Introduction to p-adic Igusa zeta functions

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Abstract. These notes constitute the content of the series of lectures *Introduction to local zeta functions* by the second author, in the Lluís Santaló Research Summer School 2019: p-Adic Analysis, Arithmetic and Singularities, June 24-28, at the Palacio de la Magdalena in Santander. We want to thank the organizers of the school for their excellent work.

The lectures were intended as an elementary introduction to p-adic Igusa zeta functions and related topics. We hope that these notes reflect that goal. Our text is complementary to the one of León-Cardenal and Zúñiga-Galindo [**LZ**]. The notes of Nicaise [**Ni**] are an introduction to p-adic and motivic zeta functions. Substantial survey articles are the 'old' Bourbaki report of Denef [**De3**] and the more recent paper of Meuser [**Me**].

Contents

1. POLYNOMIAL CONGRUENCES

1.1. Setting and examples

Consider the classical problem of counting the number of solutions of polynomial congruences. More specifically, fix a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$. Given a positive integer m, we are interested in the number of solutions of $f \bmod m$, that is, in the cardinality of the set $\{a \in (\mathbb{Z}/m\mathbb{Z})^n \mid f(a) \equiv 0 \bmod m\} = \{a \in (\mathbb{Z}/m\mathbb{Z})^n \mid f(a) = 0 \text{ in } \mathbb{Z}/m\mathbb{Z}\}$. By the Chinese Remainder Theorem, it suffices to investigate the cases where m is a prime power.

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Theorem 1.1 (Chinese Remainder Theorem). *Let $m = \prod_{j=1}^{r} p_j^{k_j}$ be the decomposition of m as product of primes (where all p_j are different).*

(1) *The map*

$$\frac{\mathbb{Z}}{m\mathbb{Z}} \xrightarrow{\cong} \prod_{j=1}^{r} \frac{\mathbb{Z}}{p_j^{k_j}\mathbb{Z}} : x \bmod m \mapsto (x \bmod p_1^{k_1}, \ldots, x \bmod p_r^{k_r})$$

is an isomorphism of rings.

(2) *Under the induced isomorphism*

$$\left(\frac{\mathbb{Z}}{m\mathbb{Z}}\right)^n \xrightarrow{\cong} \prod_{j=1}^{r} \left(\frac{\mathbb{Z}}{p_j^{k_j}\mathbb{Z}}\right)^n$$

the set $\{a \in (\mathbb{Z}/m\mathbb{Z})^n \mid f(a) \equiv 0 \bmod m\}$ is mapped to $\prod_{j=1}^{r}\{a_j \in (\mathbb{Z}/p_j^{k_j}\mathbb{Z})^n \mid f(a_j) \equiv 0 \bmod p_j^{k_j}\}$.

Given a prime number p, define

$$M_i = M_i(p) := \sharp\left\{a \in \left(\frac{\mathbb{Z}}{p^i\mathbb{Z}}\right)^n \mid f(a) \equiv 0 \bmod p^i\right\}$$

for $i \in \mathbb{Z}_{\geq 0}$ (where $M_0 = 1$). We want to study these values, especially how they vary with i.

From now on, we assume p to be a fixed prime number. All values of M_i are with respect to this prime number p.

Example 1.2. Let $f = y - x^2 \in \mathbb{Z}[x, y]$. It is not hard to see that $M_i = p^i$ for all $i \geq 0$.

Example 1.3. Let $f = xy \in \mathbb{Z}[x, y]$. We claim that $M_i = (i+1)p^i - ip^{i-1}$. To see this, we consider the following table which contains all solutions exactly once.[1]

x	**y**	**number of solutions**
$x = 0$	y is free	$1 \cdot p^i$
$x \neq 0$ but $p^{i-1} \mid x$	$p \mid y$	$(p-1) \cdot p^{i-1}$
$p^{i-1} \nmid x$ but $p^{i-2} \mid x$	$p^2 \mid y$	$(p^2 - p) \cdot p^{i-2}$
$\vdots$	$\vdots$	$\vdots$
$p^2 \nmid x$ but $p \mid x$	$p^{i-1} \mid y$	$(p^{i-1} - p^{i-2}) \cdot p$
$p \nmid x$	$y = 0$	$(p^i - p^{i-1}) \cdot 1$

Summing all values in the last column yields the stated result.

[1] Read the table as follows. We make a case distinction for x. Next, we count for each value of x the values of y for which $x \cdot y \equiv 0 \bmod p^i$.

EXAMPLE 1.4. Let $f = y^2 - x^3 \in \mathbb{Z}[x, y]$. It is not hard to see that $M_1 = p$ (consider the parametrization $t \mapsto (t^2, t^3)$, valid over any field). The next values, less 'uniform in i' than in the previous examples, are as follows.

$$\begin{array}{lll} M_2 = p(2p-1) & M_6 = p^5(p^2+p-1) & M_8 = p^7(2p^2-1) \\ M_3 = p^2(2p-1) & M_7 = p^6(p^2+p-1) & M_9 = p^8(2p^2-1) \\ M_4 = p^3(2p-1) & & M_{10} = p^9(2p^2-1) \\ M_5 = p^4(2p-1) & & M_{11} = p^{10}(2p^2-1) \end{array}$$

EXERCISE 1.5. Let $f = x^2+y^2 \in \mathbb{Z}[x, y]$. Compute the value of M_i for all $i \geq 0$. Hint: make a distinction between the cases $p = 2$, $p \equiv 1 \bmod 4$ and $p \equiv 3 \bmod 4$. In which cases is -1 a square mod p?

REMARK 1.6. In general, it is hard to compute the M_i by hand. Moreover, the examples above are somewhat misleading in the sense that their $M_i(p)$ for fixed i are 'uniform' in p, even of polynomial form. Examples as $f = y^2 - x(x-1)(x-\lambda)$ with $\lambda \neq 0, 1$ and $p \neq 2$ (elliptic curves) already exhibit a more complicated behaviour: here M_1 is of the form $p - 2\sqrt{p}\cos\theta(p)$, where $\theta(p)$ depends in a complicated way on p.

QUESTION 1.7. Is there in general some structure in the values M_i for varying i? Does it suffice to know finitely many such values in order to know all of them?

A partial answer (in easy situations) to Question 1.7 is given by Hensel's Lemma.

LEMMA 1.8 (Hensel's Lemma). *Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$.*

(1) *Let $a \in \mathbb{Z}^n$ such that $f(a) \equiv 0 \bmod p$. If there exists $j \in \{1, \ldots, n\}$ such that $\frac{\partial f}{\partial x_j}(a) \not\equiv 0 \bmod p$, then*

$$\sharp\left\{x \in \left(\frac{\mathbb{Z}}{p^i\mathbb{Z}}\right)^n \mid f(x) \equiv 0 \bmod p^i \text{ and } x \equiv a \bmod p\right\} = p^{(i-1)(n-1)}.$$

(2) *Hence, if all $a \in \mathbb{Z}^n$ for which $f(a) \equiv 0 \bmod p$ have this property, then $M_i = M_1 \cdot p^{(i-1)(n-1)}$. In particular, all M_i are then determined by M_1.*

REMARK 1.9. Example 1.2 is an illustration of Hensel's Lemma.

1.2. Poincaré series

A typical way to encode and study a countable set of numbers is via their generating series.

DEFINITION 1.10. The *Poincaré series* $P_p(f, T)$ is defined as (a slight adaptation of) the generating series of the values M_i, that is,

$$P(T) = P_p(f, T) := \sum_{i \geq 0} M_i (p^{-n}T)^i = \sum_{i \geq 0} \frac{M_i}{p^{ni}} T^i.$$

The factors p^{ni} are the cardinalities of the sets $(\mathbb{Z}/p^i\mathbb{Z})^n$. So, more precisely, the series $P(T)$ is the generating series of the 'counting measure' of the sets $\{a \in (\mathbb{Z}/p^i\mathbb{Z})^n \mid f(a) \equiv 0 \bmod p^i\}$. This interpretation will pop up naturally in the link with the Igusa zeta function.

Example 1.11. Recall Examples 1.2, 1.3 and 1.4.

(1) For $f = y - x^2$, we have

$$P(T) = \sum_{i\geq 0} \frac{M_i}{p^{2i}} T^i = \sum_{i\geq 0} \frac{1}{p^i} T^i = \frac{p}{p-T}\,.$$

More generally, it is easy to compute the Poincaré series when we can apply Hensel's Lemma. Indeed, in that case $M_i = M_1 \cdot p^{(i-1)(n-1)}$ and we obtain

$$P(T) = \sum_{i\geq 0} M_1 \cdot p^{(i-1)(n-1)} (p^{-n}T)^i = M_1 \cdot \sum_{i\geq 0} p^{-i-n+1} T^i = \frac{M_1}{p^{n-1}} \cdot \frac{p}{p-T}\,.$$

(2) For $f = xy$, one computes $P(T) = \frac{p^2 - T}{(p-T)^2}$.

(3) For $f = y^2 - x^3$, we claim that $P(T) = \frac{p^6 + (p^4 - p^3)T^2 - T^6}{(p-T)(p^5 - T^6)}$. This will be discussed later (in Exercise 4.12).

Conjecture 1.12 (Borewicz-Šafarevič, 1966 [**BS**]). The Poincaré series $P(T)$ is a rational function in T.

1.3. Relation with p-adics

Igusa proved the conjecture above in 1975 [**Ig1**]. His strategy was relating $P(T)$ to a p-adic integral and to use change of variables (geometrically given by a resolution of singularities) to study this integral. As a consequence, Question 1.7 has a positive answer!

Remark 1.13. The rings $\mathbb{Z}/p^i\mathbb{Z}$ in the polynomial congruences introduced above have a natural link with the ring of p-adic integers $\mathbb{Z}_p$ (and its fraction field $\mathbb{Q}_p$); indeed, $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i\mathbb{Z}$ via the natural projections

$$\frac{\mathbb{Z}}{p^{i+1}\mathbb{Z}} \to \frac{\mathbb{Z}}{p^i\mathbb{Z}} : a_0 + a_1 p + \cdots + a_{i-1}p^{i-1} + a_i p^i \mapsto a_0 + a_1 p + \cdots + a_{i-1}p^{i-1},$$

where all $a_\ell \in \{0, \ldots, p-1\}$, inducing also the projections

$$\mathbb{Z}_p \xrightarrow{\pi} \frac{\mathbb{Z}_p}{p^i\mathbb{Z}_p} \cong \frac{\mathbb{Z}}{p^i\mathbb{Z}} \qquad \text{and} \qquad \mathbb{Z}_p^n \xrightarrow{\pi_n} \left(\frac{\mathbb{Z}_p}{p^i\mathbb{Z}_p}\right)^n \cong \left(\frac{\mathbb{Z}}{p^i\mathbb{Z}}\right)^n.$$

Notation 1.14. In the subsequent sections we denote by $\mathrm{ord}_p(\cdot)$ and $|\cdot|_p = p^{-\mathrm{ord}_p(\cdot)}$ the p-order and the standard p-adic norm on $\mathbb{Q}_p$, respectively. That is, writing $z \in \mathbb{Q}_p, z \neq 0$, as $z = p^k u$ with $k \in \mathbb{Z}$ and u a unit in $\mathbb{Z}_p$, we have $\mathrm{ord}_p(z) = k$ and $|z|_p = p^{-\mathrm{ord}_p(z)} = p^{-k}$. Furthermore, $\mathrm{ord}_p(0) = +\infty$ and $|0|_p = 0$. Also, $|dx| = |dx_1 dx_2 \ldots dx_n|$ denotes the Haar measure on $\mathbb{Q}_p^n$, normalized such that $|dx|(\mathbb{Z}_p^n) = 1$.

Remark 1.15. Consider $C \subseteq (\mathbb{Z}_p/p^i\mathbb{Z}_p)^n \cong (\mathbb{Z}/p^i\mathbb{Z})^n$. The inverse image $\pi_n^{-1}(C)$ lives in $\mathbb{Z}_p^n$ and is called a *cylindrical set* or *cylinder*. Then

$$|dx|(\pi_n^{-1}(C)) = \frac{\sharp C}{p^{in}}.$$

Indeed, taking for each $\bar{a} \in C$ a fixed representative $a \in \mathbb{Z}_p^n$, the set $\pi_n^{-1}(C)$ is the disjoint union of the sets $a + (p^i\mathbb{Z}_p)^n, \bar{a} \in C$, all with measure p^{-in}.

2. p-ADIC IGUSA (LOCAL) ZETA FUNCTIONS

2.1. Definition and examples

Definition 2.1. Consider $f \in \mathbb{Q}_p[x_1, \dots, x_n]$ and $s \in \mathbb{C}$ with $\Re(s) > 0$. The *Igusa zeta function* of f is defined as

$$Z(s) = Z_p(f; s) := \int_{\mathbb{Z}_p^n} |f(x)|_p^s \, |dx|.$$

More generally, one can integrate over 'balls' $c + (p^e\mathbb{Z}_p)^n$ or $c + (\prod_{i=1}^n p^{e_i}\mathbb{Z}_p)$ for some $c \in \mathbb{Q}_p^n$, or even consider

$$\int_{\mathbb{Q}_p^n} \varphi(x)|f(x)|_p^s \, |dx|,$$

where φ is a *test function*, i.e., a locally constant function with compact support. Observe that $Z(s)$ is in fact this last integral with φ equal to the characteristic function of $\mathbb{Z}_p^n$.

