Group Geodesic Growth

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Definitions

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Recall that usual growth is defined as

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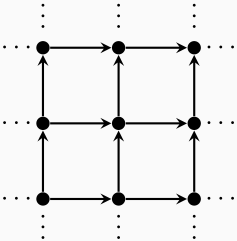
Clearly,

$$\gamma_X(n) \leqslant \Gamma_X(n) \leqslant |X| \left(|X| - 1\right)^{n-1}$$

Example 1: Different Growth Classes



Presentation:

$$\mathbb{Z}^2 = \langle a, b \,|\, [a, b] \rangle$$

Regular Growth:

$$\gamma_{\{a^{\pm 1}, b^{\pm 1}\}}(n) = 2n^2 + 2n + 1$$

Geodesic Growth:

$$\Gamma_{\{a^{\pm 1}, b^{\pm 1}\}}(n) = 2^{n+3} - 4n - 7$$

Example 2: Every Group has Exponential Geodesic Growth

Presentation:

$$\mathbb{Z} = \langle z \mid - \rangle \qquad \mathbb{Z} = \langle a, b \mid a = b \rangle$$

Usual Growth:

$$\gamma_{\{z^{\pm 1}\}}(n) = 2n + 1 \qquad \gamma_{\{a^{\pm 1}, b^{\pm 1}\}}(n) = 2n + 1$$

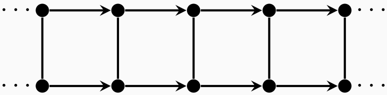
Geodesic Growth:

$$\Gamma_{\{z^{\pm 1}\}}(n) = 2n + 1 \qquad \Gamma_{\{a^{\pm 1}, b^{\pm 1}\}}(n) = 2^{n+2} - 3$$

Example 3:[1] Different Geodesic Growth Rates

Presentation:

$$\left\langle a, t \;\middle|\; t^2, [a,t] \right\rangle$$



Geodesic Growth Rate:

$$\Gamma_{\{a^{\pm 1}, t\}}(n) = n^2 + 3n \quad \textit{(for } n \geqslant 2\textit{)}$$

[1] Bridson, Burillo, M. Elder and Z. Šunić, 'On groups whose geodesic growth is polynomial', 2012.

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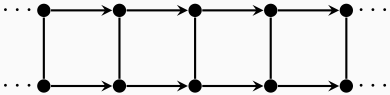

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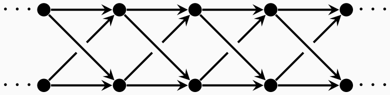
Titze Transform: (where $c = at$)

$$\left\langle a, c \;\middle|\; a^2 = c^2, [a,c] \right\rangle$$



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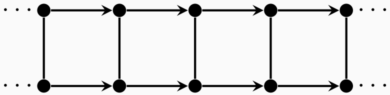

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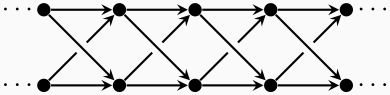

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Geodesic Growth of $\mathbb{Z}^2$

Theorem (Proposition 10[2])
$\mathbb{Z}^2$ *has exponential geodesic growth w.r.t. any finite generating set*

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Geodesic Growth of $\mathbb{Z}^2$

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Theorem (Corollary 11[2])
If G maps homomorphically onto $\mathbb{Z}^2$, then G has exponential geodesic growth w.r.t. any finite generating set

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Example 4:[3] Virtually $\mathbb{Z}^2$

Presentation:

$$\left\langle a, b, t \,\middle|\, [a, b], t^2, a^t = b \right\rangle$$

Usual Growth:

$$\gamma_{\{a^{\pm 1}, b^{\pm 1}, t\}}(n) = 4n^2 + 2 \qquad \textit{(for } n \geqslant 2\textit{)}$$

Geodesic Growth:

$$\Gamma_{\{a^{\pm 1}, b^{\pm 1}, t\}}(n) = (n + 1) \cdot 2^{n+2} - 2n^2 - 6n - 2 \qquad \textit{(for } n \geqslant 5\textit{)}$$

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Geodesic Growth:

$$\Gamma_{\{a^{\pm 1}, t^{\pm 1}\}}(n) = \frac{2n^3 - 2n + 18}{3} \qquad \textit{(for } n \geqslant 5\textit{)}$$

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Is there a group with intermediate geodesic growth?

Intermediate Usual Growth

What about Grigorchuk's group?

