On the behavior of pro-isomorphic zeta functions under base extension

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Zeta functions and motivic integration
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Subgroup growth

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Let G be a finitely generated group. For any $n \geq 1$ it has finitely many subgroups of index n.

Let $a_n^{\leq} = |\{H \leq G : [G : H] = n\}|$. Can consider variations of this sequence:

$$\begin{aligned} a_n^{\triangleleft} &= |\{H \trianglelefteq G : [G : H] = n\}| \\ a_n^{\wedge} &= |\{\widehat{H} \simeq \widehat{G} : [G : H] = n\}|, \end{aligned}$$

where $\widehat{G}$ is the profinite completion of G.

Dirichlet series

Theorem (Lubotzky-Mann-Segal)

Let G be a finitely generated residually finite group. Then there exists C such that $a_n^{\leq} \leq n^C$ for all n if and only if G is virtually solvable of finite rank.

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Example

Let $G = \mathbb{Z}$. Then

$$\zeta_G^{\leq}(s) = \zeta_G^{\triangleleft}(s) = \zeta_G^{\wedge}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}$$

is the Riemann zeta function.

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In this talk we concentrate on pro-isomorphic zeta functions.
Note that the condition $M \simeq L$ does not correspond to closure under the action of some subalgebra of $\mathrm{End}_{\mathbb{Z}}(L)$, so pro-isomorphic zeta functions do not in general fit into Roßmann's framework of subalgebra zeta functions.

Euler decomposition

Theorem (Grunewald-Segal-Smith, 1988)

Let G be a finitely generated torsion-free nilpotent group. Then

$$\zeta^*_G(s) = \prod_p \zeta^*_{G,p}(s),$$

for any $* \in \{\leq, \triangleleft, \wedge\}$, where

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We investigate the behavior of $\zeta_L^\wedge(s)$ under base extension.

Base extension

Our main question

Let Γ be a $\mathbb{Z}$-group scheme such that $\Gamma(\mathbb{Z})$ is finitely generated torsion-free nilpotent. How does $\zeta^*_{G(\mathcal{O}_K)}(s)$ behave as K varies over number fields?

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The simplest example is not encouraging. Let A be an abelian $\mathbb{Z}$-Lie ring of rank m. If $[K : \mathbb{Q}] = d$, then $A \otimes_{\mathbb{Z}} \mathcal{O}_K$ is simply an abelian $\mathbb{Z}$-Lie ring of rank md.

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$$\zeta^{\leq}_{A,p}(s) = \zeta^{\triangleleft}_{A,p}(s) = \zeta^{\wedge}_{A,p}(s) = \frac{1}{(1 - p^{-s})(1 - p^{1-s}) \cdots (1 - p^{m-1-s})}.$$

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Five proofs of this in Lubotzky-Segal, e.g. count Smith normal forms.

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Here $\mathfrak{p}$ runs over the primes of K.
$N\mathfrak{p} = |\mathcal{O}_K/\mathfrak{p}|$ is the norm of $\mathfrak{p}$.
$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1-(N\mathfrak{p})^{-s}}$ is the Dedekind zeta function of K.

Pro-isomorphic zeta functions and p-adic integrals

Our aim: if we know $\zeta^{\wedge}_{L}(s)$, to predict the structure and properties of $\zeta^{\wedge}_{L \otimes \mathcal{O}_K}(s)$. The Heisenberg example suggests that one should be able to do this in some cases; the abelian example suggests it won't be in all cases!

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Let L be a $\mathbb{Z}$-Lie ring, and let $\mathcal{G} = \mathfrak{Aut}\, L$ be its algebraic automorphism group:

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for all field extensions $K/\mathbb{Q}$.

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Theorem

Normalize the Haar measure on $\mathcal{G}(\mathbb{Q}_p)$ so that $\mu(\mathcal{G}(\mathbb{Z}_p)) = 1$ and set $\mathcal{G}^+(\mathbb{Q}_p) = \mathcal{G}(\mathbb{Q}_p) \cap M(\mathbb{Z}_p)$. Then,

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Such p-adic integrals are of independent interest and have been studied for decades (Satake, Tamagawa, Macdonald, etc.)

Algebraic automorphism groups of extensions: the bad ...

Question

Let $\mathcal{L}$ be a $\mathbb{Q}$-Lie algebra ($\mathcal{L} = L \otimes_{\mathbb{Z}} \mathbb{Q}$). Let $\mathfrak{Aut}\,\mathcal{L}$ be its algebraic automorphism group. View $\mathcal{L} \otimes_{\mathbb{Q}} K$ as a $\mathbb{Q}$-algebra. What can we say about the algebraic group $\mathfrak{Aut}\,(\mathcal{L} \otimes_{\mathbb{Q}} K)$ for a number field K?