Remark 2.2. Write $s = a + bi$ with $a, b \in \mathbb{R}$. Then

$$Z(s) = \int_{\mathbb{Z}_p^n} |f(x)|_p^a \, |f(x)|_p^{bi} \, |dx|.$$

Observe that $|f(x)|_p^{bi}$ has modulus 1. Moreover, since f is a continuous function and $\mathbb{Z}_p^n$ is compact, $|f(x)|_p^a$ is bounded for $a > 0$. Hence, $Z(s)$ is indeed well-defined if $\Re(s) > 0$.

At this point, it is also not difficult to show that $Z(s)$ is a holomorphic function in the domain $\Re(s) > 0$. We will prove a stronger result in Theorem 3.7.

Remark 2.3. The terminology in the literature is sometimes confusing. One uses the adjectives *p-adic*, *Igusa* and/or *local* in all possible combinations to indicate the zeta function $Z(s)$. Here 'local' refers to the fact that $\mathbb{Q}_p$ is a local field.

However, when it is clear that one works in the p-adic setting, the term local zeta function could also mean the integral $\int_{(p\mathbb{Z}_p)^n} |f(x)|_p^s \, |dx|$ (or, more generally, $\int_{(p^e\mathbb{Z}_p)^n} |f(x)|_p^s \, |dx|$ with $e \geq 1$), where now 'local' refers to the fact that the integration domain is a smaller neighbourhood of the origin.

Remark 2.4. In the sequel we will also consider such integrals for some $s \in \mathbb{C}$ with $\Re(s) \leq 0$. Strictly speaking, we then integrate over $\{x \in \mathbb{Z}_p^n \mid f(x) \neq 0\}$. But, since the excluded set $\{x \in \mathbb{Z}_p^n \mid f(x) = 0\}$ has measure zero, we keep notation as before.

The following example shows that this integral might be ill-defined if $\Re(s) \leq 0$.

Example 2.5. Let $f = x \in \mathbb{Q}_p[x]$ and $s = -1 \in \mathbb{C}$. In order to compute $\int_{\mathbb{Z}_p} |x|^{-1}|dx|$, we consider the partition

$$\begin{aligned} \mathbb{Z}_p \setminus \{0\} &= \{x \in \mathbb{Q}_p \mid \exists l \geq 0 \text{ such that } |x|_p = p^{-l}\} \\ &= \bigsqcup_{l \geq 0} \{x \in \mathbb{Q}_p \mid |x|_p = p^{-l}\}. \end{aligned}$$

Denote $\{x \in \mathbb{Q}_p \mid |x|_p = p^{-l}\}$ by A_l and observe that

$$|dx|(A_l) = |dx| \left(\{x \in \mathbb{Q}_p \mid |x|_p \leq p^{-l}\} \setminus \{x \in \mathbb{Q}_p \mid |x|_p \leq p^{-l-1}\}\right) = p^{-l} - p^{-l-1}.$$

Therefore, since $\{0\}$ has measure zero, we have

$$\int_{\mathbb{Z}_p} |x|_p^{-1}|dx| = \sum_{l\geq 0}\int_{A_l} |x|_p^{-1}|dx| = \sum_{l\geq 0}\int_{A_l} p^l\,|dx| = \sum_{l\geq 0} p^l \int_{A_l} |dx|$$
$$= \sum_{l\geq 0} p^l(p^{-l} - p^{-l-1}) = \sum_{l\geq 0}(1-p^{-1}) = +\infty.$$

Exercise 2.6. On the other hand, $Z(s)$ can be well defined in a larger half-plane than $\Re(s) > 0$. Verify the following basic examples; they will be important later on.

(1) $\int_{\mathbb{Z}_p} |x|_p^s\,|dx| = \frac{1-p^{-1}}{1-p^{-(s+1)}}$ when $\Re(s) > -1$.
(2) More generally, $\int_{p^e\mathbb{Z}_p} |x|_p^s\,|dx| = \frac{(1-p^{-1})p^{-e(s+1)}}{1-p^{-(s+1)}}$ when $\Re(s) > -1$.
(3) And still more generally (for positive integers N and ν),

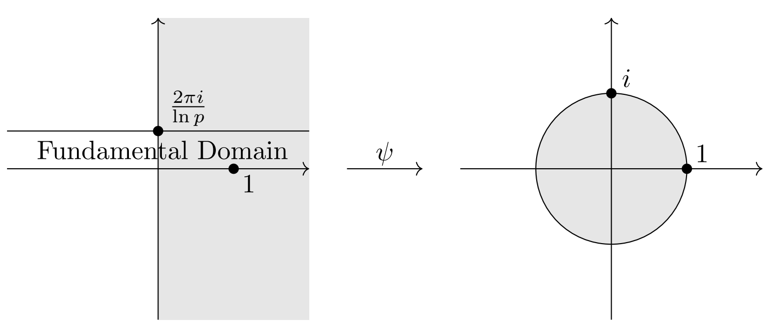
$$\int_{p^e\mathbb{Z}_p} |x|_p^{Ns+\nu-1}|dx| = \frac{(1-p^{-1})p^{-e(Ns+\nu)}}{1-p^{-(Ns+\nu)}}$$

when $\Re(s) > -\frac{\nu}{N}$.

Remark 2.7. We have

$$Z(s) = Z_p(f;s) = \int_{\mathbb{Z}_p^n} |f(x)|_p^s\,|dx| = \int_{\mathbb{Z}_p^n} \left(p^{-\mathrm{ord}_p(f(x))}\right)^s |dx|$$
$$= \int_{\mathbb{Z}_p^n} p^{-s\cdot\mathrm{ord}_p(f(x))}|dx| = \int_{\mathbb{Z}_p^n} \left(p^{-s}\right)^{\mathrm{ord}_p(f(x))} |dx|.$$

Hence, the Igusa zeta function $Z(s)$ is actually a function of p^{-s}. From now on we will write $t = p^{-s}$ and $Z'(t) = Z'(p^{-s}) := Z(s)$. The change of coordinates is given by the map $\psi : \mathbb{C} \to \mathbb{C}^* : s \mapsto t = p^{-s}$.



The relations between the parameters s and t are given in the following table.

parameter s	parameter $\mathbf{t} = \boldsymbol{\psi}(\mathbf{s}) = \mathbf{p}^{-\mathbf{s}}$
$\Re(s) > 0$	$0 < \lvert\psi(s)\rvert < 1$
$\Re(s) = 0$	$\lvert\psi(s)\rvert = 1$
$\Re(s) < 0$	$\lvert\psi(s)\rvert > 1$
$s - s' \in \frac{2\pi i}{\ln p}\mathbb{Z}$	$\psi(s) = \psi(s')$

2.2. Link with Poincaré series

PROPOSITION 2.8. *Let $f \in \mathbb{Z}_p[x_1, \ldots, x_n]$. Consider its Poincaré series $P(T)$ as given in Definition 1.10. Then*

$$Z(s) = \frac{(p^{-s} - 1)P(p^{-s}) + 1}{p^{-s}}$$

or, equivalently,

$$P(t) = \frac{tZ'(t) - 1}{t - 1}.$$

PROOF. Rewrite the Igusa zeta function in the following way:

$$Z(s) = \int_{\mathbb{Z}_p^n} |f(x)|_p^s \, |dx| = \sum_{l \geq 0} \int_{\mathrm{ord}_p f(x) = l} p^{-ls} |dx| = \sum_{l \geq 0} p^{-ls} \int_{\mathrm{ord}_p f(x) = l} |dx|$$
$$= \sum_{l \geq 0} p^{-ls} \Big(|dx|(\{x \in \mathbb{Z}_p^n \mid \mathrm{ord}_p f(x) \geq l\}) - |dx|(\{x \in \mathbb{Z}_p^n \mid \mathrm{ord}_p f(x) \geq l + 1\}) \Big).$$

Recall the projection (see Remark 1.13)

$$\mathbb{Z}_p^n \xrightarrow{\pi_n} \left(\frac{\mathbb{Z}}{p^l \mathbb{Z}} \right)^n : x \mapsto x \bmod p^l$$

and observe that $\{x \in \mathbb{Z}_p^n \mid \mathrm{ord}_p(f(x)) \geq l\} = \pi_n^{-1}(\{x \in (\mathbb{Z}/p^l\mathbb{Z})^n \mid f(x) \equiv 0 \bmod p^l\})$. By Remark 1.15 we see that

$$|dx| \left(\pi_n^{-1} \left(\left\{ x \in \left(\frac{\mathbb{Z}}{p^l \mathbb{Z}} \right)^n \mid f(x) \equiv 0 \bmod p^l \right\} \right) \right) = \frac{M_l}{p^{ln}}.$$

Hence

$$Z(s) = \sum_{l \geq 0} p^{-ls} \left(\frac{M_l}{p^{ln}} - \frac{M_{l+1}}{p^{(l+1)n}} \right) = \sum_{l \geq 0} p^{-ls - ln} M_l - \sum_{l \geq 0} p^{-ls - (l+1)n} M_{l+1}$$
$$= \sum_{l \geq 0} \left(p^{-s} p^{-n} \right)^l M_l - \sum_{l \geq 1} p^{-(l-1)s - ln} M_l = P(p^{-s}) - p^s \sum_{l \geq 1} \left(p^{-s} p^{-n} \right)^l M_l$$
$$= P(p^{-s}) - p^s (P(p^{-s}) - 1) = \frac{(p^{-s} - 1)P(p^{-s}) + 1}{p^{-s}}.$$

The other equality follows immediately. □

2.3. Igusa's Stationary Phase Formula

A possible strategy to compute p-adic integrals –which can be practical or (in)famously hard– is Igusa's Stationary Phase Formula. The starting point of this formula is the obvious equality

$$\int_{\mathbb{Z}_p^n} |f(x)|_p^s \, |dx| = \sum_{a \in (\mathbb{Z}/p\mathbb{Z})^n} \int_{a+(p\mathbb{Z}_p)^n} |f(x)|_p^s \, |dx|, \tag{2.1}$$

where we use the same notation for $a \in (\mathbb{Z}/p\mathbb{Z})^n$ and a chosen representative $a \in \mathbb{Z}_p^n$. We denote $\int_{a+(p\mathbb{Z}_p)^n} |f(x)|_p^s \, |dx|$ by I_a. We partition the values of a as follows:

(1) a such that $f(a) \not\equiv 0 \bmod p$,
(2) a such that $f(a) \equiv 0 \bmod p$ and there exists $j \in \{1, \ldots, n\}$ such that $\frac{\partial f}{\partial x_j}(a) \not\equiv 0 \bmod p$,
(3) a such that $f(a) \equiv 0 \bmod p$ and for all $j \in \{1, \ldots, n\} : \frac{\partial f}{\partial x_j}(a) \equiv 0 \bmod p$.

The first two cases are easy to compute:

(1) $I_a = \int_{a+(p\mathbb{Z}_p)^n} 1^s |dx| = |dx| \left(a + (p\mathbb{Z}_p)^n\right) = p^{-n}$,
(2)
$$\begin{aligned} I_a &= \int_{a+(p\mathbb{Z}_p)^n} |f(x)|_p^s \, |dx| \\ &= \sum_{l \geq 0} p^{-ls} |dx| \left(\{x \in a + (p\mathbb{Z}_p)^n \mid \operatorname{ord}_p f(x) = l\}\right) \\ &= \sum_{l \geq 1} p^{-ls} |dx| \left(\{x \in a + (p\mathbb{Z}_p)^n \mid \operatorname{ord}_p f(x) = l\}\right) \\ &= \sum_{l \geq 1} p^{-ls} \left(\frac{M_l}{p^{ln}} - \frac{M_{l+1}}{p^{(l+1)n}}\right) \\ &= \sum_{l \geq 1} p^{-ls} \left(\frac{p^{(n-1)(l-1)}}{p^{ln}} - \frac{p^{(n-1)l}}{p^{(l+1)n}}\right) \\ &= (p-1)p^{-n} \sum_{l \geq 1} \left(p^{-s-1}\right)^l \\ &= (p-1)p^{-n} \frac{p^{-s-1}}{1 - p^{-s-1}}. \end{aligned}$$

For the fourth equality, we used a result that is explained in the proof of Proposition 2.8, and for the fifth one, we used Hensel's Lemma (Lemma 1.8).