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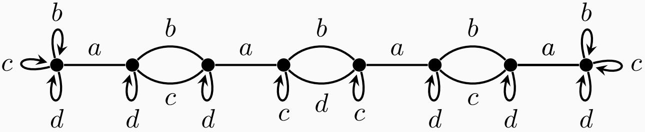
Consider the Schreier graphs



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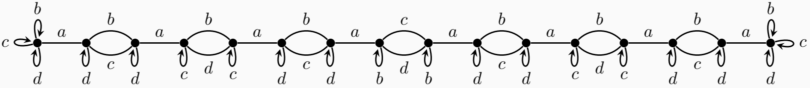

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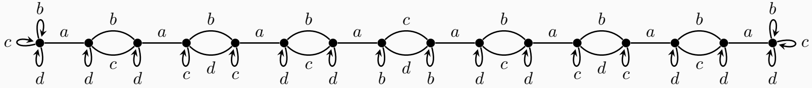

The geodesic growth is exponential[4]

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Other Candidates for this Method

Following this idea: (Brönnimann[5])

Grigorchuk G_ω:

Gupta-Sidki p-groups:

Square group:

Spinal group:

Gupta-Fabrykowski:

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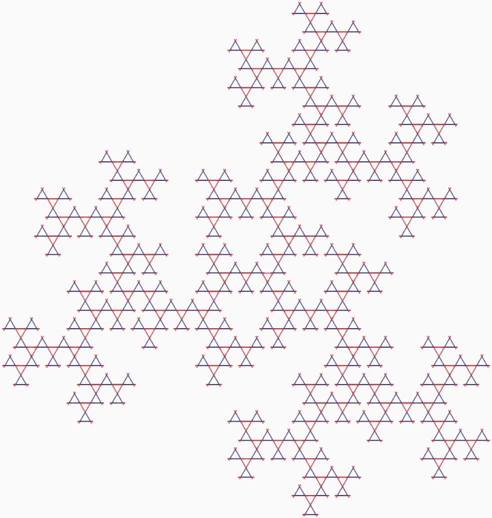
Gupta-Fabrykowski:	...technique doesn't work

[5] Brönnimann, 'Geodesic growth of groups', 2016.

Schreier Graph for Gupta-Fabrykowski



[6] Bartholdi and Grigorchuk, 'On the spectrum of Hecke type operators related to some fractal groups', 2000.

[7] Brönnimann, 'Geodesic growth of groups', 2016.

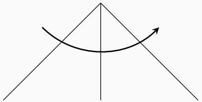
Gupta-Fabrykowski

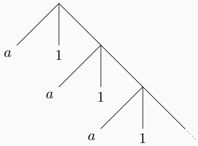
Let $X = \{1,2,3\}$ and $\mathcal{T}_3 = X^\omega$.

Considering the wreath recursion $\mathrm{Aut}(\mathcal{T}_3) = \mathrm{Aut}(\mathcal{T}_3) \wr \mathrm{Sym}(X)$ then

$$a = (1,1,1)\cdot\sigma \qquad\qquad b = (a,1,b)\cdot 1$$

where $\sigma = (1\ 2\ 3)$ is a cyclic permutation of X.





Then, together a and b generate the Gupta-Fabrykowski group.

Current Research: Experimental Mathematics

I wrote a computer program for generating the geodesics of the Gupta-Fabrykowski group.

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- try to find an exponential growth sub-family

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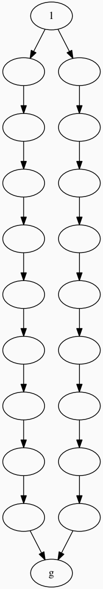
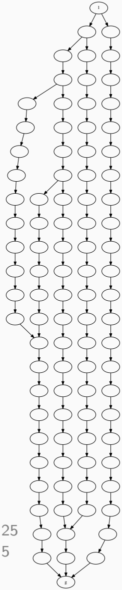
Why?

- try to find an exponential growth sub-family
- or some pattern to guide a proof of intermediate growth

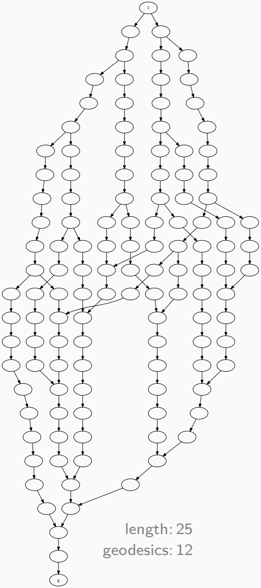
Geodesic Patterns



length: 10
geodesics: 2



length: 25
geodesics: 5



length: 25
geodesics: 12

The Program

I wrote a C++11 program which, in 2 weeks, generated to length 38 (producing over 3.7 TB of output)

Naïve brute-force method:

- add a letter to the previous length geodesics
- then, remove all non-geodesics by using the word problem

My method:

- add a letter to the previous length geodesics
- then, perform a modified *merge sort* to remove non-geodesics

Remarks:

- works as the group is contracting & self-similar

Potential Techniques

- Consider the number of geodesics per element

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- Consider the number of geodesics per element
    - Show a bounded upper bound

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- Consider the number of geodesics per element
  - Show a polynomial upper bound

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    - Show a sub-exponential upper bound

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- Consider the number of geodesics per element
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    - would require a more exotic language class

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Thanks!