipschitz equivalences just consider quasi-isometries). Give the definition of quasi-isometry types of groups (Definition 5.2.7).

Talk 2: Quasi-geodesics and quasi-isometry invariants *(Chapter 5 of [1])*
Introduce (quasi-)geodesic spaces and define quasi-isometry invariants:
Cover all of section 5.3 ((quasi-)geodesic spaces and geometric realisation of graphs)

as well as subsection 5.6.1 (quasi-isometric invariants).

Talk 3: (Quasi-)Hyperbolic spaces *(Chapter 7 of [1])*
The aim of this talk is to introduce the notion of hyperbolicity for geodesic spaces and then translate this notion to quasi-geometry:
Start by giving a brief overview of classical curvature (no details needed, just a general idea), see section 7.1.
Cover all of subsection 7.2.1 (slim triangles and hyperbolic spaces) and subsection 7.2.2 (quasi-hyperbolic spaces and quasi-invariance of quasi-hyperbolicity).
For the proof of (1) in Proposition 7.2.9 it is enough to provide a sketch, while (2) and (3) should be shown in detail.

Talk 4: Hyperbolic groups *(Chapter 7 of [1])*
Show that hyperbolicity is quasi-isometry invariant and give the definition of hyperbolic groups:
Prove Theorem 7.2.10, state Theorem 7.2.11 (you can skip the proof) and prove Corollary 7.2.13 (quasi-isometry invariance of hyperbolicity).
Cover all of subsection 7.2.4 on hyperbolic graphs.
Introduce the notion of hyperbolic groups (Definition 7.3.1), prove Proposition 7.3.2 and provide examples of hyperbolic groups (see Example 7.3.3).

Talk 5: The word problem in hyperbolic groups *(Chapter 7 of [1])*
The aim of this talk is to show a first property of hyperbolic groups: they have solvable word problem. This is done by showing that all hyperbolic groups admit a Dehn presentation:
Cover all of section 7.4 (you can skip the proof of Lemma 7.4.11).

Talk 6: Elements of infinite order *(Chapter 7 of [1])*
Show that every infinite hyperbolic group contains an element of infinite order:
Prove Theorem 7.5.1, introduce the cone type of a group element (Definition 7.5.2) and give some examples (describe the cone types of $\mathbb{Z}^2$, see Example 7.5.3). Cover Proposition 7.5.4 (sketch the proof, go into as much detail as time allows) and prove Proposition 7.5.6.

Talk 7: Centralisers in hyperbolic groups *(Chapter 7 of [1])*
The aim of this talk is to show that a hyperbolic group cannot contain $\mathbb{Z}^2$ as a subgroup:
State Theorem 7.5.9 (skip the proof) and Caveat 7.5.13, prove Theorem 7.5.10, sketch the proof of Lemma 7.5.14 and prove Corollary 7.5.15. Conclude with Example 7.5.16.

Talk 8: Small cancellation theory *(Ch.V of [2], Appendix of [3], Ch.5 of [4])*
The aim of this talk is to introduce the notions needed for the Rips construction:

Give the definition of cyclically reduced words (see [2] Chapter V, section 2), then see [3] for the definitions of symmetrization and piece (Appendix, §1.1) and give an example. Define the small cancellation conditions (Definition 3 and Definition 4 in [3], Appendix §1.1) and cover Remark 4. Show some examples (see Appendix §1.2 [3]). State Theorem 5.9 of [4] (without proof) and prove Corollary 5.10 of [4].

Talk 9: The Rips Construction *(see [5])*
Describe the Rips construction and show how it can be used to construct particular examples of hyperbolic groups:
State and prove the Theorem in [5], state the Corollary and prove points (a) and (b). Explain what these results mean when applied to the case of hyperbolic groups.

References

[1] C. Löh. *Geometric group theory. An introduction*, Universitext. Springer, 2017.

[2] R. Lyndon, P. Schupp. *Combinatorial Group Theory*, Springer, Berlin, 1977.

[3] E. Ghys, P. de la Harpe (eds.). *Sur les groupes hyperboliques d'apres Mikhael Gromov*, Birkhäuser, 1990.

[4] M. Hull. *Hyperbolic Groups, Lecture Notes*.

[5] E. Rips. *Subgroups of small cancellation groups*, Bull. London Math. Soc. 14 (1982), 45–47.