Theorem 2.9 (Igusa's Stationary Phase Formula, 1994 [**Ig3**]). *Define*

$$V := \left\{ a \in \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^n \mid f(a) \equiv 0 \bmod p \right\},$$
$$S := \left\{ a \in V \mid \frac{\partial f}{\partial x_j}(a) \equiv 0 \bmod p \textit{ for all } j \in \{1, \ldots, n\} \right\}.$$

Then

$$Z(s) = \frac{p^n - \sharp V}{p^n} + (\sharp V - \sharp S) \frac{p-1}{p^n} \frac{p^{-s-1}}{1 - p^{-s-1}} + \sum_{a \in S} \int_{a+(p\mathbb{Z}_p)^n} |f(x)|_p^s \, |dx|.$$

PROOF. Use equation (2.1) and split the sum according to the three cases discussed above. Next, use the previous computations for the cases (1) and (2). This yields the formula. $\square$

EXAMPLE 2.10. We compute the integral $I := \int_{\mathbb{Z}_p^3} |xy - z^2|_p^s \, |dxdydz|$ (for $p \neq 2$) using Theorem 2.9. First, remark that $\sharp V = p^2$. Indeed, if $x = 0$, then y is free and $z = 0$. If $x \neq 0$, then $y = x^{-1}z^2$ and hence x is free (non-zero), z is free and y is fixed. Therefore, $\sharp V = 1 \cdot p \cdot 1 + (p-1) \cdot 1 \cdot p = p^2$. Remark that $\sharp S = 1$ since $S = \{(0,0,0)\}$. Using Igusa's Stationary Phase Formula, we have

$$I = \frac{p^3 - p^2}{p^3} + (p^2 - 1)\frac{p-1}{p^3}\frac{p^{-s-1}}{1 - p^{-s-1}} + \int_{(p\mathbb{Z}_p)^3} |xy - z^2|_p^s \, |dxdydz|.$$

Since $xy - z^2$ is a homogeneous polynomial of degree two, we can rewrite the last integral, using the change of coordinates $(x, y, z) = (px', py', pz')$, as

$$\int_{\mathbb{Z}_p^3} |p^2|_p^s \, |x'y' - (z')^2|_p^s \, |p^3|_p \, |dx'dy'dz'| = p^{-2s-3} I,$$

and thus

$$I = \frac{p^3 - p^2}{p^3} + (p^2 - 1)\frac{p-1}{p^3}\frac{p^{-s-1}}{1 - p^{-s-1}} + p^{-2s-3} I.$$

An easy calculation then yields

$$I = \frac{p-1}{p} \cdot \frac{1 - p^{-s-3}}{(1 - p^{-s-1})(1 - p^{-2s-3})}.$$

There are some intriguing open problems concerning this formula.

(1) Prove for arbitrary f that $Z(s)$ is rational in p^{-s} using Igusa's Stationary Phase Formula. Recall that this result was already proven by Igusa using other techniques [**Ig1**], see Theorem 3.7.
(2) Show that one can compute $Z(s)$ explicitly for arbitrary f using finitely many iterations of Igusa's Stationary Phase Formula.

These two problems are probably strongly related, but we are not aware of a clear strategy to attack them. In concrete examples, repeated application of Theorem 2.9 often results in an explicit expression for $Z(s)$, which is a rational function in p^{-s}. Typically such an expression arises in two possible ways. Finitely many iterations either yield immediately a concrete expression, or at some point the original integral pops up again (one or more times) as in Example 2.10, and then rewriting the resulting relation yields a concrete expression.

Can this experimental fact lead to a theorem for general f?

EXERCISE 2.11. Compute in some way $\int_{\mathbb{Z}_p^2} |x^2 + y^2|_p^s \, |dxdy|$ (depending on p, there are three cases). Verify that your findings are consistent with Exercise 1.5, using Proposition 2.8.

3. p-ADIC MANIFOLDS AND RATIONALITY OF THE ZETA FUNCTION

3.1. p-adic manifolds

We only state briefly the essential features of p-adic manifolds, and refer for a detailed explanation to [**Ig4**, Section 2.4]. Anyway, this notion is analogous to classical real or complex analytic manifolds, defined via appropriate atlases.

Definition 3.1 (Informal). An n-dimensional $\mathbb{Q}_p$*-analytic manifold* or p*-adic manifold* is a Hausdorff topological space that is locally (analytically) of the form $c + (p^e\mathbb{Z}_p)^n$ for some $c \in \mathbb{Q}_p^n$.

Remark 3.2. A special and crucial feature in the p-adic case is that any n-dimensional p-adic manifold can be covered by (two by two) *disjoint* balls as above.

We want to mention the following remarkable structure theorem due to Serre (although it will not be used in the sequel).

Theorem 3.3 (Serre, 1965 [**Se**]). *Let X be a compact n-dimensional p-adic manifold. Then X is isomorphic (as a p-adic manifold) to the disjoint union of r times $\mathbb{Z}_p^n$, for a unique $r \in \{1, \ldots, p-1\}$.*

Part of the proof. For the existence of such r, observe that X is compact and thus a disjoint union of finitely many sets of the form $c + (p^e\mathbb{Z}_p)^n$ by Remark 3.2. That is, $X = \bigsqcup_{i=1}^k c_i + (p^{e_i}\mathbb{Z}_p)^n$ for some positive k.

Moreover, we have the following isomorphism of p-adic manifolds:

$$c + (p^e\mathbb{Z}_p)^n \xrightarrow{\cong} Z_p^n : c + p^e y \mapsto y.$$

Therefore, X is isomorphic to $\bigsqcup_{i=1}^k \mathbb{Z}_p^n$. Recall now that

$$\mathbb{Z}_p = p\mathbb{Z}_p \sqcup (1 + p\mathbb{Z}_p) \sqcup \cdots \sqcup ((p-1) + p\mathbb{Z}_p) \cong \bigsqcup_{j=1}^{p} \mathbb{Z}_p.$$

Similarly, we have that $\mathbb{Z}_p^n$ is isomorphic (as a p-adic manifold) to $\bigsqcup_{j=1}^p \mathbb{Z}_p^n$. (Indeed, $\mathbb{Z}_p^n \cong \mathbb{Z}_p^{n-1} \times \mathbb{Z}_p \cong \mathbb{Z}_p^{n-1} \times \bigsqcup_{j=1}^p \mathbb{Z}_p \cong \bigsqcup_{j=1}^p \mathbb{Z}_p^n$.)

Hence, if k is larger than $p-1$ in $\bigsqcup_{i=1}^k \mathbb{Z}_p^n$, we can reduce k by $p-1$ using the isomorphism above. Doing this until we obtain $1 \le k \le p-1$ yields the existence of r. □

For the proof of uniqueness of r we refer to [**Se**]. Nowadays r is called the (p-adic) Serre invariant of X.

Remark 3.4. On an n-dimensional p-adic manifold, one associates a measure to a given $\mathbb{Q}_p$-analytic differential form of degree n, see e.g. [**Ig3**, Section 7.4] or [**Se**]. For example, the Haar measure $|dx|$ on $\mathbb{Q}_p^n$ is associated to the differential form $dx_1 \wedge \cdots \wedge dx_n$. In what follows, we will use this correspondence.

3.2. Resolution: analytic version

The statement below on embedded resolution follows essentially from Hironaka's work [28]. See also [**BEV**, Section 8], [**ENV**, Section 5], [**Wł2**], [**Wł3**], and especially [**DvdD**, Theorem 2.2] for this p-adic setting.

Theorem 3.5 (Embedded Resolution of Singularities; p-adic analytic version). *Let $f \in \mathbb{Q}_p[x_1, \dots, x_n] \setminus \mathbb{Q}_p$ and consider its zero locus $\{f = 0\} \subseteq \mathbb{Q}_p^n$. Then there exists an n-dimensional p-adic manifold Y, a proper $\mathbb{Q}_p$-analytic map $h : Y \to \mathbb{Q}_p^n$, and finitely many closed $(n-1)$-dimensional submanifolds $E_i, i \in I$, of Y, equipped with* numerical data $(N_i, \nu_i) \in (\mathbb{Z}_{>0})^2, i \in I$, *satisfying*

(1) *h is an isomorphism of p-adic manifolds outside the inverse image by h of the singular locus of $\{f = 0\}$,*
(2) $h^{-1}(\{f = 0\}) = \cup_{i \in I} E_i$,
(3) *for all $b \in Y$ there exist local coordinates $(y_1, \dots, y_n)$ around b such that, if $b \in E_1, \dots, E_r$ (relabel if necessary), then $y_i = 0$ is the equation of E_i for $1 \leq i \leq r$ (i.e., $h^{-1}(\{f = 0\})$ is a normal crossings divisor), and such that*

$$f \circ h = \varepsilon(y_1, \dots, y_n) \prod_{i=1}^{r} y_i^{N_i}, \text{ and}$$

$$h^*(dx_1 \wedge \cdots \wedge dx_n) = \eta(y_1, \dots, y_n) \prod_{i=1}^{r} y_i^{\nu_i - 1} dy_1 \wedge \cdots \wedge dy_n$$

on some neighbourhood of b, where $\varepsilon(y_1, \dots, y_n)$ and $\eta(y_1, \dots, y_n)$ are invertible.

Such a map h is called an embedded resolution of f.

Remark 3.6. It is possible to construct the map h as a composition of finitely many *admissible* blow-ups. More precisely, each centre of blow-up is a closed submanifold C of codimension at least 2 in the ambient manifold, having normal crossings with the exceptional components created by previous blow-ups. That is, there are local coordinates $z_1, \dots, z_n$ around each point c of C, such that each previously created exceptional component containing c is given by $z_i = 0$ for some $i \in \{1, \dots, n\}$, and C is given by $z_{i_1} = \cdots = z_{i_d} = 0$ with $d \geq 2$ and $\{i_1, \dots, i_d\} \subseteq \{1, \dots, n\}$.

3.3. Rationality and candidate poles

Theorem 3.7 (Igusa, 1974 [**Ig1**]). *Consider $f \in \mathbb{Q}_p[x_1, \dots, x_n]$. Then the following statements hold.*

(1) *The Igusa zeta function $Z(s)$ is a rational function in $p^{-s} = t$. Therefore, it has a meromorphic continuation to the whole of $\mathbb{C}$.*
(2) *Fix an embedded resolution h of f and use notation as in Theorem 3.5. Then*

$$Z(s) = \frac{Q(p^{-s})}{\prod_{j \in I}(1 - p^{-(N_j s + \nu_j)})},$$

or, equivalently,

$$Z'(t) = \frac{Q(t)}{\prod_{j \in I}(1 - p^{-\nu_j} t^{N_j})},$$

where Q is a polynomial with rational coefficients.
(3) *The poles of $Z(s)$ or $Z'(t)$ have order at most n.*

In particular, the poles of $Z(s)$ are contained in the list

$$-\frac{\nu_j}{N_j} + \frac{2\pi k}{(\ln p) N_j} i, \ j \in I \text{ and } k \in \mathbb{Z},$$

or, equivalently, the poles of $Z'(t)$ are contained in the list

$$\xi p^{\nu_j/N_j},\ j \in I \text{ and } \xi^{N_j} = 1.$$

Proof. We use a fixed embedded resolution $h : Y \to \mathbb{Q}_p^n$ to perform a classical change of variables computation (for a proof in the p-adic setting, see e.g. [**Ig4**, Proposition 7.4.1]):

$$\begin{aligned} Z(s) &= \int_{\mathbb{Z}_p^n} |f(x)|_p^s \, |dx| \\ &= \int_{h^{-1}(\mathbb{Z}_p^n)} |(f \circ h)(y)|_p^s \, |\mathrm{Jac}\, h(y)|_p \, |dy| \\ &= \int_{h^{-1}(\mathbb{Z}_p^n)} |(f \circ h)(y)|_p^s \, |h^* dx|. \end{aligned}$$

Observe that the p-adic manifold $h^{-1}(\mathbb{Z}_p^n)$ is compact because h is proper. Therefore, we can write $h^{-1}(\mathbb{Z}_p^n)$ as a disjoint union of finitely many sufficiently small (open and compact) 'balls' B, that can be considered as neighbourhoods of points b as in Theorem 3.5. We can take those balls of the form $(p^e Z_p)^n$, but for concrete computations it is sometimes useful to allow more flexibility and to take balls B of the form $\prod_{j=1}^n p^{e_j}\mathbb{Z}_p$. We compute the integral over such a fixed ball/neighbourhood:

$$\int_B \left| \varepsilon(y) \prod_{j=1}^r y_j^{N_j} \right|_p^s \left| \eta(y) \prod_{j=1}^r y_j^{\nu_j - 1} \right|_p |dy| = \int_B |\varepsilon(y)|_p^s \, |\eta(y)|_p \left| \prod_{j=1}^r y_j^{N_j s + \nu_j - 1} \right|_p |dy|.$$

We may suppose (by taking the ball small enough) that $|\varepsilon(y)|_p$ and $|\eta(y)|_p$ are constant. Indeed, ε and η (and their norms) are continuous functions and their value at b is non-zero. Hence, since locally the p-adic norms $|\varepsilon(y)|_p$ and $|\eta(y)|_p$ take discrete values (integer powers of p), they must be constant on a sufficiently small neighbourhood. Then we can compute the integral above by separating variables as

$$|\varepsilon(b)|_p^s \, |\eta(b)|_p \prod_{j=1}^r \int_{p^{e_j} Z_p} |y_j|_p^{N_j s + \nu_j - 1} |dy_j| \prod_{j=r+1}^n \int_{p^{e_j} Z_p} |dy_j|.$$

Since $|\varepsilon(b)|$, $|\eta(b)|$ and $\prod_{j=r+1}^n \int_{p^{e_j} Z_p} |dy_j|$ are all constants (integer powers of p), the integral is equal to the product of a constant, an integer power of p^{-s}, and the expression

$$\prod_{j=1}^r \frac{(1-p^{-1})p^{-e_j(N_j s + \nu_j)}}{1 - p^{-(N_j s + \nu_j)}},$$

using Exercise 2.6. Summing these contributions over all balls B, and putting everything on a common denominator, we obtain that $Z(s)$ is a rational function in p^{-s} of the described form.

