Advanced Seminar on Group Theory - WS 2019/20

Number Theory in Function Fields

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After introducing the Representation Theory of the Symmetric Groups in our five first talks we get an insight into similarities and analogies between the classical number theory and the „Number Theory in Function Fields". The aim of these talks is to see that many results are much easier to prove in the context of function fields than in the classical number theory. For example we don't need to use zero densities/distributions of Zeta-functions to prove a prime number theorem for polynomials over finite fields. In the first three talks we derive a „Prime Number Theorem for polynomials" and a „Prime Number Theorem for polynomials in arithmetic progressions". Then we treat a reciprocity law in our context. Finally, we give an introduction to number theory in function fields and show consequences of the Riemann-Hypothesis in that context. The talks follow mainly chapters 1-5 of Michael Rosen's book [1]. For recalling the results of the classical number theory the books [2]-[4] could be useful.

Talk 1: Overview, Notations and Prime Number Theorem for polynomials

We introduce and recall some notations for polynomials in $\mathbb{F}[X]$ and give an overview about all following talks. Further, we define a Zeta-Function over $\mathbb{F}[X]$, prove a Prime Number Theorem for polynomials and point out the similarities to classical results of number theory [1, p.1-15].

Talk 2: Arithmetic functions for polynomials

Motivated by the classical arithmetic functions we define similar functions in our context and prove some interesting analogous results. Moreover, we introduce Dirichlet-series and L-series over polynomials [1. p-15-19/ p.33-35].

Talk 3: A Prime Number Theorem in arithmetic progressions for polynomials

The aim of this talk is to give an insight to the proof of a weak and a strong form for an „Prime Number Theorem in arithmetic progressions" in our context. Further, we point out the differences and similarities to the classical result [1, p.36-42].

Talk 4: A reciprocity law for polynomials

We define the d-th power residue symbol for polynomials and investigate in which cases the congruence $a \equiv x^d$ modulo an irreducible polynomial P has solutions. Finally, we are led to a reciprocity law for the d-th power residue symbol [1, p.23-30].

Talk 5: Introduction to number theory in function fields

We recall some basic definitions, which are necessary in our context [1, p.45-47]. Moreover, we show some interesting corollaries [47-51] by using the Riemann-Roch Theorem.

Talk 6: The Riemann Hypothesis for function fields

We introduce a Zeta-function for the function field K and show some similarities to the classical Zeta-function, for example an Euler-product. Further, we quote the Riemann Hypothesis for the function field and show a generalization of the Prime Number Theorem, which we had in our first talk [51-59].

References

[1] Michael Rosen, *Number Theory in Function Fields*,
Graduate Texts in Mathematics 210, Springer-Science+Business Media,LLC
https://katalog.ulb.hhu.de/Record/003517618

[2] Jörg Brüdern, *Einführung in die analytische Zahlentheorie*, Springer-Verlag

[3] Harald Scheid / Andreas Frommer, *Zahlentheorie*, Spektrum-Verlag

[4] Frazer Jarvis, *Algebraic Number Theory*, Springer-Verlag
https://link.springer.com/book/10.1007/978-3-319-07545-7