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This essentially accounts for the bad behavior of $\zeta^{\wedge}_{A_m}(s)$ under base extension.

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It turns out that $\mathfrak{Aut}\,(H \otimes K)$ contains essentially nothing else. We give this phenomenon a name.

Goodness

Definition

Let $\mathcal{L}$ be a $\mathbb{Q}$-Lie algebra and Z a characteristic ideal. We say that $\mathcal{L}$ *is Z-good* if for all finite extensions $K/\mathbb{Q}$:

$$\mathfrak{Aut}\,(\mathcal{L} \otimes_{\mathbb{Q}} K) = \mathrm{Res}_{K/\mathbb{Q}}(\mathfrak{Aut}\,(\mathcal{L})) \cdot (\ker(\mathfrak{Aut}\,\mathcal{L} \to \mathfrak{Aut}\,\mathcal{L}/Z)) \rtimes (\text{finite}).$$

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Proposition

Suppose that $\mathcal{L}$ is Z-good for a central Z. Then for all number fields K there is a fine Euler decomposition

$$\zeta^{\wedge}_{L \otimes_{\mathbb{Z}} \mathcal{O}_K}(s) = \prod_{\mathfrak{p}} \zeta^{\wedge}_{L \otimes_{\mathbb{Z}} \mathcal{O}_K, \mathfrak{p}}(s),$$

where $\mathfrak{p}$ runs over the primes of K and the local factor $\zeta^{\wedge}_{L \otimes_{\mathbb{Z}} \mathcal{O}_K, \mathfrak{p}}(s)$ depends only on the isomorphism class of the local field $K_{\mathfrak{p}}$.

Segal's criterion

A criterion for goodness: for any ideal $I \leq \mathcal{L}$ and subset $S \subset \mathcal{L}$, set

$$C_{\mathcal{L}/I}(S) = \{x \in \mathcal{L} : [s, x] \in I \text{ for all } s \in S\}.$$

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Theorem (Segal, 1989)

Let $\mathcal{L}$ be a k-Lie algebra. Let $Z \subseteq M \subseteq [\mathcal{L}, \mathcal{L}]$ be characteristic ideals of $\mathcal{L}$ such that $\dim(\mathcal{L}/M) > 1$. Set

$$\begin{aligned} \mathcal{X}(M, Z) &= \{x \in \mathcal{L} \setminus M : C_{\mathcal{L}/[M,\mathcal{L}]}(x) = M + kx\} \\ \mathcal{Y}(M, Z) &= \{x \in \mathcal{L} \setminus M : C_{\mathcal{L}/[Z,\mathcal{L}]}(C_{\mathcal{L}/[Z,\mathcal{L}]}(x)) = Z + kx\} \end{aligned}$$

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$\mathcal{X}(M, Z)$ and $\mathcal{Y}(M, Z)$ each generate $\mathcal{L}$ as Lie algebra $\Rightarrow \mathcal{L}$ is Z-good.

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$$C_{\mathcal{L}/I}(S) = \{x \in \mathcal{L} : [s,x] \in I \textit{ for all } s \in S\}.$$

Theorem (Segal, 1989)

Let $\mathcal{L}$ be a k-Lie algebra. Let $Z \subseteq M \subseteq [\mathcal{L},\mathcal{L}]$ be characteristic ideals of $\mathcal{L}$ such that $\dim(\mathcal{L}/M) > 1$. Set

$$\begin{aligned} \mathcal{X}(M,Z) &= \{x \in \mathcal{L} \setminus M : C_{\mathcal{L}/[M,\mathcal{L}]}(x) = M + kx\} \\ \mathcal{Y}(M,Z) &= \{x \in \mathcal{L} \setminus M : C_{\mathcal{L}/[Z,\mathcal{L}]}(C_{\mathcal{L}/[Z,\mathcal{L}]}(x)) = Z + kx\} \end{aligned}$$

$\mathcal{X}(M,Z)$ and $\mathcal{Y}(M,Z)$ each generate $\mathcal{L}$ as Lie algebra $\Rightarrow$ $\mathcal{L}$ is Z-good.

Moral: If $\mathcal{L}$ has many elements whose centralizer is as small as possible, it is Z-good.

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Moral: If $\mathcal{L}$ has many elements whose centralizer is as small as possible, it is Z-good. Grunewald-Segal-Smith applied this result to free nilpotent Lie algebras (note Heisenberg is the free nilpotent algebra of class two on two generators).