Moreover, it is clear that all contributions have poles of order at most n. □

Remark 3.8. For a polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ and $s \in \mathbb{C}$ with $\Re(s) > 0$, one classically studies integrals

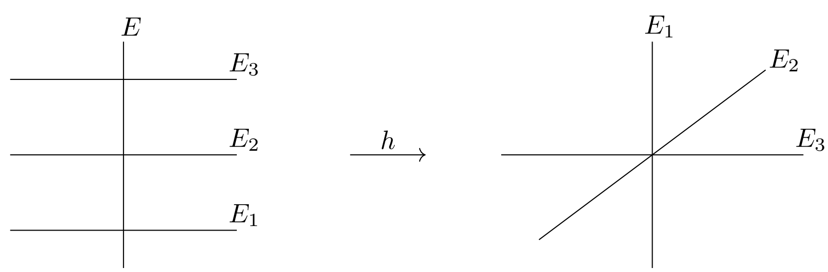
$$\int_{\mathbb{R}^n} \varphi(x) |f(x)|^s \, |dx|,$$

analogous to those in Definition 2.1, where now the test function φ is a C^∞ function with compact support and $|dx|$ the Lebesgue measure on $\mathbb{R}^n$ (and similarly for a complex polynomial). It is again easy to verify that such an integral is a holomorphic function of s in the domain $\Re(s) > 0$. I. Gel'fand asked in 1954 whether there exists a meromorphic continuation of this integral to the whole of $\mathbb{C}$. Independently, Bernstein and S. Gel'fand [**BG**] in 1969 and Atiyah [**At**] in 1970 proved this using resolution of singularities. Alternatively, Bernstein [**Be**] showed it using Bernstein-Sato polynomials in 1972. Note that both techniques were not yet available in 1954.

Remark 3.9. The possible poles of $Z(s)$, namely $\{-\frac{\nu_j}{N_j} + \frac{2\pi k}{(\ln p)N_j} i \mid j \in I, k \in \mathbb{Z}\}$, are called the candidate poles (associated to the resolution h). Intriguingly, in general most of the candidate poles turn out to be no poles of the Igusa zeta function (this is why elements of this set are called *candidate* poles). The reason for this behaviour is not yet understood in full generality. When $n = 2$, we will determine the actual poles in Section 6.

In general, it is quite difficult to obtain an explicit formula for concrete examples via the technique in the proof of Theorem 3.7. We sketch an easy case where this is doable.

Example 3.10. Consider $Z(s) = \int_{\mathbb{Z}_p^2} |xy(x-y)|_p^s \, |dxdy|$. As embedded resolution $h : Y \to \mathbb{Q}_p^2$ of $xy(x-y)$ we take the blow-up at the origin.



Then Y is covered by two affine charts, both isomorphic to $\mathbb{Q}_p^2$. The restrictions h_1 and h_2 of h to those charts can be described in affine coordinates by

$$h_1 : (x, u) \mapsto (x, y = ux) \qquad \text{and} \qquad h_2 : (y, v) \mapsto (x = vy, y),$$

respectively. The exceptional curve E is a projective line with equation $x = 0$ and $y = 0$ in the first and second chart, respectively.

A crucial point is an adequate description of $h^{-1}(\mathbb{Z}_p^2)$. Writing for example

$$\mathbb{Z}_p^2 = \{(x, y) \mid \mathrm{ord}_p y \geq \mathrm{ord}_p x \geq 0\} \sqcup \{(x, y) \mid 0 \leq \mathrm{ord}_p y < \mathrm{ord}_p x\},$$

we can interpret $h^{-1}(\mathbb{Z}_p^2)$ as the disjoint union of $\mathbb{Z}_p^2$ (with coordinates (x, u) in the first chart), and $\mathbb{Z}_p \times p\mathbb{Z}_p$ (with coordinates (y, v) in the second chart). Then, using change of variables with respect to h, we obtain that

$$Z(s) = \int_{\mathbb{Z}_p^2} |x^3u(1-u)|_p^s \, |x|_p \, |dxdu| + \int_{\mathbb{Z}_p \times p\mathbb{Z}_p} |y^3v(v-1)|_p^s \, |y|_p \, |dydv|.$$

To compute the first integral, use separation of variables:

$$\int_{\mathbb{Z}_p^2} |x^3u(1-u)|_p^s \, |x|_p \, |dxdu| = \int_{\mathbb{Z}_p} |x|_p^{3s+1} \, |dx| \cdot \int_{\mathbb{Z}_p} |u(1-u)|_p^s \, |du|.$$

The first factor can be computed using Exercise 2.6, the second one by doing a straightforward computation or using Theorem 2.9. Similarly, one can compute the second integral. We leave this as an exercise. Finally, we obtain

$$Z(s) = \frac{(p-1)p^{3s}(-1+2p-2p^{s+1}+p^{s+2})}{(p^{s+1}-1)(p^{3s+2}-1)}.$$

Remark that the numerical data associated to the exceptional divisor E and to the strict transforms E_1, E_2 and E_3 are given by $(3,2)$ and $(1,1)$, respectively.

Exercise 3.11. Verify the result in Example 3.10 using Igusa's Stationary Phase Formula.

Remark 3.12. As a corollary of Theorem 3.7, the Igusa zeta function $Z(s)$ always converges on a larger part of $\mathbb{C}$ than $\Re(s) > 0$, namely for all $s \in \mathbb{C}$ such that $\Re(s) > -\min_{i \in I} \frac{\nu_i}{N_i}$. The value $\min_{i\in I} \frac{\nu_i}{N_i}$ is an important invariant, sometimes called the *p-adic log canonical threshold* of f.

When working with polynomials over $\mathbb{C}$ (or any algebraically closed field of characteristic zero) and their embedded resolutions, the analogous minimum is a famous singularity invariant, called the *(geometric) log canonical threshold.* Note that the geometric one can be smaller than the p-adic one, due to singular points that are not defined over $\mathbb{Q}_p$. Take for example the polynomial $f = y^2 - x^3(x^2+1)^5$. Its zero locus over $\mathbb{C}$ contains the three singular points $(0,0)$, $(i,0)$ and $(-i,0)$. It is not difficult to verify that the last two induce the log canonical threshold $\frac{7}{10}$ of f. However, for $p \equiv 3 \bmod 4$, the origin is the only singular point of the zero locus of f over $\mathbb{Q}_p$, inducing the p-adic log canonical threshold $\frac{5}{6}$ of f.

Theorem 3.13 ([**Ig2**], [**VZ1**], [**VZ2**]). *Let $f \in \mathbb{Q}_p[x_1,\dots,x_n]$. Fix an embedded resolution of f and use notation as in Theorem 3.5. Write $\sigma := \min_{i\in I} \frac{\nu_i}{N_i}$. Then σ does not depend on the chosen resolution and $-\sigma$ is always a pole of $Z(s)$.*

4. DENEF'S FORMULA

There is an explicit formula, due to Denef, in terms of an algebro-geometric version of embedded resolution. This formula is however only valid for all but finitely many p. We need several preparations to state this result. We first mention such a version of embedded resolution, valid over an arbitrary field K of characteristic zero (not necessarily algebraically closed). In the formulation below all varieties and morphisms are defined over K and such a variety is called irreducible if it is irreducible as K-variety.

4.1. Resolution: algebro-geometric version

THEOREM 4.1 (Embedded Resolution of Singularities; algebro-geometric version [**Hi**]). *Let K be a field of characteristic* 0 *and $f \in K[x_1, \ldots, x_n] \setminus K$. Then there exists an n-dimensional smooth and irreducible variety Y, a proper birational morphism $h : Y \to \mathbb{A}^n_K$, and finitely many closed $(n-1)$-dimensional smooth and irreducible subvarieties $E_i, i \in I$, of Y, equipped with* numerical data *$(N_i, \nu_i) \in (\mathbb{Z}_{>0})^2, i \in I$, satisfying*

(1) *h is an isomorphism of algebraic varieties outside the inverse image by h of the singular locus of $\{f = 0\}$,*
(2) *$h^{-1}(\{f = 0\}) = \cup_{i \in I} E_i$,*
(3) *$h^{-1}(\{f = 0\})$ is a normal crossings divisor, and*

$$\operatorname{div}(f \circ h) = h^* \operatorname{div}(f) = \sum_{i \in I} N_i E_i \quad \text{and} \quad K_{Y/\mathbb{A}^n_K} = \operatorname{div}(\operatorname{Jac} h) = \sum_{i \in I} (\nu_i - 1) E_i,$$

where $K_{Y/\mathbb{A}^n_K} = K_Y - h^* K_{\mathbb{A}^n_K}$ denotes the relative canonical divisor of the map h, which in this case is just the canonical divisor K_Y, since $K_{\mathbb{A}^n_K} = 0$.

Note that writing the last part in local coordinates is compatible with the last part of Theorem 3.5.

REMARK 4.2. It is again possible to construct the map h as a composition of finitely many *admissible* blow-ups at closed smooth and irreducible centres (all defined over K), i.e., each centre has normal crossings with the previously created exceptional components.

4.2. Reduction mod p

Let $f \in \mathbb{Z}[x_1, \ldots, x_n] \setminus p\mathbb{Z}[x_1, \ldots, x_n]$. Writing f explicitly as $f = \sum_{j_1, \ldots, j_n} a_{j_1, \ldots, j_n} x_1^{j_1} \ldots x_n^{j_n}$, the reduction mod p of f is

$$\bar{f} = f \bmod p := \sum_{j_1, \ldots, j_n} \overline{a_{j_1, \ldots, j_n}} x_1^{j_1} \ldots x_n^{j_n} \in \mathbb{F}_p[x_1, \ldots, x_n],$$

where $\overline{a_{j_1, \ldots, j_n}}$ is the class of $a_{j_1, \ldots, j_n} \in \mathbb{Z}$ in $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$. Similarly, one defines the reduction mod p of $f \in \mathbb{Z}_p[x_1, \ldots, x_n] \setminus p\mathbb{Z}_p[x_1, \ldots, x_n]$, also denoted by $\bar{f}$.

More generally, let $f \in \mathbb{Q}_p[x_1, \ldots, x_n]$ be a nonzero polynomial and consider $D := \operatorname{div}(f)$ in $\mathbb{A}^n_{\mathbb{Q}_p}$. Pick the unique $e \in \mathbb{Z}$ such that $p^e f \in \mathbb{Z}_p[x_1, \ldots, x_n] \setminus p\mathbb{Z}_p[x_1, \ldots, x_n]$. Define the reduction mod p of D as $\overline{D} = D \bmod p := \operatorname{div}(\overline{p^e f})$, where $\overline{p^e f}$ is the reduction mod p of $p^e f$ as explained above. Hence $\overline{D}$ is a divisor in $\mathbb{A}^n_{\mathbb{F}_p}$, and can be considered as a subscheme of $\mathbb{A}^n_{\mathbb{F}_p}$.

EXAMPLE 4.3. Consider $f = y - p^3 x^2 \in \mathbb{Z}_p[x, y] \setminus p\mathbb{Z}_p[x, y]$. Then $D := \operatorname{div}(f)$ is a parabola in $\mathbb{A}^2_{\mathbb{Q}_p}$, but $\overline{D} = \operatorname{div}(\overline{y - p^3 x^2}) = \operatorname{div}(y)$ is a line in $\mathbb{A}^2_{\mathbb{F}_p}$. The polynomial $f = p^{-2} y - p x^2 \in \mathbb{Q}_p[x, y]$ induces the same divisors D and $\overline{D}$.

In fact, there is a well-defined notion of reduction mod p for any algebraic variety or scheme Z over $\mathbb{Q}_p$. It is a scheme defined over $\mathbb{F}_p$, denoted by $\overline{Z}$. Similarly, there is a well-defined notion of reduction mod p for any morphism g of $\mathbb{Q}_p$-schemes, being a morphism $\overline{g}$ of the corresponding $\mathbb{F}_p$-schemes. See for example [**De1**] and the references therein.

We now introduce the necessary terminology and notation to state Denef's formula. Consider an embedded resolution $h : Y \to \mathbb{A}^n_{\mathbb{Q}_p}$ of $f \in \mathbb{Q}_p[x_1, \dots, x_n] \setminus \mathbb{Q}_p$, for which we use notation as in Theorem 4.1. Then

$$E_i \hookrightarrow Y \xrightarrow{h} \mathbb{A}^n_{\mathbb{Q}_p} \hookleftarrow D = \mathrm{div}(f)$$

has a reduction mod p:

$$\overline{E}_i \hookrightarrow \overline{Y} \xrightarrow{\overline{h}} \mathbb{A}^n_{\mathbb{F}_p} \hookleftarrow \overline{D}.$$

Definition 4.4 (Denef [**De1**]). We say that h has *good reduction* mod p if

(1) $\overline{Y}$ and all $\overline{E}_i, i \in I$, are smooth,
(2) $\bigcup_{i\in I} \overline{E}_i$ is a normal crossings divisor,
(3) $\overline{E}_i$ and $\overline{E}_j$ have no common components for $i \neq j$.

