Group Theory Seminar - Summer Semester 2022
Totally disconnected locally compact groups

Totally disconnected locally compact (short tdlc) groups arise naturally as the quotients of locally compact groups by their identity component. The goal is to understand the theory of tidy subgroups and the scale function introduced by Willis, which gave a big impetus to the area.

TALK 1: Introduction and van Dantzig's Theorem *(24th May 2022)*

Introduce the notion of tdlc groups and the aims of the seminar. Recall basic properties of topological groups and focus of van Danzig's theorem, which is the classical theorem on the structure of tdlc groups. Conclude the talk by giving the first basic examples of tdlc groups and in particular the construction of automorphism groups of a connected locally finite graph.

Sources: [5, Introduction and Section 1.1]; [2, Chapter 2, Section 3.1 and 3.2]. Van Dantzig's theorem could be also found as [3, Theorem 7.7]. First four examples form [2, Section 3.3]. The construction of automorphism groups of a connected locally finite graph is also in [5, Section 1.3].

TALK 2: Cayley-Abels Graphs *(7th June 2022)*

Tdlc groups that are *compactly generated* come with a certain 'geometric' space that they act upon: the Cayley-Abels graph. Define and discuss both compact generation and Cayley-Abels graphs. Prove [5, Theorem 3.5] and its converse, [5, Proposition 3.2].

Sources: [5, Chapter 3] and [2, Section 4].

TALK 3: Neretin Groups *(14th June 2022)*

The Neretin groups are a family of simple tdlc groups. Introduce these as a class of examples. Give emphasis to their definition and prove that they are tdlc. For the remaining time, present some of the further results in [1, Chapter 8].

Sources: [2, Section 3.3] and [1, Chapter 8].

For the remainder of the seminar, we follow Möller's treatment of Willis's theory of the scale function [4]. See also [1, Chapter 9] for an additional source.

TALK 4: Willis' theory. Introduction *(21st June 2022)*

Review the introduction, emphasising on the concept of tidy subgroups and its application. Present the background material on permutation groups and the permutation topology.

Sources: [4, Introduction, Section 1].

TALK 5: Willis' theory. Tidy subgroups *(28th June 2022)*

Willis' theory (5 of [2]) part 2 and Chapter 9 of [1]
Give proofs for the theorems in Section 2 and present the results of Section 3.
Sources: [4, Sections 2 and 3].

TALK 6: Willis' theory. Tidy subgroups *(5th July 2022)*

Given a tdlc group and an element x of G, explain how to construct a compact open subgroup that is tidy for x. If time permits, compare with the construction described by Willis in [6] and [7], and show an example where this difference is highlighted. Introduce the scale function.

Sources: [4, Section 4 and 5].

TALK 7: Willis' theory. The scale function and Tidy subgroups *(12th July 2022)*

Explain the link between tidy subgroups and the scale function, giving examples.
Sources: [4, Section 6].

TALK 8: Willis' theory. More on the scale function *(19th July 2022)*

Present your favourite results of [4, Section 7], e.g. 7.1, 7.2, 7.7.
Source: [4, Section 7].

Programme discussion for next semester *(26th July 2022)*

Main references

[1] P. E. Caprace and N. Monod, *New directions in locally compact groups*, London Mathematical Society Lecture Note Series 447, Cambridge: Cambridge University Press, 2018.

[2] I. Castellano, *An Introduction to totally disconnected locally compact groups*, 2020. `https://www.icms.org.uk/downloads/castellano.pdf`.

[3] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. I, second ed., Grundlehren der Mathematischen Wissenschaften [Funda- mental Principles of Mathematical Sciences], vol. 115, Springer-Verlag, Berlin-New York, 1979.

[4] R. G. Möller, *Structure theory of totally disconnected locally compact groups via graphs and permutations*, Canad. J. Math. Vol. 54 (4), 795–827, 2002.

[5] P. Wesolek, *An Introduction to totally disconnected locally compact groups*, 2018. `https://people.maths.bris.ac.uk/~cb20249/TDLC_Groups_revised.pdf`

[6] G. Willis, *The structure of totally disconnected, locally compact groups*, Math. Ann. 300, no. 2, 341–363, 1994.

[7] G. Willis, *Totally disconnected groups and proofs of conjectures of Hofmann and Mukherjea*, Bull. Austral. Math. Soc. 51, 489–494.

Related literature

[8] D. Van Dantzig, *Zur topologischen Algebra. III. Brouwersche und Cantorsche Gruppen.* Compositio Mathematica, 3, 408–426, 1936.

[9] L. Ribes and P. Zalesskii, *Profinite groups*, Springer, Berlin, Heidelberg, 2000.

Video lecture

[10] I. Castellano, LMS Autumn Algebra school, *Introduction to totally disconnected locally compact groups*, 21 Sept 2020. `https://media.ed.ac.uk/media/Introduction%20to%20totally%20disconnected%20locally%20compact%20groups%20-%20Ilaria%20Castellano/1_jpfm8f2f`