Centrally amalgamated copies of Heisenberg I

Recall that

$$\zeta^{\wedge}_{H \otimes \mathcal{O}_K}(s) = \prod_{\mathfrak{p}} \frac{1}{(1 - (N\mathfrak{p})^{2d-2s})(1 - (N\mathfrak{p})^{2d+1-2s})}.$$

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Let H_m be the Lie ring obtained by taking m copies of H and identifying their centers. H_m is spanned by $x_1, \ldots, x_m, y_1, \ldots, y_m, z$, where

$$[x_i, y_j] = \begin{cases} z & : i = j \\ 0 & : i \neq j. \end{cases}$$

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Lemma (du Sautoy and Lubotzky, 1996)

For all $m \geq 1$ we have $\mathfrak{Aut}\, H_m \simeq \left\{ \begin{pmatrix} A & * \\ 0 & \lambda \end{pmatrix} : A\Omega A^T = \lambda \Omega \right\}$, where $\Omega = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$. Note the reductive part is GSp_{2m}.

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Lemma

For $\mathcal{L}$ a nilpotent $\mathbb{Q}$-Lie algebra of class 2, if $\dim_{\mathbb{Q}} \mathcal{L} > 2 \dim_{\mathbb{Q}} [\mathcal{L}, \mathcal{L}] + 1$, then $\mathcal{L}$ fails Segal's criterion for all pairs (M, Z).

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Proposition

Suppose $\mathcal{L}$ is nilpotent and $C_{\mathcal{L}/[Z,\mathcal{L}]}(\mathcal{L}) \subseteq [\mathcal{L}, \mathcal{L}]$. Suppose $\mathcal{L}$ is generated as an algebra by $\mathcal{Y}(Z, Z)$ and also by a finite set $\mathcal{S}$ of elements with centralizer of codimension 1, such that the non-commutation graph of $\mathcal{S}$ is connected (in particular, $\mathcal{L}$ is indecomposable). Suppose a technical condition, that E-linear automorphisms of $\mathcal{L} \otimes K$ are not hopelessly far from being $E \otimes K$-linear. Then $\mathcal{L}$ is Z-good.

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Such integrals have been studied since Satake in the 1960's. It should follow from Igusa (1989) that this is an Igusa function

$$\zeta^{\wedge}_{H_m \otimes \mathcal{O}_K, \mathfrak{p}} = \frac{1}{1-X_0} \sum_{I \subseteq [m-1]} \binom{m}{I}_{(N\mathfrak{p})^{-1}} \prod_{i \in I} \frac{X_i}{1-X_i},$$

where $X_i = (N\mathfrak{p})^{\sum_{j=1}^{i}(m+1-j)+2md-(m+1)s}$ and $d = [K : \mathbb{Q}]$.

Centrally amalgamated copies of Heisenberg IV

Macdonald has formulas for these integrals:

$$\sum_{k=0}^{m} \frac{1}{1-(N\mathfrak{p})^{(k+1)+\cdots+m-2md-(m+1)s}} \prod_{1\leq i<j\leq m} \frac{1-q_{ik}q_{jk}(N\mathfrak{p})^{-1}}{1-q_{ik}q_{jk}} \prod_{i=1}^{m} \frac{1}{1-q_{ik}},$$

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Note that, by contrast, $\zeta^{\triangleleft}_{H_m\otimes\mathcal{O}_K,p}(s)$ has no fine Euler decomposition, but it does not increase in complexity (for fixed K) as m increases, only shifts the numerical data (MMS-Voll, Bauer).

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Challenge

Does there exist a non-good Lie algebra that doesn't have an abelian direct summand?

D^*-Lie algebras

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$$\langle x_1, \dots x_m, y_1, \dots, y_{m+1}, e, f \,|\, [x_i, y_i] = e, [x_i, y_{i+1}] = f \rangle.$$

(Also have a family of even-dimensional algebras, parametrized by primitive polynomials.)

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The pro-isomorphic zeta functions of these Lie algebras were computed by Berman, Klopsch, and Onn. Knowing that these algebras are Z-good, where Z is the center, would enable us to compute the pro-isomorphic zeta functions of their base changes. The proposition above does not apply to these algebras, but a different one, weaker and more technical, does.

A family of maximal class Lie algebras

Let $c \geq 2$, and let $A_c = \langle z, x_1, \ldots, x_m | [z, x_i] = x_{i+1}, 1 \leq i \leq m-1 \rangle$.

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The functional equation has symmetry factor $(N\mathfrak{p})^{c^2+2cd-c-1-(\binom{c+1}{2}+1)s}$.

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Thus we obtain an infinite family of Lie algebras with no functional equation.

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- What does one need to know to determine the abscissa of convergence of $\zeta^{\wedge}_{\mathcal{L}\otimes K}(s)$? Does it always vary linearly with d?

Thank You!