Example 4.5 (Bad reduction). We give examples that show how these conditions might fail.

(1) The zero locus of $g = px + x^2 - y^2$ is smooth, but the zero locus of $\overline{g} = x^2 - y^2 = (x + y)(x - y)$ is not.
(2) The zero loci of $g = x$ and $g' = x + py - y^2$ have normal crossings, but those of $\overline{g} = x$ and $\overline{g'} = x - y^2$ do not.
(3) The zero loci of $g = px + y + p$ and $g' = px + y$ are disjoint, but $\overline{g} = \overline{g'}$.

Notation 4.6. For $J \subseteq I$, define $\overline{E}_J = \bigcap_{j\in J} \overline{E}_j$ and $\overset{\circ}{\overline{E}}_J = \overline{E}_J \setminus \bigcup_{k\notin J} \overline{E}_k$. Note that in particular $\overline{E}_\emptyset = \overline{Y}$ and $\overset{\circ}{\overline{E}}_\emptyset = \overline{Y} \setminus \bigcup_{i\in I} \overline{E}_i$.

4.3. The formula

Theorem 4.7 (Denef's formula, 1987 [**De1**]).

(1) *Let $f \in \mathbb{Q}_p[x_1, \dots, x_n] \setminus \mathbb{Q}_p$. If $h : Y \to \mathbb{A}^n_{\mathbb{Q}_p}$ is an embedded resolution of f with good reduction mod p, then*
$$Z(s) = \frac{1}{p^n} \sum_{J\subseteq I} \sharp(\overset{\circ}{\overline{E}}_J(\mathbb{F}_p)) \prod_{j\in J} \frac{p-1}{p^{N_j s+\nu_j} - 1}.$$
(2) *Let $f \in \mathbb{Q}[x_1, \dots, x_n] \setminus \mathbb{Q}$. Let $h : Y \to \mathbb{A}^n_{\mathbb{Q}}$ be an embedded resolution of f. For any prime p, we can also consider $f \in \mathbb{Q}_p[x_1, \dots, x_n]$ and view Y and h as defined over $\mathbb{Q}_p$. Then, for all but finitely many p, the resolution h (considered over $\mathbb{Q}_p$) has good reduction mod p.*

Note that for a fixed prime p, it is possible that there exists no embedded resolution of a given $f \in \mathbb{Q}[x_1, \dots, x_n]$ with good reduction mod p.

Remark 4.8. Traditionally, the factors in the product above are written as $\frac{(p-1)p^{-N_j s-\nu_j}}{1-p^{-N_j s-\nu_j}}$, in order to view $Z(s)$ explicitly as a function in p^{-s}. The presentation above is often somewhat more efficient.

Remark 4.9. We define the contribution of E_i to the Igusa zeta function $Z(s)$ to be
$$\frac{1}{p^n} \sum_{i\in J\subseteq I} \sharp(\overset{\circ}{\overline{E}}_J(\mathbb{F}_p)) \prod_{j\in J} \frac{p-1}{p^{N_j s+\nu_j} - 1}.$$
If this contribution has no pole at $s_0 = -\nu_i/N_i$, we say that E_i does not contribute to the candidate pole s_0 of $Z(s)$. Note that in this case $-\nu_i/N_i$ might still be a

pole of $Z(s)$; this is however only possible if there exists some other E_j satisfying $s_0 = -\nu_j/N_j = -\nu_i/N_i$ that does contribute to the pole s_0.

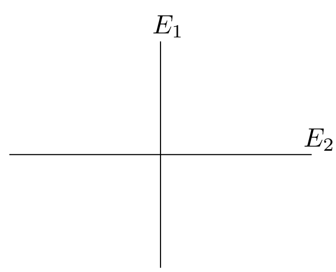
EXAMPLE 4.10. Consider $f = xy(x-y)$ as in Example 3.10. The blow-up at the origin h is also an embedded resolution in the sense of Theorem 4.1, and moreover with good reduction mod p for all primes p. Denote the exceptional curve of h by E and the strict transforms in Y of the three components of $\operatorname{div}(f)$ by E_1, E_2, E_3. Then, with the notation of Theorem 4.7, $\overline{E}$ is a projective line, and (for $i = 1,2,3$) $\overline{E}_i$ is an affine line and $\overline{E} \cap \overline{E}_i$ a point, all defined over $\mathbb{F}_p$. Hence $\sharp(\overset{\circ}{\overline{E}}_\emptyset(\mathbb{F}_p)) = p^2-(3p-2)$, $\sharp(\overset{\circ}{\overline{E}}(\mathbb{F}_p)) = (p+1)-3$ and $\sharp(\overset{\circ}{\overline{E}}_i(\mathbb{F}_p)) = p-1$ for $i = 1,2,3$. Adding everything, Denef's formula yields

$$\begin{aligned} Z(s) = \frac{1}{p^2}\Bigg(& p^2 - 3p + 2 + (p-2)\frac{p-1}{p^{3s+2}-1} \\ &+3(p-1)\frac{p-1}{p^{s+1}-1} + 3\cdot 1\frac{(p-1)^2}{(p^{3s+2}-1)(p^{s+1}-1)}\Bigg). \end{aligned}$$

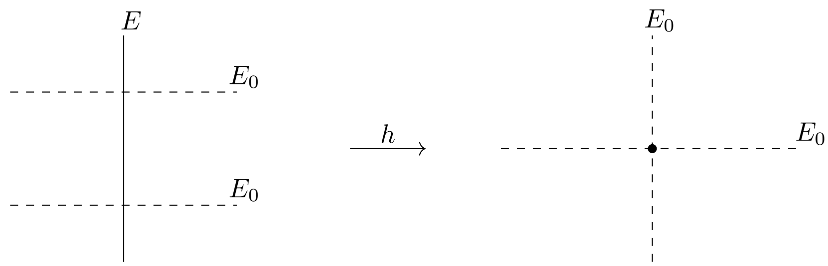
Verify that this is the same result as in Example 3.10.

EXAMPLE 4.11. Consider $f = x^2 + y^2$ as in Exercises 1.5 and 2.11.

(1) Let $p \equiv 1 \bmod 4$. Take $\alpha \in \mathbb{Z}_p$ such that $\alpha^2 = -1$. Then the zero locus of $f = (x - \alpha y)(x + \alpha y)$ consists of two affine lines E_1 and E_2 over $\mathbb{Q}_p$, intersecting transversely in the origin. Hence we can take the identity as resolution map h. Check that it has good reduction mod p. Compute $Z(s)$ using Denef's formula (there are four terms), and compare with Exercise 2.11.



(2) Let $p \equiv 3 \bmod 4$. Then $E_0 = \operatorname{div}(f)$ is irreducible as $\mathbb{Q}_p$-scheme, and its only point with coordinates in $\mathbb{Q}_p$ is the origin. In the figure below, we 'suggest' E_0 by indicating with dashed lines its two components over the quadratic extension of $\mathbb{Q}_p$ in which -1 is a square. Take the blow-up at the origin as resolution map $h : Y \to \mathbb{A}^2_{\mathbb{Q}_p}$. The exceptional curve E is a projective line over $\mathbb{Q}_p$, and the strict transform of E_0 in Y is a 1-dimensional irreducible $\mathbb{Q}_p$-scheme (now without $\mathbb{Q}_p$-points), given in one chart by $1 + u^2 = 0$.



Hence $\overline{E}$ is a projective line over $\mathbb{F}_p$, $\overline{E}_0$ in $\overline{Y}$ is an irreducible $\mathbb{F}_p$-scheme *without* $\mathbb{F}_p$*-points*, still given in one chart by $1 + u^2 = 0$, and $\overline{E} \cap \overline{E}_0$ is a 0-dimensional irreducible $\mathbb{Q}_p$-scheme, given in one chart by $x = 1+u^2 = 0$, also without $\mathbb{F}_p$-points. In particular, h has good reduction mod p. Compute $Z(s)$ using Denef's formula (there are two terms), and compare with Exercise 2.11.

(3) Let $p = 2$. Then again $\operatorname{div}(f)$ is irreducible as $\mathbb{Q}_2$-scheme, and the blow-up at the origin is a resolution map h. Verify however that h does not have good reduction mod 2. In this case we cannot apply Denef's formula.

Exercise 4.12.

(1) Compute the Igusa zeta function for $f = y^2 - x^3$, using Denef's formula. Then use Proposition 2.8 to compute its Poincaré series; you should obtain what was claimed in Example 1.11.
(2) Verify the result of Example 2.10 for $f = xy - z^2$, using Denef's formula.

Remark 4.13. The relatively easy examples that we considered up to now are somewhat misleading. In general, it is very hard to determine whether a resolution h has good reduction mod p for a given (small) prime p. Denef's formula is typically used when $f \in \mathbb{Q}[x_1, \ldots, x_n]$ and one wants to compute $Z_p(f; s)$ or prove properties of $Z_p(f; s)$ for large enough p.

Remark 4.14. Another technique to compute the Igusa zeta function is by using toric geometry. Denef and Hoornaert [**DH**] give a combinatorial formula to compute $Z(s)$ when $f \in \mathbb{Q}_p[x_1, \ldots, x_n]$ is non-degenerate (mod p) with respect to its Newton polyhedron. That formula contains typically less candidate poles than the list given by an embedded resolution.

5. BACK TO POLYNOMIAL CONGRUENCES

Recall the original problem: given $f \in \mathbb{Z}[x_1, \ldots, x_n]$, we want to study the values

$$M_i = M_i(p) = \sharp \left\{ a \in \left(\frac{\mathbb{Z}}{p^i \mathbb{Z}} \right)^n \mid f(a) \equiv 0 \bmod p^i \right\}.$$

In this section, we discuss how to use the machinery we introduced before to obtain information about the M_i.

5.1. Asymptotic behaviour of the M_i

We start with an example.

EXAMPLE 5.1. Let $f = y^2 - x^3 \in \mathbb{Z}[x, y]$. Recall from Example 1.11 or Exercise 4.12 that

$$P(T) = \frac{p^6 + (p^4 - p^3)T^2 - T^6}{(p - T)(p^5 - T^6)} = \frac{1 + (p^{-2} - p^{-3})T^2 - p^{-6}T^6}{(1 - p^{-1}T)(1 - p^{-5}T^6)}.$$

Using partial fraction decomposition, we know that there exists a constant $q \in \mathbb{Q}$ and a polynomial $Q(T) \in \mathbb{Q}[T]$ (of degree at most 5) such that

$$P(T) = \frac{q}{1 - p^{-1}T} + \frac{Q(T)}{1 - p^{-5}T^6}.$$

Combining this with the definition of $P(T)$ we obtain

$$\sum_{i \geq 0} M_i (p^{-2}T)^i = P(T) = q \sum_{i \geq 0} (p^{-1}T)^i + Q(T) \sum_{i \geq 0} (p^{-5}T^6)^i.$$

EXERCISE 5.2. Compute q and Q of Example 5.1 explicitly and give explicit formulas for the M_i (depending on the value of $i \bmod 6$). For example, when $i = 6e$ ($e \in \mathbb{Z}_{>0}$), you should obtain $M_{6e} = (p + 1)p^{7e-1} - p^{6e-1}$. Conclude that the asymptotic behaviour of the M_i is roughly $p^{\frac{7}{6}i}$.

In general, it is not difficult to conclude the following (using Theorem 3.13 to identify σ), see for instance [**VZ1**, Theorem 2.7] and [**VZ2**, Remark 3.11].

PROPOSITION 5.3. *Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$. Denote by $-\sigma$ the largest real part of a pole of $Z(s)$, i.e., σ is the p-adic log canonical threshold of f. Equivalently, p^σ is the smallest modulus of a pole of $Z'(t)$ or $P(T)$. Then the main contribution to the M_i is determined by σ: roughly, the M_i behave asymptotically as $p^{(n-\sigma)i}$. More precisely:* $\limsup_i M_i^{1/i} = p^{n-\sigma}$.

Moreover, if $-\sigma$ is a pole of $Z(s)$ of order d (where necessarily $d \leq n$), then there exists a positive constant C such that $M_i \leq C i^{d-1} p^{(n-\sigma)i}$ for all i.

Note that for the cusp (see Example 5.1 and Exercise 5.2) we have indeed $\limsup_i M_i^{1/i} = p^{2-5/6} = p^{7/6}$.

REMARK 5.4. An exact expression for the M_i, with 'dominating term' as in Proposition 5.3, involves necessarily data from *all* poles of $P(T)$. Such a complete description of the M_i in terms of the actual poles of $P(T)$ and their orders (and partial fraction decomposition) was given by Segers in full generality in [**Sg2**]. (Note that there is a crucial typo in [**Sg2**]. On the last line of page 4 the numbers involving e must be augmented by 1.) The outcome of Exercise 5.2 is a very special case of this description.

Here we should stress that determining the actual poles is in general a difficult problem. When $n = 2$, this is understood and explained in the next section.

