Advanced Seminar on Group Theory
Winter Term 2026/27

1 Overview of topics

The purpose of this seminar is to cover topics in group theory. First, we learn about representation theory of profinite group over finite fields. Further topics will be chosen as the seminar develops. How the seminar will be exactly structured is deliberately left open. The below programme outlines the basic material to be covered, providing directions in which the seminar can evolve. In particular, how much time is exactly dedicated to each topic is subject to be determined within the seminar.

1.1 Representation theory of profinite groups

This part is based on the paper [1] where we outline hereby what material is covered. Recall that profinite groups are topological groups that can be assembled from finite groups in a particular manner. For instance, they include Galois groups (with the Krull topology) and any (cartesian) product of finite groups with the product topology. They unite properties of finite and infinite groups alike. Namely, they all have Sylow subgroups, but they can also be torsion-free. Therefore, one may want to study the representation theory of profinite groups over finite fields.

More specifically, we wish to count representations of a profinite group G using a series called a zeta function. To ensure that the representations of G are comparable, we only consider absolutely irreducible representations $G \to \mathrm{Gl}_n(\mathbb{F}_{p^j})$, meaning irreducible representations that remain irreducible when taken over the algebraic closure $\overline{\mathbb{F}}_p$. The subject of the first part is the Hasse–Weil zeta function ζ_G^{HW} which runs over the number of absolutely irreducible of G in every degree n over all finite fields $\mathbb{F}_{p^j}$. This part on Hasse–Weil zeta functions has three basic goals:

- to establish a probabilistic interpretation of the Hasse–Weil zeta function,
- to determine how the abscissa of convergence of this zeta function behaves under taking subgroups, products, split extension and
- to investigate the Hasse–Weil zeta functions of some free and free abelian profinite groups.

It will be decided during the seminar whether and which further material will be covered in addition to the above goals.

Outline of talks

Talk 1: Introduction to Hasse–Weil zeta functions

This talk gives an overview over the part on representation theory of profinite groups. The basic notions are introduced: profinite groups, pro-p groups, Hasse–Weil zeta functions of profinite groups, abscissae of convergence. We briefly sketch a comparison to the Hasse–Weil zeta functions for algebraic varieties. The three main goals of the first part are spelt out. In particular, we introduce the definitions that are necessary for the probabilistic interpretation of Hasse–Weil zeta functions. This talk is based on a part of Section 1 of [1], but contains some definitions from [2] and [3].

Talk 2: Properties, examples and an interpretation of Hasse–Weil zeta functions

This talk is based on Sections 2 and 3 of [1] and has the following objectives.
First, we show that Hasse–Weil zeta functions converge on a complex half-plane (Corollary 2.3). Then we present formulae for the Hasse–Weil zeta functions of the trivial group (Example 2.4), the profinite completion of the integers (Example 2.5) and for matrix algebras over finite fields (Example 2.6). After providing a formula for the Hasse–Weil zeta function of a product of two groups (Proposition 2.9), we prove a probabilistic interpretation of Hasse–Weil zeta functions (Theorem A).

Talk 3: Finite groups have abscissa of convergence 1

This talk is based on Section 4 up to and including Corollary 4.4 from [1] and has the following objectives.
The bulk of the talk consists in proving two formulae that relate the abscissa of convergence of the Hasse–Weil zeta function of a profinite group to that of an open subgroup (Proposition 4.3). It follows as a corollary that the abscissa of convergence for a finite group is 1 (Corollary 4.4).

Talk 4: Abscissae of convergence for extensions, the zeta function of free abelian profinite groups

This talk is based on the remainder of Section 4 from Lemma 4.6 onward and Subsection 5.1 from [1], which has the following objectives.
First, we prove how the abscissa of convergence of Hasse–Weil zeta functions behaves under taking products of groups (Proposition 4.7). Then we prove how it behaves under taking split extensions (Theorem 4.8). The talk ends in providing a formula for the Hasse–Weil zeta functions of free abelian profinite groups.

Talk 5: Abscissae of convergence for free abelian pro-p groups, representations of free pro-p groups

This talk is based on Subsections 5.2 and 5.4 of [1] and has the following objectives.
The first half establishes a formulae estimating the abscissa of convergence for free abelian

pro-p groups (Proposition 5.2) while the second half provides an estimate for the number of absolutely irreducible representations of free pro-p groups.

1.2 Further directions

Further possible material on Hasse–Weil zeta functions that we can investigate includes:

- A proof that Hasse–Weil zeta functions can realise every positive real number as abscissa of convergence. More specifically, we construct for every positive real number α an explicit example of profinite group for which the abscissa of convergence is α.

- We construct infinitely many examples G that are split extensions of finite-index normal subgroups H such that the abscissa of converge for H is strictly smaller than that of for G.

- We express the Hasse–Weil zeta function of a finite group G in terms of rational irreducible G-representations and Dedeking zeta functions from number theory.

- We prove that the Hasse–Weil zeta functions of virtually abelian groups are rational functions.

1.3 Guest speaker

For this first part, we are planning on inviting a guest speaker to give a talk.

References

[1] G. Corob Cook, S. Kionke, and M. Vannacci. Weil zeta functions of group representations over finite fields. *Selecta Mathematica*, 30(46):1–57, 2024.

[2] E. Detomi and A. Lucchini. Probabilistic non-generators in profinite groups. In *Ischia Group Theory 2004: Proceedings of a Conference in Honor of Marcel Herzog*, Contemporary Mathematics–Israel Mathematical Conference Proceedings, pages 149–158. American Mathematical Society, 2006.

[3] A. Mann. Positively finitely generated groups. *Forum mathematicum*, 8(4):429–460, 1996.