5.2. Divisibility properties of the M_i

There is a remarkable theorem by Segers stating (1) divisibility properties of the M_i by powers of p, (2) a sharp lower bound for the possible real parts of poles of $Z(s)$, and (3) the relation between those two properties. In order to place this result into context, we first mention an 'easy' lower bound for the real parts of poles.

Proposition 5.5. *Let $f \in K[x_1, \ldots, x_n] \setminus K$, where K is any field of characteristic zero and $n \geq 2$. Consider an embedded resolution $h : Y \to \mathbb{A}^n_K$ of f, that is constructed as a finite composition of admissible blow-ups at smooth centres Z_i (always contained in the strict transform of the divisor of f). Each such blow-up induces an exceptional component E_i with numerical data (N_i, ν_i).*

(1) *Let Z_j be contained in $E_{i_1}, \ldots, E_{i_l}$ (and not in the other E_ℓ). Denote by μ_j the multiplicity of the strict transform of the divisor of f along Z_j. Then*

$$N_j = \sum_{k=1}^{l} N_{i_k} + \mu_j,$$

$$\nu_j = \sum_{k=1}^{l} (\nu_{i_k} - 1) + \operatorname{codim}(Z_j),$$

where $\operatorname{codim}(Z_j) = n - \dim(Z_j)$ *is the codimension of Z_j in the ambient space.*

(2) *Assume that each centre Z_j is contained in an exceptional component or in the singular locus of the strict transform. Then all $\frac{\nu_i}{N_i} \leq n - 1$.*

Proof. We explain the argument with local analytic coordinates in the analytic setting when K is $\mathbb{Q}_p$, $\mathbb{R}$ or $\mathbb{C}$. A similar argument can be given with étale coordinates in a general algebro-geometric setting.

(1) Choose local coordinates $z_1, \ldots, z_n$ around a general point b of Z_j such that E_{i_k} is given by $\{z_k = 0\}$ for $k = 1, \ldots, l$ and Z_j is given by $\{z_1 = \cdots = z_d = 0\}$ in a neighbourhood of b. Note that $d = \operatorname{codim}(Z_j) \geq l$. Suppose that, locally around b, the strict transform of f is given by $f_{j-1}(z_1, \ldots, z_n)$. With local coordinates $x_1, \ldots, x_n$ in the first chart of the blow-up at the center Z_j, this blow-up is described by the change of variables

$$\begin{aligned}&(x_1, x_2, \ldots, x_d, x_{d+1}, \ldots, x_n) \mapsto \\ &(z_1 = x_1, z_2 = x_1 x_2, \ldots, z_d = x_1 x_d, z_{d+1} = x_{d+1}, \ldots, z_n = x_n).\end{aligned}$$

Hence the pullback of

$$\varepsilon(z_1, \ldots, z_n) z_1^{N_1} z_2^{N_2} \ldots z_l^{N_l} f_{j-1}(z_1, \ldots, z_n)$$

is

$$\varepsilon'(x_1, \ldots, x_n) x_1^{N_1 + \cdots + N_l + \mu_j} x_2^{N_2} \ldots x_l^{N_l} f_j(x_1, \ldots, x_n),$$

with f_j not divisible by x_i for $i = 1, \ldots, l$ and $\varepsilon, \varepsilon'$ invertible. Therefore, N_k equals $\sum_{k=1}^{l} N_{i_k} + \mu_j$.

Similarly, the pullback of

$$\eta(z_1, \ldots, z_n) z_1^{\nu_1 - 1} z_2^{\nu_2 - 1} \ldots z_l^{\nu_l - 1} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$$

is

$$\eta'(x_1, \ldots, x_n) x_1^{\nu_1 - 1 + \nu_2 - 1 + \cdots + \nu_l - 1 + d - 1} x_2^{\nu_2 - 1} \ldots x_l^{\nu_l - 1} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n,$$

with η, η' invertible. Hence, $\nu_j = \sum_{k=1}^{l} (\nu_{i_k} - 1) + d = \sum_{k=1}^{l} (\nu_{i_k} - 1) + \operatorname{codim}(Z_j)$.

(2) Concerning the first blow-up with centre Z_1 and exceptional component E_1, we know that $(N_1, \nu_1) = (\mu_1, d_1)$ with μ_1 the multiplicity of f at Z_1 and

$d_1 = \operatorname{codim}(Z_1)$. By assumption, Z_1 is contained in the singular locus of $\{f = 0\}$, hence $\mu_1 \geq 2$. Consequently, $\frac{\nu_1}{N_1} = \frac{d_1}{\mu_1} \leq \frac{n}{2} \leq n - 1$.

Next, we suppose that Z_j is contained in at least one exceptional component. We use the notation of part (1), where we may assume by induction that $\frac{\nu_{i_k}}{N_{i_k}} \leq n-1$ for $k = 1, \dots, l$. Compute

$$\begin{aligned}\frac{\nu_j}{N_j} &= \frac{\sum_{k=1}^{l}(\nu_{i_k} - 1) + \operatorname{codim}(Z_j)}{\sum_{k=1}^{l} N_{i_k} + \mu_j}\\ &\leq \frac{\sum_{k=1}^{l}(n-1)N_{i_k} - l + n}{\sum_{k=1}^{l} N_{i_k} + \mu_j}\\ &= \frac{(n-1)\sum_{k=1}^{l} N_{i_k}}{\sum_{k=1}^{l} N_{i_k} + \mu_j} - \frac{l - n}{\sum_{k=1}^{l} N_{i_k} + \mu_j}\\ &= n - 1 - \frac{(n-1)\mu_j + l - n}{\sum_{k=1}^{l} N_{i_k} + \mu_j}.\end{aligned}$$

Since $\mu_j \geq 1$ and $l \geq 1$, we conclude that $\frac{\nu_j}{N_j} \leq n - 1$. $\square$

Note that the assumption in Proposition 5.5(2) is always satisfied if no 'stupid' blow-ups are used to construct the resolution h. A stupid example is blowing up at a smooth point of the divisor of f; this yields an exceptional component with numerical data $(N_1, \nu_1) = (1, n)$ and then $\frac{n}{1}$ is not bounded by $n - 1$.

As a corollary of Proposition 5.5, all real parts of poles of zeta functions $Z(s)$ are larger than or equal to $-(n-1)$. The following much stronger result of Segers was in fact not conjectured before, and is proven by quite elementary arguments and techniques. In fact, he first shows the divisibility properties, and derives the bound on the poles from it.

THEOREM 5.6 (Segers, 2006 [**Sg1**]). *Let $f \in \mathbb{Z}[x_1, \dots, x_n]$, where $n \geq 2$, and fix any prime p. Then*

(1) *M_i is divisible by $p^{\lceil n/2(i-1)\rceil}$ for all i,*
(2) *all real parts of poles of $Z(s)$ are larger than or equal to $-\frac{n}{2}$.*

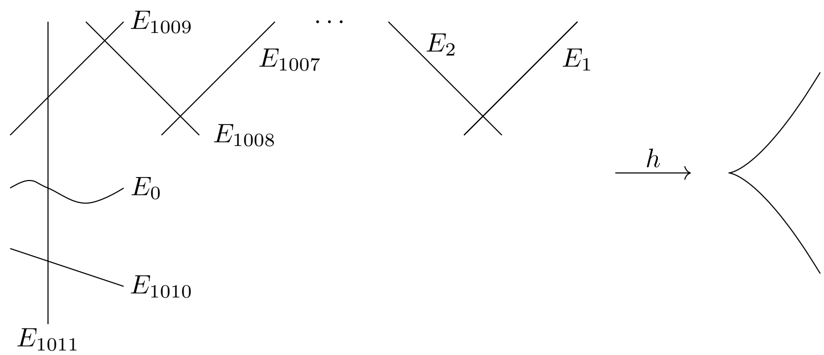
EXERCISE 5.7. Verify that for $f = x_1^2 + x_2^2 + \cdots + x_n^2$ the lower bound $-\frac{n}{2}$ is an actual pole of the zeta function $Z(s)$.

6. IGUSA ZETA FUNCTION FOR PLANE CURVES

For polynomials in two variables, the poles of its Igusa zeta function are well understood.

6.1. Example and non-contribution

EXAMPLE 6.1. Consider the polynomial $f = y^2 - x^{2019}$ and its minimal embedded resolution h, where E_0 denotes the strict transform.



The numerical data associated to this minimal resolution are as follows:

$$(N_i, \nu_i) = \begin{cases} (1,1) & \text{if } i = 0, \\ (2i, i+1) & \text{if } 1 \leq i \leq 1009, \\ (2019, 1011) & \text{if } i = 1010, \\ (4038, 2021) & \text{if } i = 1011. \end{cases}$$

Hence, there are 1012 different real values of candidate poles $-\frac{\nu_i}{N_i}$ of $Z(s)$. However, using Denef's formula, and either a lot of courage or a computer, one computes that

$$Z(s) = \frac{\text{a polynomial in } p^{-s}}{\left(1 - p^{-(s+1)}\right)\left(1 - p^{-(4038s+2021)}\right)}.$$

Remark 6.2. Using either the formula for non-degenerate polynomials of Denef and Hoornaert [**DH**], or the formula in [**Ve5**] in terms of the relative log canonical model (which is a certain partial resolution), there are only two candidate real parts of poles, and one obtains this result quite immediately.

Example 6.1 illustrates the general 'geometric determination' of the actual poles of $Z(s)$. Roughly speaking, exceptional components E_i that intersect exactly once or twice other components do not contribute to the poles, and other exceptional components or components of the strict transform do contribute. We explain the main point of the non-contribution argument, in the setting of Denef's formula.

Lemma 6.3. *Let $f \in \mathbb{Q}_p[x, y] \setminus \mathbb{Q}_p$ and $h : Y \to \mathbb{A}^2_{\mathbb{Q}_p}$ an embedded resolution of f with good reduction mod p. Let E (with numerical data (N, ν)) be an exceptional component of h, intersecting two other components, say E_1 and E_2 (with numerical data (N_1, ν_1) and (N_2, ν_2), respectively). Here we mean 'intersecting over $\mathbb{Q}_p$', i.e., E is a projective line over $\mathbb{Q}_p$ and each of the two intersection points is defined over $\mathbb{Q}_p$. Assume that E induces a candidate pole of order 1, i.e., $\frac{\nu}{N} \neq \frac{\nu_i}{N_i}$ for $i = 1, 2$. Then E does not contribute to a pole (in the sense of Remark 4.9).*

PROOF. The contribution of E to the Igusa zeta function is equal to

$$\frac{1}{p^2}\Bigg(\sharp((\overline{E}\setminus(\overline{E}_1\cup\overline{E}_2))(\mathbb{F}_p))\frac{p-1}{p^{Ns+\nu}-1}+\sharp((\overline{E}\cap\overline{E}_1)(\mathbb{F}_p))\frac{p-1}{p^{Ns+\nu}-1}\cdot\frac{p-1}{p^{N_1s+\nu_1}-1}$$
$$+\sharp((\overline{E}\cap\overline{E}_2)(\mathbb{F}_p))\frac{p-1}{p^{Ns+\nu}-1}\cdot\frac{p-1}{p^{N_2s+\nu_2}-1}\Bigg)$$
$$=\frac{1}{p^2}\left((p-1)\frac{p-1}{p^{Ns+\nu}-1}+\frac{p-1}{p^{Ns+\nu}-1}\cdot\frac{p-1}{p^{N_1s+\nu_1}-1}+\frac{p-1}{p^{Ns+\nu}-1}\cdot\frac{p-1}{p^{N_2s+\nu_2}-1}\right)$$
$$=\frac{(p-1)^2}{p^2}\left(\frac{(p^{N_1s+\nu_1}-1)(p^{N_2s+\nu_2}-1)+p^{N_2s+\nu_2}-1+p^{N_1s+\nu_1}-1}{(p^{Ns+\nu}-1)(p^{N_1s+\nu_1}-1)(p^{N_2s+\nu_2}-1)}\right)$$
$$=\frac{(p-1)^2}{p^2}\left(\frac{p^{(N_1+N_2)s+\nu_1+\nu_2}-1}{(p^{Ns+\nu}-1)(p^{N_1s+\nu_1}-1)(p^{N_2s+\nu_2}-1)}\right).$$

By Lemma 6.5, there exists some $\kappa\in\mathbb{Z}_{>0}$ such that $\kappa N=N_1+N_2$ and $\kappa\nu=\nu_1+\nu_2$. Therefore, the contribution of E is given by

$$\frac{(p-1)^2}{p^2}\left(\frac{p^{\kappa(Ns+\nu)}-1}{(p^{Ns+\nu}-1)(p^{N_1s+\nu_1}-1)(p^{N_2s+\nu_2}-1)}\right)$$
$$=\frac{(p-1)^2}{p^2}\left(\frac{\sum_{j=0}^{\kappa-1}p^{j(Ns+\nu)}}{(p^{N_1s+\nu_1}-1)(p^{N_2s+\nu_2}-1)}\right).$$

Hence, E does not contribute to a pole of $Z(s)$. □

REMARK 6.4. When E, defined over $\mathbb{Q}_p$, intersects only once another component, say E_1, then the intersection point is automatically defined over $\mathbb{Q}_p$. Assuming that $\frac{\nu}{N}\neq\frac{\nu_1}{N_1}$, the same proof (putting formally $(N_2,\nu_2)=(0,1)$) yields that E does not contribute to a pole.

The following lemma was first proven for arbitrary plane curves by Loeser [**Lo**] (preceded by some partial results by Strauss, Meuser and Igusa). We present a conceptual proof, which is the starting point of a generalized theory of relations and congruences between numerical data in arbitrary dimension n [**Ve1**], [**Ve2**].

LEMMA 6.5. *Let $h:Y\to\mathbb{A}^2$ be an embedded resolution of singularities of a plane curve C with defining equation g. Write $h^{-1}(C)=E\cup\bigcup_{i=1}^k E_i$, where E is a fixed exceptional component. Let E intersect exactly r times other components, say $E_1,\dots,E_r$. Then there exists some $\kappa\in\mathbb{Z}_{>0}$ such that $\kappa N=\sum_{i=1}^r N_i$ and $\kappa\nu=\sum_{i=1}^r(\nu_i-1)+2$. More precisely, $\kappa=-E^2$ where $E^2=E\cdot E$ is the self-intersection number of E on Y.*

PROOF. Recall that the Picard group $\mathrm{Pic}(Y)$ of Y is defined as the group of Cartier divisors on Y modulo linear equivalence. In $\mathrm{Pic}(Y)$ we have that

$$NE+\sum_{i=1}^k N_iE_i=h^*\mathrm{div}(g)=\mathrm{div}(h^*(g))=0$$

and

$$(\nu-1)E+\sum_{i=1}^k(\nu_i-1)E_i=K_Y-h^*K_{\mathbb{A}^2}=K_Y.$$

We compute the intersection product of both expressions with E (in $\mathrm{Pic}(Y)$). First,

$$0 = E \cdot 0 = E \cdot \left(NE + \sum_{i=1}^{k} N_i E_i \right) = E \cdot NE + E \cdot \left(\sum_{i=1}^{r} N_i E_i \right) = NE^2 + \sum_{i=1}^{r} N_i.$$

Hence, $\kappa N = -NE^2 = \sum_{i=1}^{r} N_i$. Similarly, we compute

$$\begin{aligned} E \cdot K_Y &= E \cdot \left((\nu - 1)E + \sum_{i=1}^{k} (\nu_i - 1) E_i \right) \\ &= E \cdot (\nu - 1)E + E \cdot \left(\sum_{i=1}^{r} (\nu_i - 1) E_i \right) = (\nu - 1)E^2 + \sum_{i=1}^{r} (\nu_i - 1). \end{aligned}$$

Using the adjunction formula, we obtain

$$-2 = \deg K_E = (K_Y + E) \cdot E = \nu E^2 + \sum_{i=1}^{r} (\nu_i - 1),$$

and hence $\kappa \nu = -\nu E^2 = \sum_{i=1}^{r} (\nu_i - 1) + 2$. □

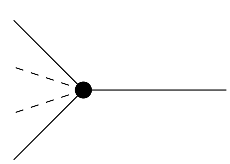
Remark 6.6. Observe that Lemma 6.5 agrees (as it should) with the numerical data in Example 6.1.

6.2. Structure of dual graph and general result

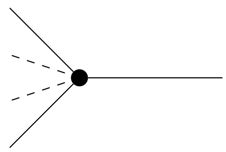
Before stating the general theorem on the determination of the poles of $Z(s)$ for $n = 2$, it is useful to know the structure of the dual graph of the minimal embedded resolution of f, decorated with the values $\frac{\nu_i}{N_i}$. This is a geometrical and 'local' result; so we assume now that f is defined over $\mathbb{C}$, and we consider only the germ of f at the origin. (We also exclude the trivial case where this germ has already normal crossings.)

Notation 6.7. Let $h : Y \to \mathbb{A}^n_{\mathbb{C}}$ be an embedded resolution of the germ at 0 of $\{f = 0\}$. The dual graph associated to this resolution is a graph that consists of the following data. The vertices of the graph correspond to the irreducible components of $h^{-1}(\{f = 0\})$. The exceptional components are represented by a dot, the (analytically irreducible) components of the strict transform by a circle. Two vertices are connected by an edge precisely when the associated components intersect. It is well known that this dual graph is a tree, where all circles are end vertices.

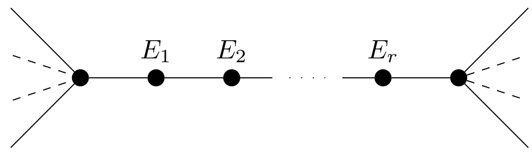
A vertex with at least three edges is depicted in the following way.



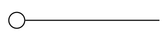
Theorem 6.8 (Veys, 1995 [**Ve4**]). *Consider the minimal embedded resolution $h : Y \to \mathbb{A}^n_{\mathbb{C}}$ of f with irreducible components $E_i, i \in I$, and associated numerical data $(N_i, \nu_i), i \in I$. Denote the locus (with edges) where $\frac{\nu_i}{N_i}$ is minimal by $\mathcal{M}$. Then $\mathcal{M}$ is connected and has one of the following possible forms, where $r \geq 0$.*



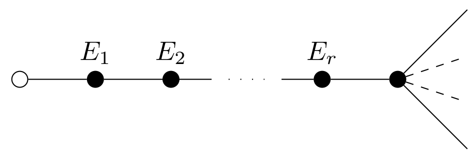
(1)



(2)



(3)



(4)

When f is reduced, the last two cases (3) and (4), involving a component of the strict transform, cannot occur.

Furthermore, starting from $\mathcal{M}$, the values $\frac{\nu_i}{N_i}$ strictly increase along each path away from $\mathcal{M}$.

When $n = 2$, we already knew that poles of $Z(s)$ have order at most 2. Theorem 6.8 implies that there is at most one real part of a pole of order 2; here is the precise statement.

COROLLARY 6.9 (Veys, 1995 [**Ve4**]). *Let $f \in \mathbb{Q}_p[x, y] \setminus \mathbb{Q}_p$. Let $h : Y \to \mathbb{Q}_p^2$ be an embedded resolution of f as in Theorem 3.5. A real number s_0 is (the real part of) a pole of order 2 of $Z(s)$ if and only if there exist $i, j \in I$ with $s_0 = -\frac{\nu_i}{N_i} = -\frac{\nu_j}{N_j}$ for* intersecting *components E_i and E_j. In that case, s_0 equals minus the p-adic log canonical threshold of f.*

REMARK 6.10. The following generalisation (to higher dimension) was conjectured by the second author in 1999 [**LV**], and proven by Nicaise and Xu in 2016 [**NX**]. Let $f \in \mathbb{Q}_p[x_1, \dots, x_n]$ and consider its associated Igusa zeta function $Z(s)$. The real part of a pole of order n of $Z(s)$ must be equal to minus the p-adic log canonical threshold of f. In particular, there is at most one possible real part of a pole of order n.

Also, the real part of a pole of order n is of the form $-\frac{1}{k}$ for some positive integer k [**LV**].

Concerning the poles of order 1, 'morally' an exceptional E_i does not contribute if and only if it intersects other components in exactly one or two points defined over $\mathbb{Q}_p$. In order to formulate this precisely, it is more convenient to state it in terms of the local equation of the strict transform before the creation of E_i in the resolution process.

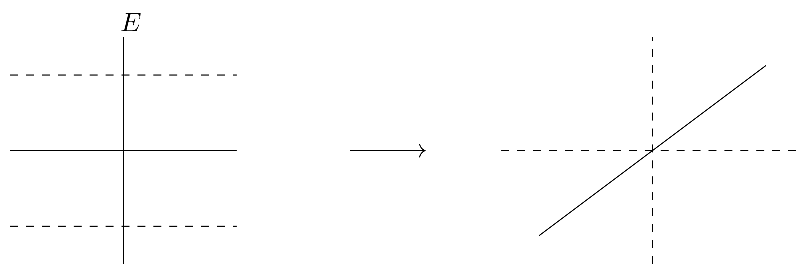
Also, as for the structure of the dual resolution graph, the result is 'local' in the sense that it is formulated for one singular point.

THEOREM 6.11 (Ibadula-Segers, 2012 [**IS**]). *Let $f \in \mathbb{Q}_p[x, y]$. Suppose that the origin is the only singular point of $\{f = 0\}$. Let $h : Y \to \mathbb{Q}_p^2$ be an embedded* *resolution of f as in Theorem 3.5, constructed as a finite composition of admissible blow-ups. Let $s_0 = -\frac{\nu_i}{N_i}$ for some $i \in I$, such that s_0 is not a pole of order 2 of $Z(s)$.*

(1) *Suppose that $s_0 \neq -\frac{\nu_i}{N_i}$ for all components E_i of the strict transform of f. Then s_0 is a pole of order 1 if and only if there is an exceptional component E_i with $s_0 = -\frac{\nu_i}{N_i}$ satisfying the following. Write the equation of the strict transform of f, at the stage of h before the blow-up with centre C where E_i is created as exceptional component, as a convergent power series in local analytic coordinates around C; then the lowest degree part of this power series is* not *a power of a linear form (over $\mathbb{Q}_p$) or a product of two such powers.*

(2) *Let $s_0 = -\frac{\nu_i}{N_i}$ for a component E_i of the strict transform of f. Then s_0 is always 'locally' a pole of order 1 in the sense that it is a pole of $\int_{(p^k\mathbb{Z}_p)^2} |f(x,y)|_p^s \, |dxdy|$ for some large enough k.*

Exercise 6.12. Let $f = y(x^2+y^2)+x^4$ and fix a prime p such that $p \equiv 3 \bmod 4$. The blow-up at the origin is an embedded resolution of f. In the figure below, we sketch the germ of f at the origin, where the dashed lines 'suggest' the tangent directions that are only defined over the quadratic extension of $\mathbb{Q}_p$ in which -1 is a square.



The exceptional curve E has numerical data $(N, \nu) = (3, 2)$. In the context of p-adic manifolds, E intersects the strict transform only in one point. However, geometrically (already over that quadratic extension of $\mathbb{Q}_p$), the exceptional curve E intersects the strict transform three times. This is reflected in the lowest degree part of f, which is over $\mathbb{Q}_p$ the product of a linear form and an irreducible quadratic form. Hence, by Theorem 6.11 we know that $-\frac{3}{2}$ is a pole of order 1 of $Z(s)$.

Remark 6.13. The determination of non-real poles is less clear. Consider for example $f = x^2 + y^2$ as in Exercises 1.5 and 2.11 and Example 4.11. When $p \equiv 3 \bmod 4$, we have that both -1 and $-1 + \frac{\pi}{\ln p} i$ are poles of order 1 of $Z(s)$. On the other hand, when $p = 2$, we have that -1 is a pole of order 1 of $Z(s)$, but $-1 + \frac{\pi}{\ln 2} i$ is not a pole.

7. TOPOLOGICAL AND MOTIVIC ZETA FUNCTION

The p-adic Igusa zeta function has two remarkable 'spin-offs'.

7.1. Topological zeta function

Recall Denef's formula for the zeta function of $f \in \mathbb{Q}[x_1, \dots, x_n]$ in terms of an embedded resolution h defined over $\mathbb{Q}$, and valid for all primes p for which h has good reduction mod p. Using the notation of Section 4, we have

$$Z_p(f;s) = \frac{1}{p^n} \sum_{J \subseteq I} \sharp(\overline{\overset{\circ}{E}}_J(\mathbb{F}_p)) \prod_{i \in J} \frac{p-1}{p^{N_i s + \nu_i} - 1}.$$

We now indicate a funny heuristic argument to compute the limit of this expression, when p tends to 1 (whatever that means ...). First, using l'Hôpital's rule, $\frac{p-1}{p^{N_i s+\nu_i}-1}$ tends to $\frac{1}{N_i s+\nu_i}$. More challenging, by Grothendieck's trace formula, $\sharp(\overline{\overset{\circ}{E}}_J(\mathbb{F}_p))$ can be written as an alternating sum of traces of the pth power Frobenius operator, acting on adequate ℓ-adic cohomology groups of $\overline{\overset{\circ}{E}}_J$ [**De2**]. The limit when p tends to 1 of this operator is morally the identity operator, and then that alternating sum tends to the alternating sum of the dimensions of the cohomology groups of $\overline{\overset{\circ}{E}}_J$, that is, to an ℓ-adic Euler characteristic. Finally, a comparison theorem motivates to consider the usual topological Euler characteristic $\chi(\overset{\circ}{E}_J)$ of the complex points of $\overset{\circ}{E}_J$ as the limit of $\sharp(\overline{\overset{\circ}{E}}_J(\mathbb{F}_p))$. (It should be clear what we mean by $\overset{\circ}{E}_J$; we make it explicit below.) We conclude that heuristically

$$\lim_{p \to 1} Z_p(f;s) = \sum_{J \subseteq I} \chi(\overset{\circ}{E}_j) \prod_{i \in J} \frac{1}{N_i s + \nu_i}.$$

Denef and Loeser were inspired by this heuristic argument to define a new singularity invariant for any $f \in \mathbb{C}[x_1, \dots, x_n]$.

Consider an embedded resolution $h : Y \to \mathbb{A}^n_{\mathbb{C}}$ of f as in Theorem 4.1 with $K = \mathbb{C}$. Furthermore, for $J \subseteq I$, we denote $E_J = \bigcap_{j \in J} E_j$ and $\overset{\circ}{E}_J = E_J \setminus \bigcup_{k \notin J} E_k$. (In particular, $\overset{\circ}{E}_\emptyset = Y \setminus \bigcup_{i \in I} E_i$.)

Definition 7.1 (Denef-Loeser, 1991 [**DL1**]). Let $f \in \mathbb{C}[x_1, \dots, x_n] \setminus \mathbb{C}$ and choose an embedded resolution $h : Y \to \mathbb{A}^n_{\mathbb{C}}$ of f. The *topological zeta function of* f is

$$Z_{top}(s) = Z_{top}(f;s) := \sum_{J \subseteq I} \chi(\overset{\circ}{E}_J) \prod_{i \in J} \frac{1}{N_i s + \nu_i}.$$

Remark 7.2. It is not clear that the topological zeta function is well-defined, i.e., that it is independent of the chosen embedded resolution. Denef and Loeser proved this by turning the heuristic discussion above into an exact argument, using ℓ-adic interpolation. An easier and more geometric approach is to use the following *weak factorisation theorem* (that was not known in 1991). It reduces the problem to showing that the defining expression of $Z_{top}(s)$ is invariant under an admissible blow-up, which is an easy calculation.

Theorem 7.3 (Weak factorisation theorem, [**AKMW**][**Wł1**]).

(1) *Consider a proper birational map between smooth, irreducible varieties $\phi : Y \dashrightarrow Y'$ and an open set $U \subseteq Y$ such that the restriction of ϕ to U is an isomorphism. Then ϕ can be factored into a sequence of blow-ups and blow-downs with smooth centres disjoint from U.*

(2) *If $Y \backslash U$ and $Y' \backslash \phi(U)$ are normal crossings divisors, then that factorisation can be chosen such that at each step the images or inverse images of these divisors are also normal crossings divisors and such that the centres of blow-up intersect these divisors transversely.*

The topological zeta function is an interesting singularity invariant on its own, and also a useful test case for studying (the poles of) the Igusa zeta function.

QUESTION 7.4. It is a challenging open problem to give an intrinsic definition of the topological zeta function.

7.2. Motivic zeta function

Kontsevich suggested the idea of motivic integration (an analogue of p-adic integration) in a lecture at Orsay in 1995. This theory has been further developed by Denef and Loeser, amongst others. The motivic zeta function is defined as an analogue of the Igusa zeta function, using motivic integration instead of p-adic integration.

DEFINITION 7.5 (Grothendieck ring of k-varieties). Let k be any field. The Grothendieck group of varieties $K_0(\mathrm{Var}_k)$ is the quotient of the free group generated by the symbols $[X]$, where X runs over all varieties over k, by the relations

$$[X] = [Y] \text{ if } X \cong Y,$$
$$[X \setminus Y] + [Y] = [X] \text{ if } Y \text{ is a closed subvariety of } X.$$

Multiplication determined by

$$[X] \cdot [Y] := [X \times_k Y]$$

turns $K_0(\mathrm{Var}_k)$ into a commutative ring, called the *Grothendieck ring of k-varieties.* We denote by $\mathbb{L}$ the class of the affine line $\mathbb{A}^1_k$ and by $(K_0(\mathrm{Var}_k))_{\mathbb{L}}$ the localization of $K_0(\mathrm{Var}_k)$ with respect to $\mathbb{L}$.

The motivic zeta function is defined as an integral over the 'arc space' $k[[t]]^n$, which is the analogue of the space of p-adic integers $\mathbb{Z}_p^n$. It is a power series in the formal variable T, which is sometimes written as $\mathbb{L}^{-s}$ (with a formal s), to stress the analogy with p^{-s}. That arc space carries a natural *motivic measure* $d\mu$, with values in $(K_0(\mathrm{Var}_k))_{\mathbb{L}}$, which is the analogue of the Haar measure on $\mathbb{Z}_p^n$. For a detailed explanation, we refer to [**DL2**] and [**DL3**].

DEFINITION 7.6 (Motivic zeta function). Let k be a field of characteristic zero and $f \in k[x_1, \ldots, x_n]$. The associated *motivic zeta function* is a formal power series over $(K_0(\mathrm{Var}_k))_{\mathbb{L}}$ in the variable $T = \mathbb{L}^{-s}$, given by the motivic integral

$$Z_{mot}(f;s) = Z'_{mot}(f;T) := \int_{k[[t]]^n} (\mathbb{L}^{-s})^{\mathrm{ord}_t f(x)} d\mu.$$

More concretely, define

$$\mathcal{X}_i = \left\{ \gamma \in \left(\frac{k[[t]]}{(t^{i+1})} \right)^n \mid \mathrm{ord}_t f_i(\gamma) = i \right\},$$

where $f_i : \left({k[[t]]}/{(t^{i+1})}\right)^n \to {k[[t]]}/{(t^{i+1})}$ is the natural extension of $'f : k^n \to k$, and for $\gamma \in {k[[t]]}/{(t^{i+1})}$ we denote by $\mathrm{ord}_t(\gamma) \in \{0, 1, \ldots, i, +\infty\}$ the highest power of t

that divides γ. Then we have

$$Z_{mot}(f;s) = \frac{1}{\mathbb{L}^n} \sum_{i\geq 0} [\mathcal{X}_i]\mathbb{L}^{-in-is}.$$

In order to see also the analogy of this expression with the Igusa zeta function, verify that, for $f \in \mathbb{Q}_p[x_1, \ldots, x_n]$, we have

$$Z_p(f;s) = \frac{1}{p^n} \sum_{i\geq 0} \sharp \left(\left\{ a \in \left(\frac{\mathbb{Z}_p}{(p^{i+1})} \right)^n \mid \mathrm{ord}_p f(a) = i \right\} \right) p^{-in-is}.$$

The following table compares the notions related to the Igusa zeta function with those related to the motivic zeta function.

Igusa zeta function	**Motivic zeta function**
$f \in \mathbb{Z}[x_1, \ldots, x_n]$	$f \in k[x_1, \ldots, x_n]$
$M_i = \{$number of solutions of $f = 0$ over $\mathbb{Z}/p^i\mathbb{Z}\}$	$\mathcal{M}_i$ is the class of $\{$solutions of $f = 0$ over $k[t]/(t^i)\}$ in $K_0(\mathrm{Var}_k)$
$P(T) := \sum_{i\geq 0} M_i(p^{-n}T)^i$	$\mathcal{P}(T) := \sum_{i\geq 0} \mathcal{M}_i(\mathbb{L}^{-n}T)^i \in K_0(\mathrm{Var}_k)_{\mathbb{L}}[[T]]$
$\mathbb{Z}_p = \varprojlim_i \mathbb{Z}/p^i\mathbb{Z}$	$k[[t]] = \varprojlim_i k[t]/(t^i)$
$Z_p(f;s) := \int_{\mathbb{Z}_p^n} (p^{-s})^{\mathrm{ord}_p f(x)} \lvert dx \rvert$	$Z_{mot}(f;s) := \int_{k[[t]]^n} (\mathbb{L}^{-s})^{\mathrm{ord}_t f(x)} d\mu$
$P(T) \rightsquigarrow Z_p(f;s)$	$\mathcal{P}(T) \rightsquigarrow Z_{mot}(f;s)$

With the last line, we mean that $Z_{mot}(f;s)$ is related to $\mathcal{P}(T)$, similarly as in Proposition 2.8 for the p-adic case.

THEOREM 7.7 (Denef-Loeser [**DL2**]). *Let k be a field of characteristic zero and $f \in k[x_1, \ldots, x_n] \setminus k$. Consider an embedded resolution $h : Y \to \mathbb{A}_k^n$ of f as in Theorem 4.1, and put again $\overset{\circ}{E}_J = (\bigcap_{j\in J} E_j) \setminus (\bigcup_{k\notin J)} E_k$ for $J \subseteq I$. Then*

$$Z_{mot}(f;s) = \frac{1}{\mathbb{L}^n} \sum_{J\subseteq I} [\overset{\circ}{E}_J] \prod_{i\in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{N_i s+\nu_i} - 1}.$$

In particular, this implies that $Z_{mot}(s)$ and $\mathcal{P}(T)$ are rational functions in $T = \mathbb{L}^{-s}$.

REMARK 7.8 ([**DL2**]).

(1) When $k = \mathbb{C}$, the motivic zeta function $Z_{mot}(f;s)$ specialises to the topological zeta function $Z_{top}(f;s)$. This gives a new proof that $Z_{top}(f;s)$ is independent of the chosen resolution.
(2) When $k = \mathbb{Q}$, the motivic zeta function $Z_{mot}(f;s)$ specialises to the Igusa zeta functions $Z_p(f;s)$ for all but finitely many p.

QUESTION 7.9. Take again $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and fix a prime number p. Define

$$M_i' := \sharp \left\{ \text{solutions of } f = 0 \text{ in } \left(\frac{\mathbb{F}_p[u]}{(u^i)} \right)^n \right\}.$$

Define $P'(T) := \sum_{i\geq 0} M_i'(p^{-n}T)^i$. It is an open problem whether $P'(T)$ is a rational function in T.

REMARK 7.10. For a given $f \in \mathbb{Z}[x_1,\dots,x_n]$, it is known that $P'(T) = P(T)$ for all large enough p.

8. MISCELLANEOUS

We briefly indicate some generalisations and problems that are not treated here.

8.1. Number fields

Everything discussed before generalises to number fields in the following way. Let F be a number field with ring of integers $\mathcal{O}$, and fix a maximal ideal $\mathfrak{p}$ of $\mathcal{O}$. Instead of counting solutions of $f \in \mathbb{Z}[x_1,\dots,x_n]$ in $({}^{\mathbb{Z}}/_{p^k\mathbb{Z}})^n$, we count solutions of $f \in \mathcal{O}[x_1,\dots,x_n]$ in $({}^{\mathcal{O}}/_{\mathfrak{p}^k\mathcal{O}})^n$.

$$\begin{tikzcd} \mathbb{Q} \arrow[d, dotted] & \mathbb{Z} \arrow[l, hook'] \arrow[d, dotted] & \text{prime } p \arrow[d, dotted] \\ \text{number field } F & \text{ring of integers } \mathcal{O} \arrow[l, hook'] & \text{maximal ideal } \mathfrak{p} \end{tikzcd}$$

Similarly, $\mathbb{Q}_p$ and $\mathbb{Z}_p$ can be replaced by a finite extension K of $\mathbb{Q}_p$ and its valuation ring $\mathcal{R}$.

$$\begin{tikzcd} \mathbb{Q}_p \arrow[d, dotted] & \mathbb{Z}_p \arrow[l, hook'] \arrow[d, dotted] \\ \text{finite field extension } K & \text{valuation ring } \mathcal{R} \arrow[l, hook'] \end{tikzcd}$$

The corresponding Igusa zeta function in this setting is (the meromorphic continuation of) $\int_{\mathcal{R}^n} |f(x)|_{\mathfrak{p}}^s\,|dx|$.

8.2. Contribution and non-contribution when n $\geq$ 3

For polynomials in $n = 2$ variables, the determination of the poles of the associated Igusa zeta function $Z(s)$ is well-understood, as discussed in Section 6. However, as soon as n is at least 3, this is an open problem. For some partial results concerning non-contribution of candidate poles, mainly for $n = 3$, we refer to [**Ve3**] and [**Ve6**]. For some contribution results in arbitrary dimension, see [**Ro**].

8.3. Congruences of multiple polynomials

Instead of counting the solutions of a single polynomial, one can generalise this to studying congruences of several polynomials. More precisely, given polynomials $f_1,\dots,f_r \in \mathbb{Z}[x_1,\dots,x_n]$ and a fixed prime number p, define $M_i = \{a \in ({}^{\mathbb{Z}}/_{p^i\mathbb{Z}})^n \mid f_k(a) \equiv 0 \bmod p^i \text{ for all } k = 1,\dots,r\}$ for $i \in \mathbb{Z}_{\geq 0}$. Once again, the goal is the understand/compute these numbers. In this setting there is also a corresponding Igusa zeta function $Z(s)$, related to the generating Poincaré series of the M_i; it is defined as

$$Z(s) := \int_{\mathbb{Z}_p^n} (\max_i |f_i|_p)^s |dx|,$$

where $\max_i |\cdot|_p$ is the usual norm on $\mathbb{Q}_p^r$. For results in this setting we refer to [**VZ1**].

8.4. Monodromy conjecture

One of the most intriguing conjectures in singularity theory is the *monodromy conjecture*. Given $f \in \mathbb{Q}[x_1, \ldots, x_n]$, it relates the poles of its associated p-adic Igusa zeta functions (for all but finitely many p), which are number theoretic invariants of f, to eigenvalues of local monodromy of f, which are geometric/topological invariants of f, considered as function on $\mathbb{C}^n$. A variant is also considered for the topological and motivic zeta functions. See for example [**Ni**] for an introduction to this topic.

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